Robin Deeley *Groupoid C*-algebras*

Exercise: Suppose X is a nonempty finite set and R is an equivalence relation on X.

(a) Prove that $C^*(R)$ is the direct sum of finitely many matrix algebras.

(b) Compute the K-theory of $C^*(R)$.

If you have extra time:

Let G be a group. Examine the structure of the following two groupoids constructed from G.

1. Let $\mathcal{G} = G$ and $\mathcal{G}^2 = G \times G$; the multiplication map $\mathcal{G}^2 \to \mathcal{G}$ is given by group multiplication, and the map $\mathcal{G} \to \mathcal{G}$ is given by taking the inverse.
   (a) Prove that $\mathcal{G}$ is a groupoid.
   (b) What is $\mathcal{G}^0$ in this case?
   (c) Prove that $\mathcal{G}$ is étale iff G is discrete.
2. Let G be a finite group, and set $\tilde{\mathcal{G}} = G \times G$, with $\tilde{\mathcal{G}}^2 = \{((g,h),(gh,k)) : g,h,k \in G\}$. Then the map $\tilde{\mathcal{G}}^2 \to \tilde{\mathcal{G}}$ is given by $((g,h),(gh,k)) \mapsto (g,hk)$ and the map $\tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ is defined via $(g,h) \mapsto (gh,h^{-1})$.
   (a) Prove that $\tilde{\mathcal{G}}$ is a groupoid.
   (b) What is $C_c(\tilde{\mathcal{G}})$? What changes if G is a countable discrete group?

Corey Jones *Subfactors and quantum symmetries*

Exercise: Let $N \subset B(\mathcal{H})$ be a II_1 factor. For $d \in \mathbb{N}$, embed $N \hookrightarrow M_d(N)$ by

$$x \mapsto \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \qquad x \in N.$$

In this exercise, you will compute $[M_d(N) : N]$.

(a) Show that $B(L^2(M_d(N))) = M_{d^2}(B(L^2(N)))$, where the entries in the latter space are indexed by pairs of pairs: $((i,j),(k,\ell))$ for $i,j,k,\ell = 1,\dots,d$.

[**Hint:** first show that $L^2(M_d(N)) \cong L^2(N)^{\oplus d^2}$.]

(b) Show that $N' \cap B(L^2(M_d(N))) = M_{d^2}(N' \cap L^2(N))$.

(c) For $X = (x_{i,j})_{i,j=1}^d \in M_d(N)$, show that

$$e_N X = \begin{pmatrix} \frac{1}{d}\sum_{i=1}^d x_{i,i} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{d}\sum_{i=1}^d x_{i,i} \end{pmatrix}.$$

as vectors in $L^2(M_d(N))$.

(d) Viewing $e_N \in M_{d^2}(N' \cap L^2(N))$, show that the $((i,j),(k,\ell))$-entry of e_N is $\frac{1}{d}\delta_{i=j}\delta_{k=\ell}$.

(e) Compute $\tau_{M_d(N)}(e_N)$ and $[M_d(N) : N]$.

(f) Show that $\langle M_d(N), e_N\rangle \cong M_{d^2}(N)$.