Chapter 1

Von Neumann Algebras

Lecture Preview: In the first leture, we will cover the Bicommutant Theorem (Theorem 1.2.6) in detail. To prepare for this, you should familiarize yourself with the strong and weak operator topologies (Definition 1.1.2), and the commutant (Definition 1.2.1). The second lecture will focus on the structure of group von Neumann algebras (Section 1.3.3).

1.1 Strong and Weak Operator Topologies

Let $\mathcal{H}$ be a Hilbert space. There is a natural (metrizable) topology on $B(\mathcal{H})$ given by the operator norm. Studying this topology amounts to studying C^*-algebras. To study von Neumann algebras, we will need to consider two new topologies on $B(\mathcal{H})$. There will be several others later on that are also important, but these first two will suffice to define a von Neumann algebra.

The formal definitions of these topologies are given below, but from an analytic perspective it is much more important to understand what it means for a net to converge in these topologies. Let $(x_i)_{i\in I} \subset B(\mathcal{H})$ be a net and let $x \in B(\mathcal{H})$. Then $(x_i)_{i\in I}$ converges to x in the *strong operator topology* (SOT) if

$$\lim_{i\to\infty} \|(x - x_i)\xi\| = 0 \qquad \forall \xi \in \mathcal{H},$$

and $(x_i)_{i\in I}$ converges to x in the *weak operator topology* (WOT) if

$$\lim_{i\to\infty} \langle (x - x_i)\xi, \eta\rangle = 0 \qquad \forall \xi, \eta \in \mathcal{H}.$$

Viewing $\mathcal{H}$ as a metric space under its norm, SOT convergence can be thought of as "pointwise convergence." Compare this to convergence under the operator norm, which should be thought of as "uniform convergence."

Remark 1.1.1. Strong operator topology convergence and weak operator topology convergence are often referred to in the literature as *strong convergence* and *weak convergence*, respectively, but in these notes we will typically avoid this terminology.

Definition 1.1.2. The **strong operator topology** (**SOT**) on $B(\mathcal{H})$ is the topology generated by the basis consisting of sets of the form

$$U(x; \xi_1, \ldots, \xi_n; \epsilon) := \{y \in B(\mathcal{H}) \colon \|(x - y)\xi_j\| < \epsilon,\ j = 1, \ldots, n\},$$

for $x \in B(\mathcal{H})$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$.

The **weak operator topology** (**WOT**) on $B(\mathcal{H})$ is the topology generated by the basis consisting of sets of the form

$$U(x; \xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n; \epsilon) := \{y \in B(\mathcal{H}) \colon |\langle (x - y)\xi_j, \eta_j\rangle| < \epsilon,\ j = 1, \ldots, n\},$$

for $x \in B(\mathcal{H})$, $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$, and $\epsilon > 0$.

Operator norm convergence implies SOT convergence, which in turn implies WOT convergence (Exercise 1.1.1), but the converses are not true. Here are some simple counter-examples:

Example 1.1.3. Let m be the Lebesgue measure on $\mathbb{R}$. For a measurable subset $S \subset \mathbb{R}$, let the characteristic function 1_S act on $B(L^2(\mathbb{R}, m))$ by pointwise multiplication. Then $(1_{[-n,n]})_{n\in\mathbb{N}}$ SOT-converges to the identity, but not in operator norm. Indeed, for any $f \in L^2(\mathbb{R}, m)$ and any ϵ, there exists $N \in \mathbb{N}$ so that

$$\left(\int_{\mathbb{R}\setminus[-N,N]} |f|^2 \, dm \right)^{1/2} < \epsilon.$$

Thus, for any $n \geq N$ we have

$$\left\| (1 - 1_{[-n,n]}) f \right\|_2 = \left(\int_{\mathbb{R}\setminus[-n,n]} |f|^2 \, dm \right)^{1/2} < \epsilon.$$

Thus this sequence of operators SOT-converges to 1. However, $1 - 1_{[-n,n]} = 1_{[-n,n]^c}$ is a projection and so $\|1_{[-n,n]} - 1\| = 1$ for all n. ■

Example 1.1.4. Consider the following unitary operator on $\ell^2(\mathbb{Z})$:

$$(U\xi)(n) := \xi(n+1) \qquad \xi \in \ell^2.$$

For $n \in \mathbb{N}$, let $x_n := U^n$. Then we claim that $(x_n)_{n\in\mathbb{N}}$ WOT-converges to the zero operator but does not SOT-converge. Indeed, fix $\xi, \eta \in \ell^2(\mathbb{Z})$. Let $\epsilon > 0$, then there exists $N \in \mathbb{N}$ sufficiently large so that

$$\left(\sum_{n \geq N} |\xi(n)|^2 \right)^{1/2} < \epsilon$$
$$\left(\sum_{n < -N} |\eta(n)|^2 \right)^{1/2} < \epsilon$$

Then for $m \geq 2N$ we have

$$\begin{aligned} |\langle x_m \xi, \eta \rangle| &\leq \sum_{n\in\mathbb{Z}} |\xi(n+m)||\eta(n)| \\ &= \sum_{n<-N} |\xi(n+m)||\eta(n)| + \sum_{n\geq m-N} |\xi(n)||\eta(n-m)| \\ &\leq \|\xi\|\epsilon + \epsilon\|\eta\|. \end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ WOT-converges to zero. However, since U is a unitary,

$$\|x_n \xi\| = \|U^n \xi\| = \|\xi\| \qquad \forall \xi \in \ell^2(\mathbb{Z}),$$

thus $(x_n)_{n\in\mathbb{N}}$ does not SOT-converge to zero. ■

You will explore how these topologies interact with the $*$-algebra structure of $B(\mathcal{H})$ in the exercises, but let us summarize things here. First, addition and scalar multiplication are both continuous with respect to both the SOT and WOT (see Exercise 1.1.5). Taking adjoints is continuous with respect to the WOT but not the SOT (see Exercises 1.1.6 and 1.1.7), though it is SOT continuous on normal operators (see Exercise 1.1.8). Finally, multiplication is not continuous with respect to either the WOT or the SOT, but on bounded subsets it is SOT continuous (see Exercises 1.1.9 and 1.1.10).

We leave the proof of the next proposition as an exercise (see Exercise 1.1.11).

Proposition 1.1.5. *Let $\{p_i : i \in I\} \subset B(\mathcal{H})$ be a set of pairwise orthogonal projections. If $\mathscr{F}$ is the collection of finite subsets of I ordered by inclusion, then then net $\left(\sum_{i\in F} p_i\right)_{F\in\mathscr{F}}$ converges in the SOT to a projection which we denote by $\sum_{i\in I} p_i$.*

Exercises

1.1.1. Show that if a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges in operator norm to some $x \in B(\mathcal{H})$, then it converges in the strong operator topology to x. Show that if a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges in the strong operator topology to some $x \in B(\mathcal{H})$, then it converges in the weak operator topology to x.

1.1.2. Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the strong operator topology. Show that

$$\|x\| \leq \limsup_{i\to\infty} \|x_i\|.$$

1.1.3. Show that $(x_i)_{\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the strong operator topology if and only if $((x - x_i)^*(x - x_i))_{i\in I}$ converges to zero in the weak operator topology.

1.1.4. Let (X, Ω, μ) be a σ-finite measure space and let $f \in L^\infty(X, \mu)$. Show that a net $(f_i)_{i\in I} \subset L^\infty(X, \mu)$ converges to f in the WOT as pointwise multiplication operators in $B(L^2(X, \mu))$ if and only if the net converges to f as elements of the dual space $L^1(X, \mu)^*$.

1.1.5. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \subset B(\mathcal{H})$ be nets indexed by the same directed set and let $x, y \in B(\mathcal{H})$.

(a) Suppose $(x_i)_{i\in I}$ and $(y_i)_{\in I}$ converge to x and y, respectively, in the SOT. Show that for any $\alpha \in \mathbb{C}$, the net $(\alpha x_i + y_i)_{i\in I}$ converges to $\alpha x + y$ in the SOT.

(b) Prove the corresponding statement for the WOT.

1.1.6. If $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the WOT, show that $(x_i^*)_{i\in I}$ converges to x^* in the WOT.

1.1.7. Consider the shift operator S on $\ell^2(\mathbb{N})$:

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Show that $((S^*)^n)_{n\in\mathbb{N}}$ converges to zero in the SOT, but $(S^n)_{n\in\mathbb{N}}$ does not.

1.1.8. In this exercise you will show that taking adjoints of normal operators is continuous with respect to the strong operator topology.

(a) Show that $y \in B(\mathcal{H})$ is normal if and only if $\|y\xi\| = \|y^*\xi\|$ for all $\xi \in \mathcal{H}$.

(b) Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ is a net of normal operators converging to $x \in B(\mathcal{H})$ in the strong operator topology. Show that x is also normal.

(c) With $(x_i)_{i\in I}$ and x as in the previous part, show that $(x_i^*)_{i\in I}$ converges to x^* in the strong operator topology.

1.1.9. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \subset B(\mathcal{H})$ be nets indexed by the same directed set that converge in the strong operator topology. Show that if $\sup_{i\in I} \|x_i\| < \infty$, then $(x_i y_i)_{i\in I}$ converges in the strong operator topology.

1.1.10. Find an example of bounded nets $(x_i)_{i\in I}, (y_i)_{i\in I} \subset B(\mathcal{H})$ converging to $x, y \in B(\mathcal{H})$, respectively, in the WOT but such that $(x_i y_i)_{i\in I}$ does not converge to xy in the WOT. [**Hint:** consider Example 1.1.4.]

1.1.11. Prove Proposition 1.1.5: For each $i \in I$ let $\mathcal{K}_i := p_i\mathcal{H}$ and define

$$\mathcal{K} := \overline{\operatorname{span}} \bigcup_{i\in I} \mathcal{K}_i.$$

Letting $p \in B(\mathcal{H})$ be the projection onto $\mathcal{K}$, show that the net $(\sum_{i\in F} p_i)_{F\in\mathscr{F}}$ converges in the SOT to p.

1.2 Bicommutant Theorem

Definition 1.2.1. Let $\mathcal{H}$ be a Hilbert space. For $x, y \in B(\mathcal{H})$, the **commutator** of x and y is denoted

$$[x, y] := xy - yx.$$

For a subset $X \subset B(\mathcal{H})$, the **commutant** of X, denoted X', is the set

$$X' := \{y \in B(\mathcal{H}) \colon [x, y] = 0 \ \forall x \in X\}.$$

The **double commutant** of X is the set

$$X'' := (X')'$$

If $X \subset Y \subset B(\mathcal{H})$ is an intermediate subset, we call $X' \cap Y$ the **relative commutant** of X in Y.

Observe that, regardless of the structure of X, X' is always a unital algebra. If X is closed under taking adjoints, then X' is a $*$-algebra. It also easily checked (algebraically) that:

$$\begin{aligned} X \subset X'' &= (X'')'' = \cdots \\ X' &= (X')'' = \cdots \end{aligned}$$

Note that inclusions are reversed under the commutant: $X \subset Y$ implies $Y' \subset X'$. Remarkably, the purely algebraic definition of the commutant has analytic implications. This culminates in The Bicommutant Theorem (Theorem 1.2.6).

Example 1.2.2. Let $\mathcal{H}$ be a Hilbert space. If $1 \in B(\mathcal{H})$ is the identity operator, then for any $\alpha \in \mathbb{C}$ and any $x \in B(\mathcal{H})$ one has $[x, \alpha 1] = 0$. Consequently, $\{\mathbb{C}1\}' = B(\mathcal{H})$.

Conversely, one also has $B(\mathcal{H})' = \mathbb{C}1$, which you will show in Exercise 1.2.1. As a special case of this, consider $\mathcal{H} = \mathbb{C}^n$ so that $B(\mathcal{H}) = M_n(\mathbb{C})$. To see that $M_n(\mathbb{C})' = \mathbb{C}1$, consider the matrices $E_{i,j} \in M_n(\mathbb{C})$ for $i, j = 1, \ldots, n$, where $E_{i,j}$ is the matrix with a one in the (i, j)-entry and zeros elsewhere. Note that $E_{i,j}E_{k,\ell} = \delta_{j=k}E_{i,\ell}$. Also, observe that that for any $A = (A_{i,j})_{i,j=1}^n \in M_n(\mathbb{C})$,

$$E_{i,i}AE_{j,j} = A_{i,j}E_{i,j}$$

Thus if $A \in M_n(\mathbb{C})'$, then

$$A_{i,j}E_{i,j} = E_{i,i}AE_{j,j} = E_{i,i}E_{j,j}A = \delta_{i=j}E_{i,j}A.$$

This implies $A_{i,j} = 0$ unless $i = j$; that is, A is diagonal. We also have for any $i, j = 1, \ldots, n$

$$A_{i,i}E_{i,i} = E_{i,i}AE_{i,i} = E_{i,j}E_{j,i}AE_{i,i} = E_{i,j}AE_{j,i} = E_{i,j}E_{j,j}AE_{j,j}E_{j,i} = A_{j,j}E_{i,j}E_{j,j}E_{j,i} = A_{j,j}E_{i,i}.$$

So all the diagonal entries of A agree and so $A = A_{1,1}1 \in \mathbb{C}1$. ∎

Example 1.2.3. For (X, Ω, μ) a σ-finite measure space, view $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ where $f \in L^\infty(X, \mu)$ acts by pointwise multiplication. Then $L^\infty(X, \mu)' = L^\infty(X, \mu)$, which you will show in Exercise 1.2.3. As a special case of this, consider $\mathbb{N}$ equipped with the counting measure. For $n \in \mathbb{N}$, let $e_n \in \ell^2(\mathbb{N})$ be the function defined by $e_n(k) = \delta_{n=k}$. Note that $e_n \in \ell^\infty(\mathbb{N})$ as well, and that for $f \in \ell^2(\mathbb{N})$ one has

$$[e_n f](k) = e_n(k)f(k) = \delta_{n=k}f(n) = [f(n)e_n](k),$$

that is: $e_n f = f(n)e_n$. Now, if $T \in \ell^\infty(\mathbb{N})'$ and $f \in \ell^2(\mathbb{N})$ we have

$$[T(f)](n) = e_n(n)[T(f)](n) = [e_nT(f)](n) = [T(e_nf)](n) = f(n)[T(e_n)](n).$$

So if we define $g \colon \mathbb{N} \to \mathbb{C}$ by $g(n) := [T(e_n)](n)$, then $T(f) = gf$. Also note that

$$|g(n)| = |[T(e_n)](n)| \leq \|T(e_n)\|_2 \leq \|T\|\|e_n\|_2 \leq \|T\|.$$

Thus $g \in \ell^\infty(\mathbb{N})$. ∎

Definition 1.2.4. Let $\mathcal{K} \subset \mathcal{H}$ be a subspace. For $x \in B(\mathcal{H})$, we say $\mathcal{K}$ is **invariant** for x if $x\mathcal{K} \subset \mathcal{K}$. We say $\mathcal{K}$ is **reducing** for x if it is invariant for x and x^*. For a subset $X \subset B(\mathcal{H})$, we say $\mathcal{K}$ is **invariant** (resp. **reducing**) for X if it is invariant (resp. reducing) for all $x \in X$.

Note that if X is closed under taking adjoints, then a subspace is invariant for X if and only if it is reducing for X.

Lemma 1.2.5. *Let $M \subset B(\mathcal{H})$ be a $*$-subalgebra. Let $\mathcal{K} \subset \mathcal{H}$ be a closed subspace with $p \in B(\mathcal{H})$ the projection onto $\mathcal{K}$. Then $\mathcal{K}$ is reducing for M if and only if $p \in M'$.*

Proof. Assume $\mathcal{K}$ is reducing M. Let $x \in M$ and $\xi \in \mathcal{K}$. Then $x\xi \in \mathcal{K}$ so that

$$xp\xi = x\xi = px\xi.$$

If $\eta \in \mathcal{K}^\perp$, we have

$$\langle x\eta, \xi \rangle = \langle \eta, x^*\xi \rangle = 0,$$

since $x^*\xi \in \mathcal{K}$. Thus $x\eta \in \mathcal{K}^\perp$ and so $xp\eta = 0 = px\eta$. It follows that $xp = px$ so that $p \in M'$.

Conversely, suppose $p \in M'$. Let $x \in M$ and $\xi \in \mathcal{K}$. Then for $\eta \in \mathcal{K}^\perp$ we have

$$0 = \langle x\xi, p\eta \rangle = \langle px\xi, \eta \rangle = \langle xp\xi, \eta \rangle = \langle x\xi, \eta \rangle .$$

Thus $x\xi \in (\mathcal{K}^\perp)^\perp = \mathcal{K}$. Hence $M\mathcal{K} \subset \mathcal{K}$ so that $\mathcal{K}$ is reducing for M. $\square$

We have the following theorem due to von Neumann from 1929.

Theorem 1.2.6 (The Bicommutant Theorem)**.** *For a unital $*$-subalgebra $M \subset B(\mathcal{H})$, one has*

$$\overline{M}^{SOT} = \overline{M}^{WOT} = M''$$

Proof. We will show the following series of inclusions:

$$\overline{M}^{SOT} \subset \overline{M}^{WOT} \subset M'' \subset \overline{M}^{SOT}.$$

The first inclusion follows the fact that SOT-convergence implies WOT-convergence.

Now, suppose $x \in \overline{M}^{WOT}$, say with a net $(x_i)_{i\in I} \subset M$ converging to x in the WOT. Let $y \in M'$, then for any $\xi, \eta \in \mathcal{H}$ we have

$$\langle xy\xi, \eta \rangle = \lim_{i\to\infty} \langle x_i y\xi, \eta \rangle = \lim_{i\to\infty} \langle y x_i \xi, \eta \rangle = \langle yx\xi, \eta \rangle .$$

Since $\xi, \eta \in \mathcal{H}$ were arbitrary, we have $xy = yx$ and thus $x \in M''$.

Finally, suppose $x \in M''$. Note that to show $x \in \overline{M}^{SOT}$, it suffices to show for all $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$ that there exists $y \in M$ with

$$\|(x-y)\xi_j\| < \epsilon \qquad j = 1, \ldots, n.$$

Fix $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$. For $y \in M$, define $\pi(y) \in B(\mathcal{H}^{\oplus n})$ by

$$\pi(y)(\eta_1, \ldots, \eta_n) := (y\eta_1, \ldots, y\eta_n).$$

If you view $\mathcal{H}^{\oplus n}$ as column vectors over $\mathcal{H}$ of height n, then $\pi(y)$ corresponds to the matrix

$$\begin{pmatrix} y & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & y \end{pmatrix}.$$

With this perspective, one can show that $\pi(M)'$ consists of matrices of the form

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

where $a_{i,j} \in M'$ for $i, j = 1, \ldots, n$, and that $\pi(x)$ commutes with all such matrices since $x \in M''$ (see Exercise 1.2.6).

Now, let $\mathcal{S}$ denote the closure of the subspace $\{\pi(y)(\xi_1, \ldots, \xi_n) : y \in M\} \subset \mathcal{H}^{\oplus n}$. Then $\mathcal{S}$ is reducing for $\pi(M)$, and so if $p \in B(\mathcal{H}^{\oplus n})$ is the projection onto $\mathcal{S}$, then Lemma 1.2.5 implies $p \in \pi(M)'$ and so $p\pi(x) = \pi(x)p$. Note that $1 \in M$ implies $(\xi_1, \ldots, \xi_n) \in \mathcal{S}$. Thus we have

$$\pi(x)(\xi_1, \ldots, \xi_n) = \pi(x)p(\xi_1, \ldots, \xi_n) = p\pi(x)(\xi_1, \ldots, \xi_n) \in \mathcal{S}.$$

The definition of $\mathcal{S}$ then implies there exists $y \in M$ with

$$\|\pi(x)(\xi_1, \ldots, \xi_n) - \pi(y)(\xi_1, \ldots, \xi_n)\| < \epsilon.$$

Unpacking our notation, we see that

$$\|\pi(x)(\xi_1, \ldots, \xi_n) - \pi(y)(\xi_1, \ldots, \xi_n)\| = \|((x-y)\xi_1, \ldots, (x-y)\xi_n)\| = \left(\sum_{j=1}^{n} \|(x-y)\xi_j\|^2 \right)^{1/2}.$$

Combining this with the previous inequality yields $\|(x-y)\xi_j\| < \epsilon$ for each $j = 1, \ldots, n$. $\square$

The double commutant is given by a purely algebraic definition, whereas the SOT and WOT closures are purely analytic. Their equality in the above theorem tells us the following objects lie in the confluence of algebra and analysis:

Definition 1.2.7. We say a unital $*$-subalgebra $1 \in M \subset B(\mathcal{H})$ is a **von Neumann algebra** if $M = M''$ (equivalently, $M = \overline{M}^{SOT}$ or $M = \overline{M}^{WOT}$).

Recall from Example 1.2.2 that $B(\mathcal{H})' = \mathbb{C}1$ and that $\mathbb{C}1' = B(\mathcal{H})$. Hence $B(\mathcal{H})'' = B(\mathcal{H})$ and $\mathbb{C}1'' = \mathbb{C}1$ are examples of von Neumann algebras. Another example is $L^\infty(X, \mu)$ for a σ-finite measure space (X, Ω, μ), since

$$L^\infty(X, \mu)'' = L^\infty(X, \mu)' = L^\infty(X, \mu)$$

by Example 1.2.3. We will explore these and other examples in greater detail in the next section, but first we must define a few related concepts.

From the observation following Definition 1.2.1, we see that for M a von Neumann algebra, M' is also a von Neumann algebra. Consequently, so is $M \cap M'$ we which we give a name to here:

Definition 1.2.8. For M a von Neumann algebra, the **center of** M, denoted $\mathcal{Z}(M)$, is the von Neumann subalgebra $M \cap M'$. If $\mathcal{Z}(M) = \mathbb{C}1$, we say M is a **factor**. If $\mathcal{Z}(M) = M$, we say M is **abelian**.

For a Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ is a factor by Example 1.2.2, while for a σ-finite measure space (X, Ω, μ), $L^\infty(X, \mu)$ is abelian. There are examples where $\mathbb{C}1 \subsetneq \mathcal{Z}(M) \subsetneq M$, so factors and abelian von Neumann algebras only represent the two extremes on how much commutativity a von Neumann algebra permits.

We conclude by presenting a notion of what it means for two von Neumann algebras to be isomorphic.

Definition 1.2.9. We say two von Neumann algebras $M_1 \subset B(\mathcal{H}_1)$ and $M_2 \subset B(\mathcal{H}_2)$ are **spatially isomorphic** if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $UM_1U^* = M_2$. In this case we call $M_1 \ni x \mapsto UxU^* \in M_2$ a **spatial isomorphism**.

Exercises

1.2.1. Let $\mathcal{H}$ be a Hilbert space. Given $\xi, \eta \in \mathcal{H}$, recall that the rank one operator $\xi \otimes \bar{\eta} \in B(\mathcal{H})$ is defined by

$$(\xi \otimes \bar{\eta})(\zeta) := \langle \zeta, \eta \rangle \, \xi.$$

(a) Show that $x \in B(\mathcal{H})$ commutes with $\xi \otimes \bar{\eta}$ if and only if there exists $\lambda \in \mathbb{C}$ with $\xi \in \ker(x - \lambda)$ and $\eta \in \ker(x^* - \bar{\lambda})$.

(b) Show that $FR(\mathcal{H})' = \mathbb{C}$ and that $B(\mathcal{H})' = \mathbb{C}$.

1.2.2. For (X, Ω, μ) a σ-finite measure space and $f \in L^\infty(X, \mu)$, show that

$$L^2(X, \mu) \ni g \mapsto fg$$

defines a bounded linear operator on $L^2(X, \mu)$ with norm equal to $\|f\|_\infty$.
[**Hint:** for $\epsilon > 0$ consider $\{x \in X : |f(x)| \geq \|f\|_\infty - \epsilon\}$.]

1.2.3. For (X, Ω, μ) a σ-finite measure space, view $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ where $f \in L^\infty(X, \mu)$ acts by pointwise multiplication. Then $L^\infty(X, \mu)' = L^\infty(X, \mu)$. [**Hint:** first consider the case when μ is finite.]

1.2.4. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K} \subset \mathcal{H}$ a closed subspace, and $p \in B(\mathcal{H})$ the projection onto $\mathcal{K}$.

(a) Show that $\mathcal{K}$ is invariant for $x \in B(\mathcal{H})$ if and only if $pxp = xp$.

(b) Show that $\mathcal{K}$ is reducing for $x \in B(\mathcal{H})$ if and only if $xp = px$.

1.2.5. Let $\mathcal{H}$ be a Hilbert space and fix $n \in \mathbb{N}$. For all $T \in B(\mathcal{H}^{\oplus n})$, show that there exist $T_{i,j} \in B(\mathcal{H})$ for $i, j = 1, \ldots, n$ such that

$$T(\xi_1, \ldots, \xi_n) = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \cdots & T_{n,n} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

(In the above we are not distinguishing between row and column vectors.) Thus $B(\mathcal{H}^{\oplus n})$ can be identified with $n \times n$ matrices with entries in $B(\mathcal{H})$.

1.2.6. For $x \in B(\mathcal{H})$ and $A = (A_{i,j})_{i,j=1}^n \in M_n(\mathbb{C})$, define $x \otimes A \in B(\mathcal{H}^{\oplus n})$ by

$$x \otimes A := \begin{pmatrix} A_{1,1}x & \cdots & A_{1,n}x \\ \vdots & \ddots & \vdots \\ A_{n,1}x & \cdots & A_{n,n}x \end{pmatrix}.$$

Let $X \subset B(\mathcal{H})$, and for each $i, j = 1, \ldots, n$ let $E_{i,j} \in M_n(\mathbb{C})$ be the matrix with a one in the (i, j)-entry and zeros elsewhere.

(a) Show that

$$\{x \otimes I_n : x \in X\}' = \left\{ \sum_{i,j=1}^n y_{i,j} \otimes E_{i,j} : y_{i,j} \in X' \ i, j = 1, \ldots, n \right\}.$$

(b) Show that

$$\left\{ \sum_{i,j=1}^n y_{i,j} \otimes E_{i,j} : y_{i,j} \in X' \ i, j = 1, \ldots, n \right\}' = \{x \in \otimes I_n : x \in X''\}.$$

1.2.7. Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces, and for each $j = 1, \ldots, n$ define $\pi_j : B(\mathcal{H}_j) \to B(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$ by

$$\pi_j(x)(\xi_1, \ldots, \xi_n) = (0, \ldots, 0, x\xi_j, 0, \ldots, 0) \qquad (\xi_1, \ldots, \xi_n) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n.$$

(You can also think of $\pi_j(x)$ as an $n \times n$ matrix with x in the (j, j)-entry and zeros elsewhere).

(a) Show that π_j is an isometric $*$-homomorphism for each $j = 1, \ldots, n$.

(b) Let $M_j \subset B(\mathcal{H}_j)$ be a von Neumann algebra for each $j = 1, \ldots, n$. Show that

$$M_1 \oplus \cdots \oplus M_n = \left\{ \sum_{j=1}^n \pi_j(x_j) : x_j \in M_j, \ j = 1, \ldots, n \right\}$$

is a von Neumann algebra. (It called the **direct sum** of $M_1, \ldots, M_n$.)

(c) Show that $\mathcal{Z}(M_1 \oplus \cdots \oplus M_n) = \mathcal{Z}(M_1) \oplus \cdots \oplus \mathcal{Z}(M_n)$.

(d) Show that $M_1 \oplus \cdots \oplus M_n$ is **not** a factor for $n \geq 2$.

1.2.8. Show that a von Neumann algebra M is abelian if and only if $M \subset M'$.

1.2.9. An abelian von Neumann algebra $A \subset B(\mathcal{H})$ is called **maximal abelian** if $A \subset B \subset B(\mathcal{H})$ for another abelian von Neumann algebra B implies $A = B$. Show that an abelian von Neumann algebra A is maximal abelian if and only if $A' = A$.

1.3 First Examples

1.3.1 $B(\mathcal{H})$ and Matrix Algebras

For any Hilbert space $\mathcal{H}$, we saw above that $B(\mathcal{H})$ is always a von Neumann algebra and a factor. In particular, if $\mathcal{H}$ is finite dimensional with $d := \dim(\mathcal{H})$, then $B(\mathcal{H})$ is simply the matrix algebra $M_d(\mathbb{C})$. Though an elementary example, $M_d(\mathbb{C})$ will eventually inform a great deal of our intuition about von Neumann algebras. We highlight a few important features below.

As factors, matrix algebras are as noncommutative as a von Neumann algebra can be. They also contain a lot of projections. For each pair $i, j = 1, \ldots, d$ let $E_{i,j} \in M_d(\mathbb{C})$ be the matrix with a one in the (i, j)-entry and zeros elsewhere. Then $E_{i,i}$ is projection for each $i = 1, \ldots, n$ and so is any sum of these matrices (see also Exercise 1.3.1).

Recall that the *unnormalized trace* on $M_d(\mathbb{C})$ is a linear functional $\mathrm{Tr}\colon M_d(\mathbb{C}) \to \mathbb{C}$ defined as

$$\mathrm{Tr}(A) = \sum_{i=1}^{d} A_{i,i}.$$

The trace is invariant under cyclic permutation: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in M_d(\mathbb{C})$. In fact, up to a scalar, it is the unique linear functional on $M_d(\mathbb{C})$ with this property (see Exercise 1.3.2). Note that if $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{C}^d$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{d} \langle Ae_i, e_i \rangle .$$

In fact, the standard basis in the above formula can be replaced with *any* orthonormal basis $\{f_1, \ldots, f_d\}$ for $\mathbb{C}^d$. This is because if U is the unitary matrix whose columns are $f_1, \ldots, f_d$, then $Ue_i = f_i$ for each $i = 1, \ldots, d$. Consequently

$$\sum_{i=1}^{d} \langle Af_i, f_i \rangle = \sum_{i=1}^{d} \langle AUe_i, Ue_i \rangle = \sum_{i=1}^{d} \langle U^*AUe_i, e_i \rangle = \mathrm{Tr}(U^*AU) = \mathrm{Tr}(AUU^*) = \mathrm{Tr}(A).$$

One can even define a trace for $B(\mathcal{H})$ when $\mathcal{H}$ is infinite dimensional, but it will only be well-defined on the trace-class operators, which we revisit in Section 3.1.

1.3.2 Measure Spaces

For (X, Ω, μ) a σ-finite measure space, we saw above that $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ is an abelian von Neumann algebra. In fact, it is *maximal abelian* in the sense that if $L^\infty(X, \mu) \subset A \subset B(\mathcal{H})$ for an abelian von Neumann algebra A, then $A = L^\infty(X, \mu)$ (see Exercises 1.2.3 and 1.2.9). As with matrix algebras, $L^\infty(X, \mu)$ will also eventually inform a great deal of our intuition. Indeed, it turns out that *all* abelian von Neumann algebras are of this form and we will see a partial proof of this in Section 2.2.

Despite the fact that $L^\infty(X, \mu)$ and $M_d(\mathbb{C})$ are radically different in terms of commutativity, there are still important similarities. $L^\infty(X, \mu)$ also has an abundence of projections. Indeed, for any measurable $E \subset X$, $1_E \in L^\infty(X, \mu)$ is a projection. In fact any projection in $L^\infty(X, \mu)$ is of this form (see Exercise 1.3.3). Consequently, the linear span of projections is exactly the set of μ-measurable simple functions, which we know from measure theory are $\|\cdot\|_\infty$ norm dense in $L^\infty(X,\mu)$. Using Exercise 1.2.2, we can then deduce that the linear span of projections is actually operator norm dense in $L^\infty(X,\mu)$. Additionally, when μ is a finite measure, $L^\infty(X,\mu) \subset L^1(X,\mu)$ and so

$$L^\infty(X,\mu) \ni f \mapsto \int_X f \, d\mu$$

is a natural linear functional on this von Neumann algebra, similar to the trace on $M_d(\mathbb{C})$.

1.3.3 Group von Neumann Algebras

Let Γ be a countable discrete group, which we can use to define a Hilbert space $\ell^2(\Gamma)$. Consider the left regular representation $\lambda\colon \Gamma \to B(\mathcal{H})$:

$$[\lambda(g)\xi](h) = \xi(g^{-1}h) \qquad \xi \in \ell^2(\Gamma),\ h \in \Gamma$$

Equivalently, if for $g \in \Gamma$ we let $\delta_g \in \ell^2(\Gamma)$ be the function $\delta_g(h) = \delta_{g=h}$, then $\lambda(g)\delta_h = \delta_{gh}$ for all $h \in \Gamma$. The operators $\lambda(g)$ are in fact unitary operators with $\lambda(g)^* = \lambda(g^{-1})$, and in particular if $e \in \Gamma$ is the identity then $\lambda(e) = 1$. Denote $\mathbb{C}[\lambda(\Gamma)] := \mathrm{span}\lambda(\Gamma)$, which we note is a unital $*$-subalgebra of $B(\ell^2(\Gamma))$.

Definition 1.3.1. The **group von Neumann algebra** for Γ is $L(\Gamma) := \mathbb{C}[\lambda(\Gamma)]''$.

These von Neumann algebras can be abelian, factors, or something in between. If Γ is an abelian group, then $\mathbb{C}[\lambda(\Gamma)]$ and consequently $L(\Gamma)$ are abelian. To understand when $L(\Gamma)$ is a factor, we require a definition:

Definition 1.3.2. We say that Γ is an **infinite conjugacy class (i.c.c.)** group if the conjugacy class $\{h^{-1}gh\colon h \in h \in \Gamma\}$ is infinite for all $g \in \Gamma \setminus \{e\}$.

Example 1.3.3.

(1) For $n \in \mathbb{N}$, the free group with n generators, $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$, is an i.c.c. group.

(2) Let S_∞ denote the group of bijections $\pi\colon \mathbb{N} \to \mathbb{N}$ such that $\pi(n) = n$ for all but finitely many $n \in \mathbb{N}$. This group can be viewed as the union of all permutation groups S_n, $n \in \mathbb{N}$, where $S_n \hookrightarrow S_{n+1}$ by fixing $n+1$. Then S_∞ is an i.c.c. group.

(3) Any finite or abelian group is not an i.c.c. group. ■

You will show in Exercise 1.3.7 that $L(\Gamma)$ is a factor if and only if Γ is an i.c.c. group.

As with our previous examples, $L(\Gamma)$ admits a natural linear functional $\tau\colon L(\Gamma) \to \mathbb{C}$ defined by

$$\tau(x) = \langle x\delta_e, \delta_e \rangle .$$

Since $\tau(\lambda(g)) = \delta_{g=e}$, τ encodes the group relations; that is, $g_1 g_2 \cdots g_n = e$ for $g_1, \ldots, g_n \in \Gamma$ if and only if $\tau(\lambda(g_1)\cdots\lambda(g_n)) = 1$. Also, like the trace on $M_d(\mathbb{C})$, τ is invariant under cyclic permutations: $\tau(xy) = \tau(yx)$ for all $x, y \in L(\Gamma)$ (see Exercise 1.3.8). Because if this we call τ the **trace** on $L(\Gamma)$.

In constructing the group von Neumann algebra, one could instead use the right regular representation:

$$[\rho(g)\xi](h) = \xi(hg) \qquad \xi \in \ell^2(\Gamma),\ h \in \Gamma,$$

in which case one denotes by $R(\Gamma) := \mathbb{C}[\rho(\Gamma)]''$. There is a very natural relationship between $L(\Gamma)$ and $R(\Gamma)$ (see Theorem 1.3.7), but in order to witness it we require some additional terminology. Recall that for $\xi, \eta \in \ell^2(\Gamma)$ their convolution is defined by

$$(\xi * \eta)(g) = \sum_{h \in \Gamma} \xi(h)\eta(h^{-1}g).$$

From the Cauchy–Scwarz inequality, we have $|(\xi * \eta)(g)| \leq \|\xi\|_2 \|\eta\|_2$ for all $g \in \Gamma$. So $\xi * \eta \in \ell^\infty(G)$ with $\|\xi * \eta\|_\infty \leq \|\xi\|_2 \|\eta\|_2$.

Definition 1.3.4. We say $\xi \in \ell^2(\Gamma)$ is a **left** (resp. right) **convolver** if $\xi * \eta \in \ell^2(\Gamma)$ (resp. $\eta * \xi \in \ell^2(\Gamma)$) for all $\eta \in \ell^2(\Gamma)$. Denote the linear operator $\eta \mapsto \xi * \eta$ (resp. $\eta \mapsto \eta * \xi$) by $\lambda(\xi)$ (resp. $\rho(\xi)$). Denote $LC(\Gamma) := \{\lambda(\xi) \colon \xi \text{ is a left convolver}\}$ and $RC(\Gamma) := \{\rho(\xi) \colon \xi \text{ is a right convolver}\}$.

Observe that $\lambda(\delta_g) = \lambda(g)$ and $\rho(\delta_g) = \rho(g)$. We claim that $\lambda(\xi)$ is bounded for any left convolver ξ. By the Closed Graph Theorem, it suffices to show that if $(\eta_n)_{n\in\mathbb{N}} \subset \ell^2(\Gamma)$ satisfies $\eta_n \to 0$ and $\lambda(\xi)(\eta_n) \to \zeta$, then $\zeta = 0$. Since the $\|\cdot\|_2$ norm dominates the $\|\cdot\|_\infty$ norm, we see

$$\|\zeta\|_\infty = \lim_{n\to\infty} \|\lambda(\xi)(\eta_n)\|_\infty = \lim_{n\to\infty} \|\xi * \eta_n\|_\infty \leq \limsup_{n\to\infty} \|\xi\|_2\|\eta_n\|_2 = 0.$$

Thus $\zeta = 0$ and so $\lambda(\xi)$ is bounded. A similar argument shows that $\rho(\xi)$ is bounded for any right convolver. Hence $LC(\Gamma), RC(\Gamma) \subset B(\ell^2(\Gamma))$.

Lemma 1.3.5. *$\xi \in \ell^2(\Gamma)$ is left (resp. right) convolver if and only if there exists $c > 0$ so that $\|\xi*\kappa\|_2 \leq c\|\kappa\|_2$ (resp. $\|\kappa * \xi\|_2 \leq c\|\kappa\|_2$) for all finitely supported $\kappa \in \ell^2(\Gamma)$.*

Proof. We will consider only left convolvers, since the proof for right convolvers is similar. The "only if" direction follows from the discussion preceding the lemma, where $c = \|\lambda(\xi)\|$.

Conversely, define for finitely supported $\kappa \in \ell^2(\Gamma)$ define $x\kappa := \xi * \kappa$. The hypothesis implies that x can be extended to a bounded operator on $\ell^2(\Gamma)$, which we also denote by x. Fix $\eta \in \ell^2(\Gamma)$. Given $\epsilon > 0$ there is a finite subset $F \subset \Gamma$ satisfying

$$\sum_{g\in\Gamma\setminus F} |\eta(g)|^2 < \epsilon^2.$$

In other words, if $\kappa := \eta 1_F$, then $\|\eta - \kappa\|_2 < \epsilon$. Since κ is finitely supported, we have $x\kappa = \xi * \kappa$ and so we estimate

$$\begin{aligned}
\|\xi * \eta - x\eta\|_\infty &\leq \|\xi * \eta - x\kappa\|_\infty + \|x(\kappa - \eta)\|_\infty \\
&\leq \|\xi * (\eta - \kappa)\|_\infty + \|x(\kappa - \eta)\|_2 \\
&\leq \|\xi\|_2\|\eta - \kappa\|_2 + \|x\|\|\kappa - \xi\|_2 < (\|\xi\|_2 + \|x\|)\epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we have $\xi * \eta = x\eta \in \ell^2(\Gamma)$. Hence ξ is a left convolver. □

Proposition 1.3.6. *$LC(\Gamma)$ and $RC(\Gamma)$ are von Neumann algebras.*

Proof. We will only consider $LC(\Gamma)$, the proof for $RC(\Gamma)$ being similar. We also leave checking that $LC(\Gamma)$ is a unital $*$-algebra as an exercise (see Exercise 1.3.9). By the Bicommutant Theorem, it suffices to show $LC(\Gamma)$ is SOT closed. Let $(\xi_i)_{i\in I} \subset \ell^2(\Gamma)$ be a net of left convolvers such that $(\lambda(\xi_i))_{i\in I}$ converges to some $x \in B(\ell^2(\Gamma))$ in the SOT. Observe that $\lambda(\xi_i)\delta_e = \xi_i$, so if we set $\xi := x\delta_e$ then $\xi_i = \lambda(\xi_i)\delta_e \to x\delta_e = \xi$. Using this and the SOT convergence of $(\lambda(\xi_i))_{i\in I}$ to x, we have for any $\eta \in \ell^2(\Gamma)$ that

$$\|\xi * \eta - x\eta\|_\infty \leq \|\xi * \eta - \xi_i * \eta\|_\infty + \|\xi_i * \eta - x\eta\|_\infty \leq \|\xi - \xi_i\|_2\|\eta\|_2 + \|(\lambda(\xi_i) - x)\eta\|_2 \to 0.$$

Thus $\xi * \eta = x\eta \in \ell^2(\Gamma)$, which implies ξ is a left convolver and that $\lambda(\xi) = x$. Hence $x \in LC(\Gamma)$ and $LC(\Gamma)$ is SOT closed. □

Theorem 1.3.7. *$R(\Gamma) = L(\Gamma)'$ and $L(\Gamma) = R(\Gamma)'$.*

Proof. We begin by showing

$$L(\Gamma) \subset LC(\Gamma) \subset RC(\Gamma)' \subset R(\Gamma)' \subset LC(\Gamma).$$

For any $g \in \Gamma$, $\lambda(g) = \lambda(\delta_g) \in LC(\Gamma)$. Hence $L(\Gamma) \subset LC(\Gamma)'' = LC(\Gamma)$ by Proposition 1.3.6, which gives the first inclusion. Note that a symmetric argument implies $R(\Gamma) \subset RC(\Gamma)$, and so taking commutants yields the third inclusion. The second inclusion follows from Exercise 1.3.10. Let $x \in R(\Gamma)'$ and set $\xi := x\delta_e$. Then for any $g \in \Gamma$ we have

$$x\delta_g = x(\rho(g)\delta_e) = \rho(g)(x\delta_e) = \rho(g)\xi = \xi * \delta_g,$$

where the last equality follows from a direct computation. Consequently, for any finitely supported $\kappa \in \ell^2(\Gamma)$ we have $\|\xi * \kappa\|_2 = \|x\kappa\|_2 \leq \|x\|\|\kappa\|_2$. Lemma 1.3.5 therefore implies that ξ is a left convolver. The above computation shows $x\delta_g = \xi * \delta_g = \lambda(\xi)\delta_g$, and since such vectors densely span $\ell^2(\Gamma)$ we have $x = \lambda(\xi)$. This gives the last inclusion.

The inclusions established above show, $LC(\Gamma) = RC(\Gamma)' = R(\Gamma)'$. A symmetric argument yields $RC(\Gamma) = LC(\Gamma)' = L(\Gamma)'$. Using the Bicommutant Theorem, these equalities imply

$$R(\Gamma) = (R(\Gamma)')' = LC(\Gamma)' = L(\Gamma)'.$$

Taking commutants then gives $R(\Gamma)' = L(\Gamma)$. □

Remark 1.3.8. If G is a locally compact group (e.g. $\mathbb{R}$), it is still possible to define $L(G)$ using the left regular representation of G on $L^2(G, \mu)$, where μ the left-invariant Haar measure on G. However, in the mini-courses we will restrict ourselves to the discrete case.

Group von Neumann algebras remain far from fully understood. On the one hand, by a deep result of Alain Connes, all amenable i.c.c. groups yield the same group von Neumann algebra. This von Neumann algebra (which we will define by other means in a later chapter) is called the *hyperfinite* II_1 *factor*, but we will not have time in the mini-course to delve into Connes' proof.

On the other hand, the famous *Free Group Factor Isomorphism Problem*, which is still open, asks whether or not $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$, where $\mathbb{F}_k$ is the free group with k generators. A very active area of research in von Neumann algebras is focused on how much of Γ is "rememebered" by $L(\Gamma)$. The best results to date have relied on a collection of techniques known as Popa's deformation/ridigity theory.

Exercises

1.3.1. Consider the following 2×2 matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

Show that they are all projections and that their span is all of $M_2(\mathbb{C})$. Can you find 9 projections in $M_3(\mathbb{C})$ that span?

1.3.2. Suppose $\varphi\colon M_d(\mathbb{C}) \to \mathbb{C}$ is a linear functional satisfying $\varphi(AB) = \varphi(BA)$ for all $A, B \in M_d(\mathbb{C})$. Show that $\varphi = \varphi(1)\frac{1}{n}\mathrm{Tr}$. [**Hint:** show that $\varphi(E_{i,j}) = 0$ for $i \neq j$ and that $\varphi(E_{i,i})$ does not depend on $i = 1, \ldots, n$.]

1.3.3. For $f \in L^\infty(X, \mu) \subset B(\mathrm{L}^2(X, \mu))$, show that f is a projection if and only if $f = 1_E$ for some measurable $E \subset X$. [**Hint:** show that $\mu\{x \in X\colon f(x) \notin \{0, 1\}\} = 0$.]

1.3.4. Let Γ be a discrete group with left regular representation $\lambda\colon \Gamma \to B(\ell^2\Gamma)$. For $g \in \Gamma$, show that $\lambda(g)$ is a unitary operator with $\lambda(g)^* = \lambda(g^{-1})$.

1.3.5. Verify the claims in Example 1.3.3.

1.3.6. Let Γ be an infinite countable discrete group. Let $(g_n)_{n\in\mathbb{N}} \subset \Gamma$ be a sequence that never repeats. Show that the sequence of unitaries $(\lambda(g_n))_{n\in\mathbb{N}}$ converges to zero in the WOT.

1.3.7. Let Γ be a countable discrete group.

(a) For $x \in L(\Gamma)$ and $g \in \Gamma$, show that
$$\Gamma \ni h \mapsto \langle x\delta_{g^{-1}h}, \delta_h \rangle$$
is a constant map.

(b) Denote the value of the constant map in the previous part by $c_g(x)$. Show that
$$x\delta_e = \sum_{g\in\Gamma} c_g(x)\delta_g,$$
and hence $\sum_g |c_g(x)|^2 < \infty$.

(c) For $x \in \mathcal{Z}(L(\Gamma))$, show that $c_g(x) = c_{h^{-1}gh}(x)$ for all $g, h \in \Gamma$, and that $c_g(x) = 0$ whenever $\{h^{-1}gh \colon h \in \Gamma\}$ is infinite.

(d) Prove that $L(\Gamma)$ is a factor if and only if Γ is an i.c.c. group.

1.3.8. Let Γ be a countable discrete group and let τ be the trace on $L(\Gamma)$.

(a) Show that $\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g))$ for all $g, h \in \Gamma$.

(b) Show that τ is WOT continuous.

(c) Prove that $\tau(xy) = \tau(yx)$ for all $x, y \in L(\Gamma)$.

1.3.9. Let Γ be a countable discrete group. In this exercise, you will show that $LC(\Gamma)$ and $RC(\Gamma)$ are $*$-algebras.

(a) Show that $1 = \lambda(e) \in LC(\Gamma) \cap RC(\Gamma)$ where $e \in \Gamma$ is the identity.

(b) If $\xi, \eta \in \ell^2(\Gamma)$ are left (resp.) convolvers, show that $\xi * \eta$ is a left (resp.) convolver.

(c) For $\lambda(\xi), \lambda(\eta) \in LC(\Gamma)$, show that $\lambda(\xi)\lambda(\eta) = \lambda(\xi * \eta) \in LC(\Gamma)$.

(d) For $\rho(\xi), \rho(\eta) \in RC(\Gamma)$, show that $\rho(\xi)\rho(\eta) = \rho(\xi * \eta) \in RC(\Gamma)$.

1.3.10. For a left convolver ξ and a right convolver η, show that $\lambda(\xi)\rho(\eta) = \rho(\eta)\lambda(\xi)$.