# Problem Session

July 1st, 2020

**C*.1** (Gelfand-Mazur) If $A$ is a simple, unital, abelian Banach algebra, then $A = \mathbb{C}$.

[**Hint:** For each $a \in A$, consider $aA := \{ab : b \in A\}$]

**C*.2** Here's an exercise to build intuition:

(a) Show that all maximal ideals in $C([0,1])$ are of the form $\{f \in C([0,1]) \ : \ f(t) = 0\}$ for some $t \in [0,1]$.

(b) For each $t \in [0,1]$, define the map $ev_t : C([0,1]) \to \mathbb{C}$ by $ev_t(f) = f(t)$. Show that $\widehat{C([0,1])} = \{ev_t \ : \ t \in [0,1]\}$.

(c) Recall that for $A = C_0((0,1])$, its unitization is $\tilde{A} := C([0,1])$. That means we can identify $C_0((0,1])$ with a maximal ideal inside $C([0,1])$. To which character $\phi \in \hat{\tilde{A}}$ does this ideal correspond?
Show that this character agrees with the functional $\phi_0 : \tilde{A} \to \mathbb{C}$ given by $\phi(f + \lambda 1) = \lambda$ for all $f \in A$.

**W*.1** Consider the shift operator $S$ on $\ell^2(\mathbb{N})$:

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Show that $((S^*)^n)_{n\in\mathbb{N}}$ converges to zero in the SOT, but $(S^n)_{n\in\mathbb{N}}$ does not.

**W*.2** Let $\mathcal{H}$ be a Hilbert space. Given $\xi, \eta \in \mathcal{H}$, recall that the rank one operator $\xi \otimes \bar{\eta} \in B(\mathcal{H})$ is defined by

$$(\xi \otimes \bar{\eta})(\zeta) := \langle \zeta, \eta \rangle \, \xi.$$

(a) Show that $x \in B(\mathcal{H})$ commutes with $\xi \otimes \bar{\eta}$ if and only if there exists $\lambda \in \mathbb{C}$ with $\xi \in \ker(x - \lambda)$ and $\eta \in \ker(x^* - \bar{\lambda})$.

(b) Show that $FR(\mathcal{H})' = \mathbb{C}$ and that $B(\mathcal{H})' = \mathbb{C}$.