3. Positive elements

Preview of Lecture:

- Exercise 3.3 is a cornerstone of the theory of C*-algebras; see if you can figure out why it's true before lecture!
- In lecture, we will prove Proposition 3.6 and Example 3.9.
- We will not prove Corollary 3.7 in lecture.
- Theorem 3.10 is really important, and the proof uses all of the exercises that precede it in this section, but it's otherwise pretty straightforward. We won't discuss the proof.
- We will discuss the proofs of Proposition 3.12 and Corollary 3.13 in lecture.

The Functional Calculus is an incredibly powerful tool for handling normal elements. Of course, not every element in a C*-algebra is normal. Nonetheless, by associating to each element $a \in A$ the self-adjoint element $a^*a \in A$, we have been able to spread the influence of the functional calculus to an entire non-commutative C*-algebra. It turns out that elements of the form a^*a take on an even more important structural role in C*-algebras, which we will explore now.

Definition 3.1. A self-adjoint element a in a C*-algebra A is *positive* if $\sigma(a) \subset [0, \infty)$. We denote this by $a \geq 0$.

This allows us to define a partial ordering on the self-adjoint elements of A: for a and b self-adjoint, we say $a \leq b$ if $b - a \geq 0$.

Example 3.2. The positive elements in $C_0((0,1])$ are exactly the ones whose range (i.e. spectrum) lies in $[0, \infty)$.

Let's start with a few observations using the functional calculus:

Exercise 3.3. Each positive element in a C*-algebra has a unique positive square root.

Exercise 3.4. If $a \in A$ is a self-adjoint element, then there exist positive elements a_+ and a_- such that $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$.

Exercise 3.5. Let $a \in A$ be self-adjoint, a_+ and a_- its positive and negative parts as in Exercise 3.4, and $\sqrt{a_+}$ and $\sqrt{a_-}$ their respective unique positive square roots. Show that $a_+\sqrt{a_-} = 0$ and $\sqrt{a_+}\sqrt{a_-} = 0$.

The following proposition is mostly technically useful.

Proposition 3.6. *Let a be a self-adjoint element in a unital* C*-*algebra A. Then the following are equivalent.*

(1) $a \geq 0$;
(2) $a = b^2$ *for some self-adjoint* $b \in A$;
(3) $\|\alpha 1 - a\| \leq \alpha$ *for all* $\alpha \geq \|a\|$;
(4) $\|\alpha 1 - a\| \leq \alpha$ *for some* $\alpha \geq \|a\|$.

Proof. We assume A is unital or pass to its unitization.

That (1) ⇒ (2) follows from the functional calculus, and that (3) ⇒ (4) is clear.

Assume (2). Let $f \in C(\sigma(b))$ be given by $f(z) = z^2$. Then

$$\|f\|_{\sup} = \|b^2\| = \|a\|,$$

and so $0 \leq f \leq \|a\|$. Then $0 \leq \alpha - f \leq \alpha$ for any $\alpha \geq \|a\|$. Then (identifying α with the constant function on $\sigma(b)$ when appropriate), we compute

$$\|\alpha 1 - a\| = \|\alpha(b) - f(b)\| = \|(\alpha - f)(b)\| = \|\alpha - f\|_{\sup} \leq \alpha.$$

It remains to show (4) ⇒ (1). Suppose $\alpha \geq \|a\|$ is such that $\|\alpha 1 - a\| \leq \alpha$. Let $h(z) = z$ denote the identity function on $\sigma(a)$. Then we have

$$\alpha \geq \|\alpha 1 - a\| = \|(\alpha - h)(a)\| = \|\alpha - h\|_{\sup} = \sup_{\lambda \in \sigma(a)} |\alpha - \lambda|.$$

It follows that $\sigma(a) \subset [0, \infty)$. Since a was assumed to be self-adjoint, this means $a \geq 0$. □

Some concluding notation: The collection of positive elements in a C*-algebra A is denoted by A_+, and the self-adjoints are often denoted by $A_{s.a.}$

Corollary 3.7. *For a* C*-*algebra* A, *the sets* $A_{s.a.}$ *and* A_+ *are both closed.*

Proof. Suppose x_n is a sequence in $A_{s.a.}$ converging to $x \in A$. Then

$$\|x_n^* - x^*\| = \|x_n - x\| \to 0,$$

and so $x_n = x_n^* \to x^*$. Hence $x^* = x$. Now, suppose $(a_n) \in A_+$ converges to $a \in A$. Then we know $a = a^*$ and $\|a_n\| \to \|a\|$. Assume A is unital or unitize. Let $\alpha = \sup_n \|a_n\| \geq \|a\|$. Then $\alpha 1 - a_n \to \alpha 1 - a$, and $\|\alpha 1 - a_n\| \leq \alpha$ for all n by Proposition 3.6. It follows that $\|\alpha 1 - a\| \leq \alpha$, which again by Proposition 3.6 implies that a is positive. □

Exercise 3.8. If $a, b \in A$ are positive, then so is $a + b$. (Note that we are not assuming they commute – use the previous exercise.) If a and b moreover commute, then $ab \geq 0$. Can you think of two positive elements in a C*-algebra whose product is not positive? (Hint: For the first part, you can assume A is unital or work in $\tilde{A}$ (why?). Then use Proposition 3.6. For the second part consider operators in $M_2(\mathbb{C})$.)

Example 3.9. The positive operators in $B(\mathcal{H})$ are exactly the positive semi-definite operators.

Suppose $T \in B(\mathcal{H})$. By the preceeding proposition, if $T \geq 0$, then there exists a self-adjoint $S \in B(\mathcal{H})$ such that $T = S^2 = S^*S$. Then for any $x \in \mathcal{H}$, we have

$$\langle Tx, x\rangle = \langle S^*Sx, x\rangle = \langle Sx, Sx\rangle = \|Sx\|^2 \geq 0.$$

Now, suppose $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. By Exercise 7.42 from Day 1 lecture notes, $T = T^*$ and so $\sigma(T) \subset \mathbb{R}$. So, given $\lambda < 0$ we want to show that $T - \lambda I$ is invertible. If $\lambda < 0$, then for every nonzero $x \in \mathcal{H}$,

$$\begin{aligned}
\|(T - \lambda I)x\|^2 &= |\langle (T - \lambda I)x, (T - \lambda I)x\rangle| \\
&= |\|Tx\|^2 + 2|\lambda|\langle Tx, x\rangle + |\lambda|^2\|x\|^2| \\
&= \|Tx\|^2 + 2|\lambda|\langle Tx, x\rangle + |\lambda|^2\|x\|^2 \\
&\geq |\lambda|^2\|x\|^2.
\end{aligned}$$

That means that for every $x \in \mathcal{H}$, $\|(T - \lambda I)x\| \geq |\lambda|\|x\|$. In other words, the operator $T - \lambda I$ is bounded below, which means it is injective (Exercise 7.46 from Day 1 lecture notes). So, by the Open Mapping Theorem, to show that $T - \lambda I$ is invertible, it remains to show that it is surjective.

For any operator $S \in B(\mathcal{H})$, $\ker(S) = (S^*(\mathcal{H}))^\perp$ (Exercise 7.44 from Day 1 lecture notes). Since $(T - \lambda I) = (T - \lambda I)^*$, the above argument shows that $\ker(T - \lambda I) = 0 = ((T - \lambda I)(\mathcal{H}))^\perp$, which means $T - \lambda$ is surjective and thus invertible.

Theorem 3.10. *For any* $a \in A$, *the element* a^*a *is positive.*

Proof. Suppose $b = a^*a \in A$. Then b is self-adjoint, and hence by Exercise 3.4, we can write it as $b = b_+ - b_-$ for some $b_+, b_- \geq 0$ with $b_+b_- = 0$. We want to show that $b_- = 0$. Since it is self-adjoint, we know $\|b_-\| = r(\sigma(b_-))$, and so it suffices to show that $\sigma(b_-) = \{0\}$. Now, for notational ease, we write $c = a\sqrt{b_-}$, where $\sqrt{b_-}$ is its unique positive square root. By Exercise 3.5, we have that $\sqrt{b_-}b_+ = 0$, and so we compute

$$-c^*c = -\sqrt{b_-}a^*a\sqrt{b_-} = -\sqrt{b_-}b\sqrt{b_-} = -\sqrt{b_-}(b_+ - b_-)\sqrt{b_-} = b_-^2.$$

Then $-c^*c = b_-^2 \geq 0$, which means $\sigma(-c^*c) \subset [0, \infty)$.

Write $c = \mathrm{Re}(c) + i\mathrm{Im}(c)$ as in (1.1). Then we compute

$$\begin{aligned}
cc^* &= [c^*c + cc^*] - c^*c \\
&= [(\mathrm{Re}(c) + i\mathrm{Im}(c))^*(\mathrm{Re}(c) + i\mathrm{Im}(c)) + (\mathrm{Re}(c) + i\mathrm{Im}(c))(\mathrm{Re}(c) + i\mathrm{Im}(c))^*] - c^*c \\
&= 2(\mathrm{Re}(c)^2 + \mathrm{Im}(c)^2) + b_-^2.
\end{aligned}$$

Then cc^* is the sum of positive elements, and hence is positive. Since[2] $\sigma(cc^*) \cup \{0\} = \sigma(cc^*) \cup \{0\}$, it follows that both cc^* and c^*c have non-negative spectra, which means both are positive. But then we've shown that

[2]This is a more general ring theoretic fact that $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$ for any x, y in a complex unital ring. Indeed, if $0 \neq \lambda \notin \sigma(xy)$, then there exists z such that $z(\lambda - xy) = 1 = (\lambda - xy)z$. Then $\lambda^{-1}(\lambda + yzx)$ is the inverse of $\lambda - yx$. Check this if you haven't seen it before!

$\pm c^*c$ are both positive. It follows that $\sigma(c^*c) = \{0\}$. Since c^*c is self-adjoint, its norm is its spectral radius, and so

$$0 = \|c^*c\| = \| - c^*c\| = \|b_-^2\| = \|b_-\|^2,$$

and we are done. □

Exercise 3.11. Let A be a C*-algebra. Show the following:

(1) If $a, b \in A$ are self-adjoint and $c \in A$, then $c^*ac \leq c^*bc$. (Hint: Take a square root and use the previous theorem.)
(2) Assuming A is a unital C*-algebra and $a \in A$ positive, show that $a \leq \|a\|1$. Moreover, $\|a\| \leq 1$ iff $a \leq 1$. In this case we also have $1 - a \leq 1$ and $\|1 - a\| \leq 1$.
(3) If A is unital and $a \in A$ is invertible, then so is a^*, a^*a, and $\sqrt{a^*a}$. Moreover, the inverses are in $C^*(a)$.

3.1. **Polar decomposition.** For each $a \in A$, we define the positive operator $|a|$ to be the unique positive square root of a^*a, i.e.

$$|a| = \sqrt{a^*a}.$$

Proposition 3.12. *For each operator $T \in B(\mathcal{H})$, there is a unique partial isometry $U \in B(\mathcal{H})$ with* $\ker(U) = \ker(T)$ *and $U|T| = T$. Moreover $|T| \in C^*(T)$ and $U \in C^*(T)''$. If T is invertible, then U is a unitary.*

The description $T = U|T|$ is called the *polar decomposition* of T, in analogy with the fact that every complex number z can be written as a norm-1 element e^{it}, times a non-negative real number r.

Proof. Note that for all $\xi \in H$, we have

$$\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle = \langle |T|^2\xi, \xi\rangle = \langle |T|\xi, |T|\xi\rangle = \||T|\xi\|^2. \tag{3.1}$$

It follows that the linear map $U_0 : |T|\mathcal{H} \to T\mathcal{H}$ given by $|T|x \mapsto Tx$ is isometric, and hence extends to an isometry $\overline{|T|\mathcal{H}} \to \overline{T\mathcal{H}}$ (also denoted U_0). We define $U \in B(\mathcal{H})$ to be U_0 on $\overline{|T|\mathcal{H}}$ and 0 on $(|T|\mathcal{H})^\perp$. It follows from Exercise 7.39 from the Day 1 Lecture Notes that U is a partial isometry with $U^*|_{\overline{T\mathcal{H}}} = U_0^{-1}$ and $\ker(U^*) = (T\mathcal{H})^\perp$, and by definition $U|T| = T$. Moreover, we have from (3.1) and Exercise 7.44 from the Day 1 Lecture Notes that $\ker(U) = |T|(\mathcal{H})^\perp = \ker(|T|) = \ker(T)$. For uniqueness, suppose $V \in B(\mathcal{H})$ is another partial isometry with $\ker(V) = \ker(T)$ and $V|T| = T$. Since $V|_{|T|\mathcal{H}} = U|_{|T|\mathcal{H}}$, it follows from continuity that they also agree on $\overline{|T|\mathcal{H}}$. As $\ker(V) = \ker(T) = \ker(U) = (|T|(\mathcal{H}))^\perp$ by construction, the fact that $\mathcal{H} = \overline{|T|\mathcal{H}} \oplus (|T|\mathcal{H})^\perp$ implies that $V\xi = U\xi$ for any $\xi \in \mathcal{H}$.

It follows from the functional calculus that $|T| \in C^*(T)$. Now, suppose $S \in C^*(T)'$. If $\xi \in \ker(T) = \ker(U)$, then $TS\xi = ST\xi = 0$ and so $S\xi \in \ker(T) = \ker(U)$. Then $US\xi = 0 = SU\xi$ for every $\xi \in \ker(T) = (|T|\mathcal{H})^\perp$. For $\xi = |T|\eta \in |T|\mathcal{H}$, we have

$$US\xi = US|T|\eta = U|T|S\eta = TS\eta = ST\eta = SU|T|\eta = SU\xi.$$

Since $|T|\mathcal{H}$ is dense in $\overline{|T|\mathcal{H}}$, it follows that $US = SU$ on $\overline{|T|\mathcal{H}}$ and on $(|T|\mathcal{H})^\perp$. Then it follows by a linearity argument as above that S and U commute. Hence $U \in C^*(T)''$.

Finally, if T is invertible, then so is $\sqrt{T^*T}$. Then we have

$$U = T(T^*T)^{-1/2},$$

and one checks that $U^*U = UU^* = I$. □

As the range space of U is $T\mathcal{H}$, and U is a partial isometry, it follows that $UU^* = \text{proj}_{T\mathcal{H}}$. Similarly, the source projection of U is $U^*U = \text{proj}_{|T|\mathcal{H}}$.

Corollary 3.13. *Let $T \in B(\mathcal{H})$ with polar decomposition $T = U|T|$. Then $|T^*| = U|T|U^*$ and $T^* = U^*|T^*|$.*

Proof. Observe that $U|T|U^*$ is positive, and since $T^* = (U|T|)^* = |T|U^*$

$$(U|T|U^*)(U|T|U^*) = U|T|^2U^* = TT^*.$$

By the uniqueness of the square root, we have $U|T|U^* = (TT^*)^{1/2} = |T^*|$. From this we further deduce

$$U^*|T^*| = U^*U|T|U^* = |T|U^* = (U|T|)^* = T^*.$$ □

Exercise 3.14. Where possible, give geometric as well as algebraic explanations for the following statements about the polar decomposition:

(1) $U^*U|T| = |T|$,
(2) $U^*T = |T|$, and
(3) $UU^*T = T$.

Exercise 3.15. Show that T is compact iff $|T|$ is compact.

Exercise 3.16. Describe the projections in $C_0(X)$ where X is

(1) $(0, 1]$,
(2) $[0, 1]$,
(3) $[0, 1/3] \cup [1/3, 1]$.

Exercise 3.17. Let $\pi : A \to B$ be a surjective $*$-homomorphism between C*-algebras and $b \in B$ a positive element. Show that b lifts to a positive element $a \in A$ – that is, there is $a \in A$ with $\pi(a) = b$ – such that $\|a\| = \|b\|$.

4. Ideals, Approximate Units, and $*$-homomorphisms

Preview of Lecture: To help guide your reading, we indicate here which of the following material we will address in lecture and which we will assume familiarity with:

The lecture for this section will focus on Theorem 4.11. The techniques in the proofs of Lemma 4.6 and Theorem 4.9 do not translate well to lecture, but that does not detract from their importance. In fact, they showcase a powerful yet technical tool: an approximate unit. Many C*-algebraists (guilty!) are intimidated by these at first. But the first time you use them in your own research, you'll love them for life.

Definition 4.1. An *approximate identity* in a C*-algebra A is an increasing net $(e_\lambda)_{\lambda\in\Lambda}$ of positive contractive elements (i.e. $0 \le e_\gamma \le e_\lambda$ and $\|e_\lambda\| \le 1$ for all $\lambda, \gamma \in \Lambda$ with $\lambda \ge \gamma$) such that

$$\lim_\lambda \|e_\lambda a - a\| = \lim_\lambda \|a e_\lambda - a\| = \lim_\lambda \|e_\lambda a e_\lambda - a\| = 0.$$

Theorem 4.2. *Every* C**-algebra has an approximate identity. Moreover, if the* C**-algebra is separable, the identity can be chosen to be countable.*

The proof relies heavily on the functional calculus. We will not give it here, though it is not too sophisticated. Instead, we point to the proofs given in [4, Theorem 1.4.8] or [6, Theorem 3.1.1].

Remark 4.3. A C*-algebra with a countable approximate identity is called *σ-unital*. Any separable C*-algebra is σ-unital, but there exist non-separable σ-unital C*-algebras. A silly example is $B(\ell^2)$ since it's actually unital; a non-silly example is $C_0(X)$ where X is a locally compact but not σ-compact Hausdorff space. Many results that hold in the separable setting can be generalized to the σ-unital setting.

There are a few interesting characterizations of a σ-unital C*-algebra, such as containing a *strictly positive element*, which is an element $h \in A$ such that $\phi(h) > 0$ for every nonzero positive $\phi \in A^*$. For more on this, see [9, Section 3.10].

Example 4.4. In $K(\ell^2)$, the projections

$$P_n : (\xi_n)_n \mapsto (\xi_1, ..., \xi_n, 0, 0, ...),$$

form an approximate identity.

For a general Hilbert space, we form the approximate identity for the compact operators by nets of projections with finite rank where the order is given by the natural order on the projections, i.e. $p \le q$ iff $pq = qp = p$.

Exercise 4.5. Determine an approximate identity for $C_0((0, 1])$. (A sketch will do.)

Here is a quick application of approximate units.

Lemma 4.6. *Every closed two-sided ideal in a* C**-algebra is self-adjoint.*

Proof. Let J be a closed two-sided ideal in A. Then $B = J \cap J^*$ is a C*-subalgebra of A such that $x^*x, xx^* \in B$ for all $x \in J$. Let (e_λ) be an approximate identity for B. Then for any $x \in J$, we have $x^*x - xx^*e_\lambda \in J$ and hence

$$\begin{aligned}\lim_\lambda \|x^* - x^*e_\lambda\|^2 &= \lim_\lambda \|(x - e_\lambda x)(x^* - x^*e_\lambda)\| \\ &= \lim_\lambda \|(xx^* - xx^*e_\lambda) - e_\lambda(xx^* - xx^*e_\lambda)\| = 0.\end{aligned}$$

Since $x^*e_\lambda \in J$, it follows that $x^* \in J$ and so $J = J^*$. $\square$

This means that every ideal in a C*-algebra is a C*-subalgebra, which means that each ideal has an approximate unit. In fact, more is true. We say a net (a_λ) in a C*-algebra A is *quasi-central* if $\lim_\lambda \|a_\lambda b - ba_\lambda\| = 0$ for every $b \in A$. We have the following extension of the above theorem ([4, Theorem I.9.16]).

Theorem 4.7. *Every ideal of a* C**-algebra has a quasi-central approximate unit.*

Exercise 4.8. Suppose A is a C*-algebra with closed two-sided ideal $J \triangleleft A$ and C*-subalgebra $I \subset A$ such that $I \triangleleft J$. Show that $I \triangleleft A$.

An approximate identity will also enable us to prove that the quotient of any C*-algebra by a closed two-sided ideal is again a C*-algebra.

Theorem 4.9. *Let A be a* C**-algebra and $J \triangleleft A$. Then A/J is a* C**-algebra.*

Proof. Since $J \subset A$ is a Banach subalgebra, a basic result from functional analysis (cf. [3, Theorems III.4.2 and VII.2.6]) implies that A/J is a Banach algebra under the norm $\|a + J\| = \inf_{x\in J} \|a + x\|$. (**Exercise:** Prove it!) Moreover, from the fact that $\|b\| = \|b^*\|$ for all $b \in A$, a two-line calculation shows that $\|a + J\| = \|a^* + J\|$ for all $a \in A$. So, we just check the C*-identity for $\|a + J\| = \inf_{x\in J} \|a + x\|$. Let $a \in A$ and (e_λ) an approximate identity for J. First, we claim that $\|a + J\| = \lim_\lambda \|a - ae_\lambda\|$. Since $ae_\lambda \in J$ for each λ, the $\leq$ inequality is clear. For the other direction, let $\epsilon > 0$ and $x \in J$ such that $\|a + J\| + \epsilon > \|a - x\|$. By possibly passing to $\tilde{A}$, we assume A is unital. Then by Exercise 3.11, $\|1 - e_\lambda\| \leq 1$, and

$$\begin{aligned}\lim_\lambda \|a - ae_\lambda\| &\leq \lim_\lambda \|(a - x)(1 - e_\lambda)\| + \|x - xe_\lambda\| \\ &\leq \lim_\lambda \|a - x\| + \|x - xe_\lambda\| \\ &\leq \|a - x\| < \|a + J\| + \epsilon.\end{aligned}$$

Now, we can check the C*-norm:

$$\begin{aligned}\|(a + J)^*(a + J)\| = \|a^*a + J\| = \lim_\lambda \|a^*a(1 - e_\lambda)\| &\geq \lim_\lambda \|1 - e_\lambda\|\|a^*a(1 - e_\lambda)\| \\ &\geq \lim_\lambda \|(1 - e_\lambda)aa^*(1 - e_\lambda)\| = \lim_\lambda \|a(1 - e_\lambda)\|^2 = \|a + J\|^2 \\ &= \|a^* + J\|\|a + J\| \geq \|(a + J)^*(a + J)\|.\end{aligned}$$

$\square$

Exercise 4.10. Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras. Check that $\ker(\pi)$ is a closed two-sided ideal in A and the quotient map $q : A \to A/\ker(\pi)$ is a $*$-homomorphism.

Now, we are ready to build on Proposition 1.21 to get a very powerful theorem for $*$-homomorphisms.

Theorem 4.11. *An injective $*$-homomorphism between* C**-algebras is isometric. The image of any $*$-homomorphism between* C**-algebras is a* C**-algebra (in particular, the range of any $*$-homomorphism between* C**-algebras is closed.*

Proof. Recall from Proposition 1.21 that a $*$-homomorphism $\phi : A \to B$ between C*-algebras is contractive and for any $a \in A$, $\phi(\sigma(a)) \subset \sigma(a)$. We give the proof under the assumption that our C*-algebras and our maps are all unital and leave the adaption to the non-unital setting as an exercise.

Let $\phi : A \to B$ be an injective $*$-homomorphism. Note that for any $a \in A$, $\|a\|^2 = \|a^*a\|$ and $\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\|^2$, so by Theorem 3.10, it suffices to prove that $\|\phi(a)\| = \|a\|$ for $a \in A$ positive. Suppose $\|\phi(a)\| < \|a\|$ for some positive $a \in A$. Note that $\phi(a) \geq 0$ since $a = b^*b$ for some $b \in A$, and so $\phi(a) = \phi(b)^*\phi(b)$. So, the assumption that $\|\phi(a)\| < \|a\|$ is equivalent to the assumption that $r(a) := \alpha > \beta := r(\phi(a))$. Using the continuous functional calculus, we identify $C^*(a) = C_0(\sigma(a)) \subset C_0((0, \alpha])$ and

$C^*(\phi(a)) = C_0(\sigma(\phi(a))) \subset C((0,\beta])$. Now, define $f \in C((0,\alpha])$ so that $f|_{(0,\beta]} = 0$, $f(\alpha) = 1$, and f is affine on $[\beta, \alpha]$.

Then
$$\|f(a)\| = \sup_{\lambda \in \sigma(a)} |f(\lambda)| = 1,$$
but
$$\|f(\phi(a))\| = \sup_{\lambda \in \sigma(\phi(a))} |f(\lambda)| = 0.$$

In particular, $f(a) \neq 0$ and $f(a) \in \ker \phi$, contradicting ϕ being injective.

Now, suppose $\pi : A \to B$ is a $*$-homomorphism with kernel $J = \ker(\pi)$. Then A/J is a C*-algebra by Theorem 4.9. Let $q : A \to A/J$ be the quotient map. Then q is a $*$-homomorphism and π factors through the quotient A/J, i.e. there exists a bijective $*$-homomorphism $\rho : A/J \to \pi(A)$ given by $\rho(q(a)) = \pi(a)$. (Indeed, this is just the first isomorphism theorem for algebras. The map ρ is $*$-preserving because q and π are: $\rho(q(a)^*) = \rho(q(a^*)) = \pi(a^*) = \pi(a)^* = \rho(q(a))^*$.)

So, it follows that $\rho : A/J \to B$ is an injective $*$-homomorphism between C*-algebras, which by the first part of this theorem, means that it is isometric. It follows from this that its image $\pi(A)$ is closed in B. □

Exercise 4.12. Extend this to the general case where the assumptions that A, B, and ϕ are not unital. Here's an idea of what to check. If A is not unital, then we can extend ϕ to $\tilde{A}$ as we did in Proposition 1.18 to map $1 \in \tilde{A}$ to $1 \in B$ or $1 \in \tilde{B}$ depending on whether or not B is unital. If A is unital, then check that $\phi(1)$ is the unit in the C*-subalgebra $C^*(\phi(A)) \subset B$, and we can just replace B with this C*-subalgebra in the proof.

Remark 4.13. There is a class of C*-subalgebras called *hereditary subalgebras*, which generalizes the notion of ideal. A C*-subalgebra $A \subset B$ is hereditary if for any positive elements $a \in A$ and $b \in B$, if $b \leq a$, then $b \in A$. It turns out that ideals are always hereditary ([**?**, Theorem 1.5.3]).

Definition 4.14. A *representation* of a C*-algebra A is a $*$-homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We say a representation π is *non-degenerate* if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$.

A paradigm example of a degenerate representation is where $\mathcal{H}$ decomposes as a nontrivial direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\pi(A)$ can be realized as a $*$-subalgebra of operators on $B(\mathcal{H}_1)$ identified with the operators whose kernels contain $\mathcal{H}_2$.

Remark 4.15. Non-degeneracy is a regular assumption, which avoids some obnoxious pitfalls. Many times theorems which are phrased for non-degenerate representations still hold without this assumption. The trick usually amounts to taking a degenerate representation $\pi : A \to B(\mathcal{H})$ and to define its restriction to the closure of $\pi(A)\mathcal{H}$. Though some delicacy may be required after this, depending on what statement you are trying to prove. We will point out an example later. (Theorem 11.13)

Exercise 4.16. We say a family of representations $\{\pi_i : A \to B(\mathcal{H}_i)\}_{i \in I}$ for a C*-algebra A is *separating* if for any $a, b \in A$, there exists $i \in I$ such that $\pi_i(a) \neq \pi_i(b)$. Define $\pi : A \to B(\oplus_i \mathcal{H}_i)$ by $\pi(a) = \oplus_i \pi_i(a)$. Show that π is a *faithful* representation, i.e. an isometric $*$-representation, if the family $\{\pi_i\}_{i \in I}$ is separating.

Now, suppose $\{a_j\}_{j \in J}$ is a dense subset of A. We cannot conclude from knowing that $\{\pi_i\}_{i \in I}$ is separating for $\{a_i\}_{i \in I}$ that π is faithful (why?). However, if we know that for each $j \in J$, there exists $i \in I$ such that $\|\pi_i(a_j)\| = \|a_j\|$, then we can conclude that π is faithful (why?).