C*.1 Let A be a C*-algebra. Show the following:

(a) If $a, b \in A$ are self-adjoint elements such that $a \leq b$ and $c \in A$, then $c^*ac \leq c^*bc$.
[**Hint:** Take a square root and use the fact that elements of the form x^*x are positive.]

(b) Assuming A is a unital C*-algebra and $a \in A$ positive, show that $a \leq \|a\|1$. Moreover, $\|a\| \leq 1$ iff $a \leq 1$. In this case we also have $1 - a \leq 1$ and $\|1 - a\| \leq 1$.

C*.2 Suppose A is a C*-algebra with closed two-sided ideal $J \triangleleft A$ and C*-subalgebra $I \subset A$ such that $I \triangleleft J$. Show that $I \triangleleft A$.

W*.1 Prove Lemma 2.1.2: Let $\mathcal{H}$ be a Hilbert space and suppose $q\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is linear in the first coordinate, conjugate linear in the second coordinate, and there exists $C > 0$ such that $|q(\xi, \eta)| \leq C\|\xi\|\|\eta\|$ for all $\xi, \eta \in \mathcal{H}$. Then there exists a unique $x \in B(\mathcal{H})$ satisfying

$$\langle x\xi, \eta\rangle = q(\xi, \eta) \qquad \forall \xi, \eta \in \mathcal{H},$$

and $\|x\| \leq C$.

[**Hint:** First fix $\xi \in \mathcal{H}$ and show for all $\eta \in \mathcal{H}$ that $q(\xi, \eta) = \langle \xi_1, \eta\rangle$ for some $\xi_1 \in \mathcal{H}$. Then show that $x(\xi) := \xi_1$ defines a bounded operator $x \in B(\mathcal{H})$.]

W*.2 Let $\mathcal{H}$ be a Hilbert space and let $p \in B(\mathcal{H})$ be a non-trivial projection: $p \neq 0$ and $p \neq 1$. Show that the algebra $A := pB(\mathcal{H})p$ has no cyclic vectors.