# 1. A first look at C*-algebras

**Preview of Lecture:** To help guide your reading, we indicate here which of the following material we will address in lecture and which we will assume familiarity with:

The main goal in this lecture is proving the Gelfand-Naimark theorem for commutative C*-algebras (Theorem 2.1) and introducing the Functional Calculus (Corollary 2.18).

To that end, we will use without proof all of the results in Section 1. We will introduce the unitization from Section 1, but with more focus on the intuition in Remark 1.17.

From Section 3, we use without proof the correspondence between maximal ideals and characters established in Definition 2.2 - Corollary 2.6. We will also use without proof the fact (Proposition 2.7) that the character space (i.e. spectrum) of a C*-algebra is a weak$*$-compact subset of the unit ball of the dual of the C*-algebra.

We will prove Lemma 2.12 and assume its corollary, Lemma 2.13, to complete the proof of Theorem 2.1. However, the proof in the lecture will look a little different from the notes. In particular, we will consider the theory in the unital setting first and then explain how to get to the non-unital setting at the end.

Proposition 2.16 and Corollary 2.17 establish the important fact that the spectrum of an element in a C*-algebra is independent of the ambient unital C*-algebra. However, we will bypass this argument in lecture and go straight for a description of the correspondence in the Functional Calculus (Corollary 2.18).

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In a Banach space, there is often additional algebraic structure, in particular multiplication.

**Definition 1.1.** A *Banach $*$-algebra $A$* is a multiplicative involutive Banach space whose norm satisfies the following:

$$\|ab\| \leq \|a\|\|b\|$$

for all $a, b \in A$.

Ideally, we'd like involution to also be isometric. This and other magical results follow from the additional assumption that the norm $\|\cdot\|$ on $A$ satisfies the C*-*identity*:

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$. It follows from this that

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|,$$

and hence that $\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|$.

**Definition 1.2.** A C*-*algebra* is a Banach $*$-algebra whose norm satisfies the C*-identity.

*Remark* 1.3. Calling these C*-algebras is already highly suggestive. In fact, when they were first introduced, they were called $B^*$-algebras, and the notion of C*-algebra was reserved for norm **c**losed $*$-subalgebras of $B(\mathcal{H})$. In the coming days, we shall justify calling these C*-algebras, but for the sake of not encouraging archaic terminology, we take the privilege before we earn it.

Recall from Exercise 7.32 in the Day 1 lectures that the norm on $B(\mathcal{H})$ satisfies the C*-identity, meaning any closed self-adjoint subspace of $B(\mathcal{H})$ is a C*-algebra. These are known as *concrete* C*-*algebras*.

**Example 1.4.** Recall the unilaterial shift $S \in B(\ell^2(\mathbb{N}))$ from Example 7.19 in the Prerequisite Notes. The norm closure of the $*$-algebra generated by $S$ in $B(\ell^2(\mathbb{N}))$ is a C*-algebra often called the *Toeplitz algebra*.

**Exercise 1.5.** Let $X$ be a locally compact Hausdorff topological space. We denote by $C_0(X)$ the space of all continuous functions on $X$ vanishing at infinity. Show this is a C*-algebra with involution given by complex conjugation and norm given by the sup norm.

**Example 1.6.** Consider the C*-algebra $C(\mathbb{T})$ consisting of all continuous functions on the compact Hausdorff space $\mathbb{T} = \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}$ (sometimes denoted $S^1$). (Why don't we say $C_0(\mathbb{T})$?) It follows from the Stone-Weierstraß approximation theorem ([3, I.5,6]) that Laurent polynomials, i.e. polynomials of the form $\sum_{k=-n}^{n} \alpha_n z^n$, are dense in $C(\mathbb{T})$. So, $C(\mathbb{R})$ is actually the C*-algebra generated by the function $f \in C(\mathbb{R})$ given by $f(z) = z$.

As is often the case, C*-algebras are a little more friendly to work with when they have an identity element (also called a unit). If $1 \in A$ is the identity, then

(1) $1^* = 1^*1 = 11^* = 1$, and
(2) $\|1\| = 1$.

Analogously with elements in $B(\mathcal{H})$ (in fact, we will see soon that it is more than an analogy), we call an element $a$ in a C*-algebra $A$

- *Normal* if $a^*a = aa^*$,
- *Self-Adjoint* if $a = a^*$,
- a *Projection* if $a = a^* = a^2$,
- a *Unitary* if $a^*a = aa^* = 1$,
- an *Isometry* if $a^*a = 1$,
- a *partial isometry* if $a = aa^*a$.

Note (Check) that for any element $a$ in a C*-algebra is the sum of two self-adjoint operators, its real and imaginary parts:

$$\mathrm{Re}(a) = \frac{1}{2}(a + a^*) \qquad \mathrm{Im}(a) = \frac{1}{2i}(a - a^*). \tag{1.1}$$

This useful decomposition lets us reduce many questions to the case of self-adjoint operators.

**Proposition 1.7.** *A linear map between* C**-algebras is* $*$*-preserving iff it maps self adjoint elements to self adjoint elements.*

*Proof.* Let $\phi : A \to B$ be a linear map and $a \in A$, and write $a = \mathrm{Re}(a) + i\mathrm{Im}(a)$ and $a^* = \mathrm{Re}(a) - i\mathrm{Im}(a)$. By linearity,

$$\phi(a) = \phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a))$$
$$\phi(a^*) = \phi(\mathrm{Re}(a)) - i\phi(\mathrm{Im}(a)).$$

Since $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self-adjoint, $\phi(\mathrm{Re}(a))$ and $\phi(\mathrm{Im}(a))$ are self adjoint by assumption. So the above computation shows that

$$\phi(a^*) = \phi(\mathrm{Re}(a) + i\mathrm{Im}(a)) * .$$

□

1.1. **Unitizations and Spectra.** Let us briefly recap and expand on some facts about the spectrum of an operator in a Banach algebra– now with C*-algebras.

An element $a$ in a unital algebra is invertible when there exitst another element $b$ in the algebra that acts as a left and right inverse, i.e. $ab = ba = 1$. Sometimes, when you have a left inverse, it is automatically a right inverse. In particular, this is the case for matrix algebras. In fact, a matrix $T \in \mathrm{M}_n(\mathbb{C})$ is invertible if and only if it is injective, i.e. if and only if $\ker(T) = \{0\}$. In infinite dimensions, this is certainly still a necessary condition, but it is no longer sufficient alone.

**Exercise 1.8.** Give an example of an operator on $B(\ell^2(\mathbb{N}))$ that is injective but not invertible.

Fortunately, the Open Mapping Theorem gives us some guidance on what needs to be satisfied:

**Corollary 1.9** (to OMT/Inverse Function Theorem)**.** *For a Hilbert space* $\mathcal{H}$*,* $T \in B(\mathcal{H})$ *is invertible iff* $T$ *is bijective.*

**Example 1.10.** Unitary operators are important classes of invertible operators. In fact, the group of unitaries in a C*-algebra $A$ forms a subgroup $\mathcal{U}(A)$ of the group of invertible elements, $GL(A)$.

With the notion of invertibility, we can define the spectrum of a given element $a$ in a unital C*-algebra $A$.

$$\sigma(a) := \{\lambda \in \mathbb{C} \ : \ \lambda 1 - a \notin GL(A)\}$$

*Remark* 1.11. Unlike when $A = \mathrm{M}_n(\mathbb{C})$, these are not all eigenvalues.

**Example 1.12.** If $A$ is a unital C*-algebra and $u \in A$ is a unitary, then $\sigma(u) \subset \mathbb{T}$.

Indeed, first note that for any invertible operator $a \in A$, the spectrum of the inverse is the inverse of the spectrum. To see this, fix an invertible $a$, so that $\lambda = 0$ is not in $\sigma(a)$. For $\lambda \neq 0$, if $\lambda - a$ is invertible, then so is $\lambda^{-1}a^{-1}(\lambda - a) = a^{-1} - \lambda^{-1}$ and vice versa.

Then for any $\lambda \in \sigma(u)$, we have that $\lambda^{-1} \in \sigma(u^{-1}) = \sigma(u^*)$. Since $u^*$ is also a unitary, we know $\|u\| = \|u^*\| = 1$, which means $|\lambda| \leq 1$ and $|\lambda^{-1}| \leq 1$, which means $|\lambda| = 1$.

**Exercise 1.13.** Recall (Example 3.12 in Prerequisite Material) that continuous function $f$ on a $X$ locally compact and Hausdorff space $X$ is invertible if $1/f$ is continuous on $X$. What is the spectrum of $f(z) = z$ in $C(\mathbb{T})$?

But not all C*-algebras have units. One important example is $\mathcal{K}(\mathcal{H})$, and another important class of examples comes from spaces of continuous functions.

**Exercise 1.14.** For a locally compact topological Hausdorff space $X$, when is the C*-algebra $C_0(X)$ unital? What is the unit? Can you think of interesting classes of non-unital algebras? For the C*-algebra $C(\mathbb{T})$, what type of operator is the generator $f(z) = z$?

So, how can we make sense of a spectrum in this setting? We just add a unit! Well, technically, we embed $A$ into a unital C*-algebra.

The "smallest" unital C*-algebra containing $A$ is called its *unitization*, $\tilde{A}$. We define $\tilde{A}$ as follows:

$$\tilde{A} := A \oplus \mathbb{C}$$

with algebraic operations given by

$$\begin{aligned}(a, \alpha)(b, \beta) &= (ab + \alpha b + \beta a, \alpha\beta) \\ (a, \alpha)^* &= (a^*, \bar{\lambda}) \\ \|(a, \alpha)\| &= \sup_{b \in A, b \leq 1} \|ab + \alpha b\|\end{aligned}$$

This definition does not feel intuitive the first time around. To get an idea of where this came from, consider the following examples.

**Example 1.15.**

(1) If $A \subset B(\mathcal{H})$ is a C*-subalgebra of $B(\mathcal{H})$ that does not contain a unit, you can "unitize" it by just taking the C*-algbra generated by $A$ and $1_{\mathcal{H}}$.
$$C^*(A, 1_{\mathcal{H}}) = \{a + \lambda 1_{\mathcal{H}} \ : \ \lambda \in \mathbb{C}, a \in A\}.$$
What would multiplication/ scalar addition look like here? For the norm, it will turn out that $\|(a, \alpha)\| = \|a + \alpha 1_{\mathcal{H}}\|$, but the argument is faster after a little more theory.

(2) Identify
$$C_0((0,1]) := \{f \in C([0,1]) \ : \ f(0) = 0\}.$$
By taking the closure of the algebra generated by $C_0((0,1])$ and the constant function 1, we get its unitization $C([0,1])$. For $f \in C_0((0,1])$ and $a \in \mathbb{C}$, what is the norm of $f + a$ in the sup norm for $C([0,1])$?

Because of the example from $B(\mathcal{H})$, even in an abstract setting, elements of $\tilde{A}$ are often written as $a + \lambda 1_{\tilde{A}}$ as opposed to $(a, \lambda)$.

**Proposition 1.16.** *Any* C*-*algebra $A$ embeds into the unital* C*-*algebra $\tilde{A}$ as an ideal of codimension* 1, *i.e. no other proper ideal of $\tilde{A}$ contains $A$ and $\tilde{A}/A = \mathbb{C}$.*

*Proof.* That $\tilde{A}$ is a unital $*$-algebra is readily verified. To see that the norm is a Banach algebra norm, notice that it is exactly the norm induced from $B(A)$ where we identify $a \in A$ with the left multiplication operator $L_a \in B(A)$ given by $L_a(b) = ab$, and we identify $(a, \alpha)$ with $L_a + \alpha \mathrm{id}_A$. In other words, the norm on $\tilde{A}$ is the norm induced from $B(A)$ on the $*$-subalgebra of operators $\{L_a + \alpha \mathrm{id}_a : a \in A, \alpha \in \mathbb{C}\}$. Moreover, note that the identification $a \mapsto L_a$ is isometric. Indeed, using the C*-identity, we have for any nonzero $a \in A$,

$$\|a\| = \|a\left(\frac{a^*}{\|a\|}\right)\| \leq \sup_{\|b\| \leq 1} \|ab\| \leq \|a\|.$$

So, $\|(a,0)\| = \|a\|$, and the embedding of $A$ into $\tilde{A}$ is isometric. Since $A$ is complete, $\{L_a + \alpha \mathrm{id}_A : a \in A, \alpha \in \mathbb{C}\}$ is complete, and so $\tilde{A}$ is a Banach algebra. By design, $A$ is an ideal of codimension 1.

It remains to show that the given norm satisfies the C*-identity. To that end, we compute for $a \in A$ and $\alpha \in \mathbb{C}$

$$\begin{aligned}\|(a,\alpha)\|^2 &= \sup_{\|b\|\leq 1} \|ab + \alpha b\|^2 \\ &= \sup_{\|b\|\leq 1} \|b^*(a^*ab + \alpha a^* b + \bar{\alpha} ab + |\alpha|^2 b)\| \\ &\leq \sup_{\|b\|\leq 1} \|a^*ab + \alpha a^* b + \bar{\alpha} ab + |\alpha|^2 b\| \\ &= \|(a,\alpha)^*(a,\alpha)\| \leq \|(a,\alpha)^*\|\|(a,\alpha)\|.\end{aligned}$$

So $\|(a,\alpha)\| \leq \|(a,\alpha)^*\|$, and a symmetric argument yields $\|(a,\alpha)^*\| = \|(a,\alpha)\|$. Then the above inequality gives

$$\|(a,\alpha)\|^2 \leq \|(a,\alpha)^*(a,\alpha)\| \leq \|(a,\alpha)\|^2.$$

□

Therefore, if $a$ is an element of a non-unital C*-algebra $A$, then we define its *spectrum* to be the spectrum of $a$ as an element of $\tilde{A}$.

This fits well with what we've already seen in $B(\mathcal{H})$. If $x \in B(\mathcal{H})$, then its spectrum is defined with respect to the unit in $B(\mathcal{H})$, regardless to what closed $*$-subalgebra $x$ belongs to.

*Remark* 1.17. Suppose $A$ is a non-unital C*-subalgebra of a unital C*-algebra $B$. Then there is a clear $*$-preserving bijective homomorphism between $\tilde{A}$ and $C^*(A, 1)$ given by $(a, \alpha) \mapsto a + \alpha$. By appealing to the same subspace $\{L_a + \alpha\mathrm{id}_A \;:\; a \in A, \alpha \in \mathbb{C}\} \subset B(A)$, one can show that this is isometric. That means that, when a unit is available in an ambient C*-algebra, the unitization of $A$ is just adjoining that unit. Of course, there is now the problem that for any $a \in A$, its spectrum in $A$ might be larger than its spectrum in $B$ (an element has more potential inverses in $B$). We will see later that this is not the case.

*Remark* 1.18. There are two conventions you will see in the literature for $\tilde{A}$ when $A$ is already unital. The first is to assume that $A = \tilde{A}$ when $A$ is unital, and the second is to have a "forced unitization" where $A$ is still embedded as a maximal ideal in $A \oplus \mathbb{C}$, and the unit of $A$ becomes just the projection $1_A \oplus 0$. The choice in a given paper is often due to technical considerations (e.g. when you just want to make sure your C*-algebra has a unit vs. when you want to control where a map sends the unit) and is (hopefully) addressed somewhere in the preliminaries.

One thing that makes unitizations nice to work with is that a $*$-homomorphism always has a unique and natural extension to the unitization.

**Proposition 1.19.** *Let $A, B$ be C*-algebras with $B$ unital and $A$ non-unital and $\pi : A \to B$ a $*$-homomorphism. Then there is a unique extension of $\pi$ to a unital $*$-homomorphism $\tilde{\pi} : \tilde{A} \to B$ given by $\tilde{\pi}(a + \lambda 1_{\tilde{A}}) = \pi(a) + \lambda 1_B$.*

Note that this works also when we have $\pi : A \to B$ with $B$ non-unital but identified with its copy inside $\tilde{B}$.

*Proof.* We just need to check that this is a $*$-homomorphism. Linearity and $*$-preserving are immediate. For $a, b \in A$ and $\lambda, \eta \in \mathbb{C}$, we compute

$$\begin{aligned}\tilde{\pi}(a + \lambda 1_{\tilde{A}})\tilde{\pi}(b + \eta 1_{\tilde{A}}) &= (\pi(a) + \lambda 1_B)(\pi(b) + \eta 1_B) \\ &= \pi(ab) + \lambda\pi(b) + \eta\pi(a) + \lambda\eta 1_B = \tilde{\pi}(ab + \lambda b + \eta a + \lambda\eta 1_{\tilde{A}}).\end{aligned}$$

The uniqueness is forced by the fact that we require $\tilde{\pi}$ to be linear and $1_{\tilde{A}} \mapsto 1_B$. Indeed, if $\psi : \tilde{A} \to B$ is another unital extension of $\pi$, then for each $a + \lambda 1_{\tilde{A}} \in \tilde{A}$, we have

$$\psi(a + \lambda 1_{\tilde{A}}) = \psi(a) + \psi(\lambda 1_{\tilde{A}}) = \pi(a) + \lambda 1_B = \tilde{\pi}(a + \lambda 1_{\tilde{A}}).$$

□

Now that we have a notion of spectra for unital and nonunital C*-algebras, we are ready to see two consequences of the C*-identity that are, quite frankly, magic.

First we recall Theorems 3.16 and 3.20 from the pre-requisite material:

**Theorem.** *For any element $a$ in Banach algebra $A$, $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$. Moreover, the spectrum of $a$ is contained in the closed ball $\{x \in A \,:\, \|x\| \le \|a\|\}$. In particular, this means that $r(a) \le \|a\|$ where $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$ is the spectral radius of $a$.*

*Remark* 1.20. This implies the very useful fact that for any element $a$ in a unital Banach algebra with $\|a\| < 1$, the element $1 - a$ is invertible with inverse $\sum_{n\ge 0} a^n$.

**Theorem.** *For any element $a$ in Banach algebra $A$,*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

When our Banach algebra $A$ is a C*-algebra, it turns out the norm of any normal element *is* its spectral radius.

**Lemma 1.21.** *For any normal element $a$ in a C*-algebra $A$,*

$$\|a\| = r(a).$$

*Proof.* First, we assume that $a = a^*$. Then repeated use of the C*-identity for $a$, i.e. $\|a\|^2 = \|a^2\|$, tells us that

$$r(a) = \lim_n \|a^{2^n}\|^{2^{-n}} = \|a\|.$$

Now, suppose $a$ is normal. Then $a^*a$ is self-adjoint, and so

$$\begin{aligned} r(a)^2 = \|a\|^2 = \|a^*a\| &= r(a^*a) \\ &= \lim_n \|(a^*a)^n\|^{1/n} = \lim_n \|(a^n)^*a^n\|^{1/n} = \lim_n \|a^n\|^{2/n} \\ &= r(a)^2. \end{aligned}$$

□

As a Banach $*$-algebras, we consider C*-algebras "the same" when they are $*$-isomorphic, i.e. there exists a $*$-preserving homomorphism between them. Normally, for a Banach space, you'd also request that the bijective linear map be isometric. For $*$-isomorphisms between C*-algebras, this will be automatic, thanks again to the C*-identity.

**Proposition 1.22.** *A $*$-homomorphism $\pi : A \to B$ between C*-algebras is contractive (i.e. $\|\pi\| \le 1$) and hence continuous. A $*$-isomorphism between C*-algebras is isometric.*

*Proof.* Suppose $\pi : A \to B$ is a $*$-isomorphism. Let $a \in A$. Then $a^*a$ is a normal element in $A$, which means $\|a^*a\| = r(a^*a)$. Since homomorphisms preserve invertibility, $r(\pi(a^*a)) \le r(a^*a)$. This is where the C*-norm comes in:

$$\|a\|^2 = \|a^*a\| = r(a^*a) \ge r(\pi(a^*a)) = r(\pi(a)^*\pi(a)) = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.$$

Now, assume $\pi$ is injective. If $\pi$ is a $*$-isomorphism, then the inequality above is an equality. □

So, in C*-algebras, the algebraic structure determines the norm:

$$\|x\| = \sqrt{\|x^*x\|} = \sqrt{r(x^*x)}$$

(Compare with the same fact for matrices.) It follows from this that a C*-algebra carries a unique norm making it a C*-algebra.

*Remark* 1.23. What this is saying is that if $(A, \|\cdot\|)$ is a C*-algebra and $\|\cdot\|'$ is another C*-norm on $A$ (without assuming $A$ is complete with respect to $\|\cdot\|'$), then $\|\cdot\| = \|\cdot\|'$.

There's a subtlety here that can sometimes be a little tricky. If $B$ is just a $*$-algebra, then we can often define multiple distinct C*-norms on $B$ so that the completion of $B$ with respect to these norms becomes a C*-algebra.

We will be able to say more about $*$-homomorphisms once we have established more on C*-ideals.

## 2. Commutative C*-algebras

Some of you may have heard of the study of C*-algebras described as "non-commutative topology" or "non-commutative continuous functions". This perspective is really what jump-started the interest in C*-algebras in the first place, and it comes from the following theorem, which is the focal point of this section:

**Theorem 2.1** (Gelfand Naimark Theorem)**.** *Any commutative* C*-*algebra $A$ is $*$-isomorphic to the* C*-*algebra $C_0(X)$ for some locally compact Hausdorff space $X$. Moreover, when $A$ is unital, $X$ is compact.*

**Definition 2.2.** A nonzero homomorphism into the base field of an algebra is called a *character*. The *spectrum* of a commutative Banach algebra $A$, denoted $\hat{A}$, is the set of all nonzero characters from $A$ into $\mathbb{C}$. Hence $\hat{A}$ is often called the *character space* for $\hat{A}$.

*Remark* 2.3. We assume for now that these are just homomorphisms. In fact, much of the theory we develop on our way to the Gelfand Naimark theorem holds in general for Banach algebras. A consequence of the Gelfand Naimark theorem for commutative C*-algebras will show that characters on a commutative C*-algebra are automatically $*$-preserving.

Notice that the kernel of a character is a closed ideal in $A$ of co-dimension 1, and so it is automatically a maximal ideal, i.e. it is not contained in any other proper ideal. It turns out there is a one-to-one correspondence between maximal ideals in $A$ and ideals of co-dimension 1 (and hence characters).

**Exercise 2.4.** A maximal ideal in a unital C*-algebra is automatically closed.
(Hint: If $J \subset A$ is a proper ideal, consider $\overline{J} \cap B(1_A, 1)$.)

**Exercise 2.5** (Gelfand-Mazur)**.** If $A$ is a simple, unital, abelian Banach algebra, then $A = \mathbb{C}$.

**Corollary 2.6.** *If $A$ is a unital abelian Banach algebra, then any maximal ideal in $A$ has co-dimension* 1*, i.e. if $J \subset A$ is a maximal ideal, then $A/J \simeq \mathbb{C}$.*

*Proof.* If $J \subset A$ is a maximal ideal, then $A/J$ is simple. The rest follows from Gelfand-Mazur. □

From Theorem 3.8 in the Prerequisite notes, we have for each maximal ideal $J \triangleleft A$, a continuous homomorphism $\phi_j : A \to \mathbb{C}$.

**Proposition 2.7.** *Let $A$ be a commutative* C*-*algebra. Then $\hat{A} \cup \{0\}$ is a weak-$*$ compact subset of the unit ball of $A^*$. When $A$ is unital, $\hat{A}$ is weak-$*$ compact.*

*In particular, $\hat{A}$ is a locally compact Hausdorff space, which is compact when $A$ is unital.*

*Proof.* Let $\phi \in \hat{A}$. Suppose $\|\phi\| > 1$ and $a \in A$ with $\|a\| < 1$ and $\phi(a) = 1$. Since $\|a\| < 1$, its spectrum is in the unit ball, meaning $1 - a$ is invertible. So, we compute

$$1 = \phi((1-a)(1-a)^{-1}) = (\phi(1) - \phi(a))\phi((1-a)^{-1}) = (0)\phi((1-a)^{-1}) = 0,$$

which is an obvious contradiction.

Now, since $\hat{A} \cup \{0\}$ is contained in the unit ball of $A^*$, by Alaoglu's theorem (Theorem 2.20 in the Prereqs), all we need to show is that it is weak-$*$ closed. To that end, suppose we have a net $(\phi_i)_{i \in I}$ of characters (multiplicative linear functionals) that converges weak-$*$ to some bounded linear functional $\phi \in A^*$. We need to check that $\phi$ is multiplicative, but this follows from the fact that pointwise multiplication is continuous. Indeed, for any $a, b \in A$, we have

$$\phi(ab) = \lim_i \phi_i(ab) = \lim_i \phi_i(a)\phi_i(b) = \lim_i \phi_i(a) \lim_i \phi_i(b) = \phi(a)\phi(b).$$

It follows that $\hat{A} \cup \{0\}$ is a compact Hausdorff space (with respect to the weak-$*$ topology).

Note that if $A$ is unital, then for any $\phi \in \hat{A}$, we have $\phi(1) = 1$, and so $\|\phi\| \geq 1$. It follows by the preceeding argument that $\hat{A}$ is itself a weak-$*$ closed subset of the unit ball in $A^*$. □

Recall that when $A$ is communtative but not unital, it embeds into $\tilde{A}$ as an ideal with co-dimension 1, which means it's the kernel of a character $\phi_0 : \tilde{A} \to \tilde{A}/A = \mathbb{C}$. Notice that when restricted to $A$, this is exactly the 0 homomorphism. It turns out there is a one-to-one correspondence between $\hat{A}$ and $\hat{\tilde{A}} \backslash \{\phi_0\}$. In particular, $\hat{\tilde{A}}$ is (also) the one-point compactification of $\hat{A}$.

**Proposition 2.8.** *Suppose $A$ is a non-unital commutative* C$^*$*-algebra, and let* $\phi_0 : \tilde{A} \to \tilde{A}/A = \mathbb{C}$. *Then, there is a one-to-one correspondence between* $\hat{A}$ *and* $\hat{\tilde{A}}\backslash\{\phi_0\}$.

*Proof.* Suppose $\tilde{\phi} \in \hat{\tilde{A}}\backslash\{\phi_0\}$. Since $\tilde{A}/\ker(\tilde{\phi}) = \mathbb{C}$, $\ker(\tilde{\phi})$ is a maximal ideal in $\tilde{A}$. Similarly, $A$ is also a maximal ideal, and so $\ker(\tilde{\phi}) \cap A \subsetneq A$. Then $\ker(\tilde{\phi}) \cap A$ is an ideal of co-dimension 1 in $A$, which means the map $\phi : A \mapsto A/(A \cap \ker(\tilde{\phi}))$ gives a character in $\hat{A}$.

On the other hand, if $\phi \in \hat{A}$, define $\tilde{\phi} : \tilde{A} \to \mathbb{C}$ by $\tilde{\phi}(a, \lambda) = \phi(a) + \lambda$. Then (as per Proposition 1.19) $\tilde{\phi} \in \hat{\tilde{A}}$ is the unique extension of $\phi$ to a character on $\tilde{A}$. With that, we have established the desired bijective correspondence. □

**Definition 2.9.** For a commutative C$^*$-algebra $A$, we define the *Gelfand transform* $\Gamma : A \to C_0(\hat{A})$ by $\Gamma(a)(\phi) = \phi(a)$, i.e. $\Gamma(a)$ is the point evaluation at $a$.

**Exercise 2.10.** Here's an exercise to build intuition:

(1) Show that all maximal ideals in $C([0, 1])$ are of the form $\{f \in C([0, 1]) \ : \ f(t) = 0\}$ for some $t \in [0, 1]$.
(2) For each $t \in [0, 1]$, define the map $ev_t : C([0, 1]) \to \mathbb{C}$ by $ev_t(f) = f(t)$. Show that $\widehat{C([0, 1])} = \{ev_t \ : \ t \in [0, 1]\}$.
(3) Recall that for $A = C_0((0, 1])$, its unitization is $\tilde{A} := C([0, 1])$. That means we can identify $C_0((0, 1])$ with a maximal ideal inside $C([0, 1])$. To which character $\phi \in \hat{\tilde{A}}$ does this ideal correspond?
Show that this character agrees with the functional $\phi_0 : \tilde{A} \to \mathbb{C}$ given by $\phi(f + \lambda 1) = \lambda$ for all $f \in A$.

Here is our goal theorem:

**Theorem 2.11** (Gelfand-Naimark)**.** *For any commutative* C$^*$*-algebra $A$, the Gelfand transform is an isometric $*$-isomorphism*[1] *of $A$ onto* $C_0(\hat{A})$.

Notice that if $A$ is unital, then $C_0(\hat{A}) = C(\hat{A})$. If $A$ is not unital, then the one point compactification of $\hat{A}$ is $\hat{\tilde{A}} = \hat{A} \cup \{\phi_0\}$, which means $C_0(\hat{A})$ is exactly the continuous functions on $\hat{\tilde{A}}$ that vanish at $\phi_0$.

Before we prove the Gelfand-Naimark theorem, we will establish a few lemmas, which are interesting in their own right.

**Lemma 2.12.** *For any commutative* C$^*$*-algebra $A$, the Gelfand transform is a contractive (and hence continuous) homomorphism. Moreover, if $A$ is unital, then for any $a \in A$,*

$$\sigma(a) = \sigma(\Gamma(a)) = \{\phi(a) \ : \ \phi \in \hat{A}\} = ran(\Gamma(a)),$$

*and $\Gamma$ is isometric.*

*Proof.* Multiplicativity follows from multiplicativity of characters. Notice that $\Gamma(a)$ is automatically continuous because the topology on $\hat{A}$ is the weak-$*$ topology. When $A$ is nonunital, $\Gamma(a)(\phi_0) = \phi_0(a) = 0$ for each $a \in A$, which, by the above remarks, means $\Gamma(A) \subset C_0(\hat{A})$.

Since each character is contractive and the norm on $C_0(\hat{A})$ is the sup norm, it follows that $\Gamma$ is contractive.

Now, suppose $A$ is unital. First, we show that $a \in A$ is invertible iff $\Gamma(a) \in C(\hat{A})$ is invertible. The forward direction follows immediately from the fact that $\Gamma$ is a homomorphism. On the other hand, if $a \in A$ is not invertible, then it lives in some maximal ideal, meaning it is in the kernel of some nonzero character $\phi \in \hat{A}$. Then $\Gamma(a)(\phi) = \phi(a) = 0$, meaning $\Gamma(a)$ is not invertible. It follows that $\sigma(a) = \sigma(\Gamma(a))$ for all $a \in A$.

Now, suppose $\lambda \in \sigma(a)$. Then there exists $\phi \in \hat{A}$ such that $\Gamma(\lambda 1 - a)(\phi) = 0$, i.e. $\Gamma(a)(\phi) = \lambda$. It follows that $\|\Gamma(a)\|_\infty = r(a)$.

Since $A$ is commutative, all elements of $A$ are normal. Hence it follows from Lemma 1.21 that for any $a \in A$,

$$\|a\| = r(a) = \|\Gamma(a)\|_\infty.$$

So, $\Gamma$ is isometric.

□

[1] $*$-preserving isomorphism

Notice that the above argument shows that when $A$ is not unital, its Gelfand transform extends to the Gelfand transform on its unitization.

**Lemma 2.13.** *Let $A$ be a commutative* C*-*algebra. If $a \in A$ is self-adjoint, then $\sigma(a) \subset \mathbb{R}$.*

*Proof.* Suppose $a \in A$ is self-adjoint, and assume $A \subset \tilde{A}$. For each $t \in \mathbb{R}$, the power series

$$\sum_{n \geq 0} \frac{(ita)^n}{n!}$$

converges to some element $\exp(ita)$ in $\tilde{A}$. One checks that

$$\exp(ita)^* = \sum_{n \geq 0} \frac{(-ita)^n}{n!} = \exp(-ita) = \exp(ita)^{-1},$$

which means $\exp(ita)$ is a unitary in $\tilde{A}$. Now, consider the Gelfand map $\Gamma : \tilde{A} \to C(\hat{\tilde{A}})$. By the preceeding lemma, we know $\sigma(a) = \mathrm{ran}(\Gamma(a)) = \{\phi(a) : \phi \in \hat{A}\}$. So, it suffices to show that $\phi(a) \in \mathbb{R}$ for each $\phi \in \hat{A}$. Fix $\phi \in \hat{A}$. Since $\phi$ is a character (i.e. continuous, linear, multiplicative), it follows that for any $t \in \mathbb{R}$,

$$\phi(\exp(ita)) = \phi(\sum_{n \geq 0} \frac{(ita)^n}{n!}) = \sum_{n \geq 0} \frac{(it\phi(a))^n}{n!} = e^{it\phi(a)}.$$

Since $\exp(ita)$ is a unitary, we know from Example 1.12 that $e^{it\phi(a)} \in \mathbb{T}$ for all $t \in \mathbb{R}$. It follows that $\phi(a) \in \mathbb{R}$ as desired. □

*Remark* 2.14. We shall see soon that we did not need to assume $A$ was commutative in Lemma 2.13. The same argument would work by just considering the Gelfand transform on $\mathrm{C}^*(a, 1)$. However, we will need to first establish that the spectrum of $a$ in $\mathrm{C}^*(a, 1)$ is the same as its spectrum in $A$.

Now we are ready to prove the theorem.

*Proof of Gelfand Naimark Theorem.* First, we assume that $A$ is unital. We know from Lemma 2.12 that $\Gamma$ is isometric, which means its image in $C(\hat{A})$ is closed.

For any self-adjoint $a \in A$, we have $ran(\Gamma(a)) \subset \mathbb{R}$, which means $\Gamma(a) = \overline{\Gamma(a)}$ is self-adjoint. So Proposition 1.7, tells us $\Gamma$ is $*$-preserving.

So, altogether, $\Gamma(A)$ is a unital, norm closed self-adjoint subalgebra of $C(\hat{A})$ where $\hat{A}$ is compact and Hausdorff. Then the Stone-Weierstrass Theorem ([Conway, I.5,6]) says that $\Gamma(A) = C(\hat{A})$ provided that it separates the points of $\hat{A}$. But if $\phi$ and $\psi$ are distinct points in $\hat{A}$, then they have distinct kernels, and so $\Gamma(A)$ separates the points of $\hat{A}$.

Now suppose that $A$ is not unital. Then $\Gamma$ extends to the isometric $*$-isomorphism $\tilde{\Gamma} : A \to C(\hat{A})$. Since $A$ is an ideal of co-dimension one, $\tilde{\Gamma}(A)$ is a maximal ideal in $C(\hat{A})$ contained in the maximal ideal $\{f \in C(\hat{\tilde{A}}) : f(\phi_0) = 0\}$. Then $\tilde{\Gamma}(A) = \{f \in C(\hat{\tilde{A}}) : f(\phi_0) = 0\}$, and it follows that $\Gamma(A) = C_0(\hat{A})$ from the aforementioned identifications. □

**Corollary 2.15.** *Characters on commutative* C*-*algebras are $*$-homomorphisms.*

*Proof.* It suffices to prove that they map self-adjoint elements to real numbers. For any $\phi \in \hat{A}$, and $a \in A$ self-adjoint, we have $\Gamma(a)(\phi) = \phi(a) \in \mathbb{R}$. □

For any element $a$ in a C*-algebra $A$, we write $\mathrm{C}^*(a)$ for the C*-algebra generated by $a$. When $A$ is unital, $\mathrm{C}^*(a, 1)$ can be identified with the closure of the set of all polynomials on $a, a^*, 1$ (aka $*$-polynomials on $a$).

When $a$ is a normal, $B := \mathrm{C}^*(a)$ is a commutative C*-algebra, and so it is $*$-isomorphic to $C_0(\hat{B}) \subset C(\hat{\tilde{B}})$. Moreover, any character $\phi \in \hat{B}$ is determined by where it maps $a$. So, the map $\hat{\tilde{B}} \to \mathbb{C}$ given by $\phi \mapsto \phi(a)$ is a homeomorphism onto $\Gamma(a)(\hat{\tilde{B}})$, which we know is equal to $\sigma(a)$. Moreover, the Gelfand map then identifies $a$ with the identity function $z \mapsto z$ on $C(\sigma(a))$. When $a$ is not invertible, $C_0(\hat{B})$ corresponds to the ideal consisting of functions that vanish at 0. If $a$ is invertible, then $0 \notin \sigma(a)$, so either way, we can say

$$\mathrm{C}^*(a) \simeq C_0(\sigma(a) \backslash \{0\}).$$

**Problem:** What do we mean by $\sigma(a)$ here? By design, this must be the set of $\lambda \in \mathbb{C}$ such that $\lambda 1 - a$ is not invertible **in (the unitization of)** $B$, i.e. this is $\sigma_B(a)$, not $\sigma_A(a)$. In general, $\sigma_A(a)$ is smaller (there are more potential inverses for $a - \lambda 1$ in $A \supseteq B$), and we have no reason to suspect that these are the same set. But for C*-algebras, they are.

For now, we just establish the following.

**Proposition 2.16.** *Let $a$ be a normal element of a* C*-*algebra $A$ and $B = \mathrm{C}^*(a)$. Then $B \simeq C_0(\sigma_A(a)\backslash\{0\})$ and $\sigma_A(a) = \sigma_B(a)$.*

*Proof.* We have already established that $B \simeq C_0(\sigma_B(a)\backslash\{0\})$.

Suppose $\lambda \in \sigma_B(a)\backslash\{0\}$. Then for each $\epsilon > 0$, there exists $b \in B$ with $\|\Gamma(b)\| = 1$ and $\|\lambda\Gamma(b) - \Gamma(a)\Gamma(b)\| < \epsilon$. That means $\|b\| = 1$ and $\|\lambda b - ab\| < \epsilon$, which means $\lambda 1 - a$ is not invertible in $\tilde{A}$. (Indeed, if $c(\lambda 1 - x) = 1$, then $1 = \|b\| = \|c(\lambda 1 - x)b\| < \|c\|\epsilon$ for all $\epsilon$.) ☐

This justifies the terminology "spectrum" for the space of characters on a commutative C*-algebra.

Before moving too far away from Proposition 2.16, we remark that it yields a more general corollary.

**Corollary 2.17.** *If $a$ is a normal element in a unital* C*-*algebra $A$ and $B$ is any unital* C*-*subalgebra of $A$ containing $a$, then $\sigma_A(a) = \sigma_B(a)$.*

Now we come to an incredibly powerful tool, with which we conclude the section: *The Functional Calculus.* Let $A$ be a unital C*-algebra, $a \in A$ a normal element, and $f \in C(\sigma(a))$. We denote by $f(a)$ the inverse image of $f$ under the Gelfand transform of $\mathrm{C}^*(a, 1)$ (the isometric $*$-isomorphism between $\mathrm{C}^*(a, 1)$ and $C(\sigma(a))$).

**Corollary 2.18** (The Functional Calculus)**.** *Let $a$ be a normal element of a unital* C*-*algebra $A$ and $f, g \in C(\sigma(a))$. Then*
*(1) $f(\sigma(a)) = \sigma(f(a))$,*
*(2) $g(f(\sigma(a)) = (g \circ f)(a)$, and*
*(3) if $0 \in \sigma(a)$ and $f(0) = 0$, then $f(a)$ is in the non-unital* C*-*algebra,* $\mathrm{C}^*(a)$.

*Proof.* Since $f(a) \in \mathrm{C}^*(a, 1)$, we have

$$\sigma(f(a)) = \sigma(\Gamma(f(a))) = \sigma(f) = f(\sigma(a)).$$

Since $\Gamma$ is a homomorphism, the second claim holds immediately when $g$ is a Laurent polynomial (i.e. a polynomial in $z$ and $\overline{z}$). Then the general case follows by approximating $g$ uniformly with Laurent polynomials.

The third claim follows immediately from Proposition 2.16. ☐

**Exercise 2.19.** Check for understanding. If $a \in A$ is a normal element in a unital C*-algebra and $\Gamma : \mathrm{C}^*(a) \to C_0(\sigma(a))$ the Gelfand transform,
(1) What is its image $\Gamma(a) \in C_0(\sigma(a))$?
(2) If $a$ is invertible, is $a^{-1} \in C^*(a)$?

We will see this applied repeatedly in the section on positive elements.

**Exercise 2.20.** Suppose $A$ and $B$ are commutative unital C*-algebras and $\phi : A \to B$ a unital $*$-homomorphism. Then for any $a \in A$ and $f \in C(\sigma(a))$, we have $\phi(f(a)) = f(\phi(a))$.

**Exercise 2.21.** Let $\pi : A \to B$ be a surjective $*$-homomorphism between C*-algebras and $b \in B$ a self-adjoint element. Show that $b$ lifts to a self-adjoint element $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.