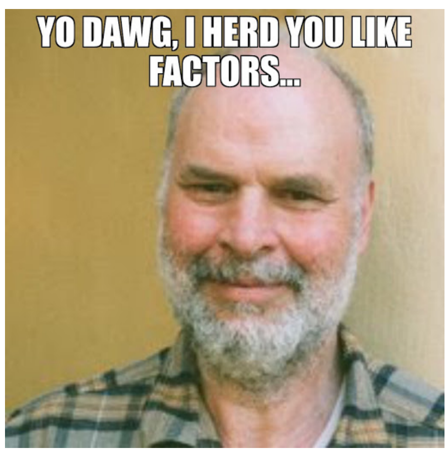
Chapter 6

Subfactors

In this chapter we will study *subfactors*: an inclusion of factors $N \leq M$ satisfying $1_M \in N$. We will restrict ourselves to the case when N and M are both II_1 factors, though more general inclusions have been studied extensively in the literature. Note that if τ_M is the unique trace on M, then $\tau_M|_N$ is necessarily the unique trace on N. Despite the starting point sounding like a an Xzibit meme, subfactors result in an incredibly rich theory with deep connections to knot polynomials and tensor categories.

This chapter will not be as thorough as the other chapters, and part of the reason is because entire books can be written about this subject alone. We present here only a starting point for learning about subfactors, though we will strive to present complete details whenever possible.



6.1 Index for Subfactors

Let $1_M \in N \subset M$ be an inclusion of II_1 factors, and let τ_M and τ_N be the unique traces on M and N, respectively. We will identify M (and consequently N) with its representation on $L^2(M)$. In this context we will denote $N' \cap B(L^2(M))$ simply by N', which satisfies $N' \supset M'$. Note that N' is a factor, and by Remark 4.3.9 we know that N' is type II. Consequently, N' is either a II_1 factor or a II_∞ factor. In the former case, we will denote its unique trace by $\tau_{N'}$.

Noting that $\tau_M|_N = \tau_N$, we see that the closure of $N\hat{1}$ in $L^2(M)$ is a copy of $L^2(N)$. Thus we can view $L^2(N)$ as a closed subspace of $L^2(M)$ and we let $e_N \in B(L^2(M))$ be the projection onto $L^2(N)$. Since $L^2(N)$ is reducing for N, we have $e_N \in N'$ by Lemma 1.2.5.

Definition 6.1.1. Let $1_M \in N \subset M$ be an inclusion of II_1 factors. We define the **index** of N inside M as the quantity

$$[M : N] := \frac{1}{\tau_{N'}(e_N)}$$

when N' is a II_1 factor, and otherwise set $[M : N] := \infty$.

Assuming N' is a II_1 (i.e. finite) factor, we have $\tau_{N'}(e_N) \leq 1$ and consequently $[M : N] \geq 1$. In particular, we have $[M : N] = 1$ if and only if $\tau_{N'}(e_N) = 1$. Since $\tau_{N'}$ is a faithful state this is further equivalent to $e_N = 1$, which means $L^2(N) = L^2(M)$ and $N = M$. Thus $[M : N] = 1$ if and only if $N = M$. Roughly speaking, $[M : N]$ measures how much larger M is than N. The notation should remind of you the notation for group indices, and the following example makes this explicit.

Example 6.1.2. Let $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ be a free ergodic p.m.p action of a countably infinite discrete group on a probability space (X, μ). Then $M := L^\infty(X, \mu) \rtimes_\alpha \Gamma$ is a II_1 factor by Example 4.3.16. Let $\Lambda \leq \Gamma$ be a subgroup such that $\alpha|_\Lambda$ is still ergodic (it is automatically free and p.m.p.). Then $N := L^\infty(X, \mu) \rtimes_{\alpha|_\Lambda} \Lambda$ is a II_1 subfactor of M. In this case, we have

$$[M : N] = [\Gamma : \Lambda].$$

We provide only a sketch of the proof. Assume $[\Gamma : \Lambda] = n < \infty$ so that

$$\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$$

for some $g_2, \ldots, g_n \in \Gamma \setminus \Lambda$. By Exercise 4.3.13, we have $L^2(M) = \ell^2(\Gamma) \otimes L^2(X, \mu)$ and $L^2(N) = \ell^2(\Lambda) \otimes L^2(X, \mu)$. Consequently

$$\begin{aligned} L^2(M) &= \left[\ell^2(\Lambda) \otimes L^2(X,\mu)\right] \oplus \left[\ell^2(\Lambda g_2) \otimes L^2(X,\mu)\right] \oplus \cdots \oplus \left[\ell^2(\Lambda g_n) \otimes L^2(X,\mu)\right] \\ &= L^2(N) \oplus \left[\ell^2(g_2\Lambda) \otimes L^2(X,\mu)\right] \oplus \cdots \oplus \left[\ell^2(g_n\Lambda) \otimes L^2(X,\mu)\right]. \end{aligned}$$

It can be shown that the projections onto each of the remaining direct summands is equivalent to e_N in N' (see Exercise 6.1.3). Consequently, $\tau_{N'}(e_N) = \frac{1}{n}$ and so $[M : N] = n = [\Gamma : \Lambda]$. ■

Remark 6.1.3. There is an alternate formula for the index. Suppose $M \subset B(\mathcal{H})$ is a finite factor such that $M' \cap B(\mathcal{H})$ is also finite. Denote their respective traces by τ_M and $\tau_{M'}$. For any non-zero $\xi \in \mathcal{H}$, $M\xi$ and $M'\xi$ are reducing for M' and M, respectively, and so $[M\xi] \in M'$ and $[M'\xi] \in M$ by Lemma 1.2.5. Murray and von Neumann defined the *coupling constant* of M over $\mathcal{H}$ to be the ratio

$$\frac{\tau_M([M'\xi])}{\tau_{M'}([M\xi])},$$

and they showed that it is independent of the choice of ξ. When $1_M \in N \subset M \subset B(\mathcal{H})$ is a subfactor, it can be shown that the ratio of the coupling constants for N and M

$$\frac{\tau_N([N'\xi])}{\tau_{N'}([N\xi])} \frac{\tau_{M'}([M\xi])}{\tau_M([M'\xi])} \tag{6.1}$$

is further independent of the representation $M \subset B(\mathcal{H})$. This expression is in fact Jones' original definition for $[M : N]$, and since it does not depend on either $\mathcal{H}$ or ξ we can check that it matches with Definition 6.1.1. Indeed, take $\mathcal{H} = L^2(M)$ and $\xi = \hat{1}$, then $\hat{1}$ being cyclic and separating for M implies $[M\hat{1}] = [M'\hat{1}] = [N'\hat{1}] = 1$. Consequently

$$\frac{\tau_N([N'\hat{1}])}{\tau_{N'}([N\hat{1}])} \frac{\tau_{M'}([M\hat{1}])}{\tau_M([M'\hat{1}])} = \frac{1}{\tau_{N'}([N\hat{1}])} = [M : N].$$

Thus (6.1) gives us a more flexible definition for the $[M : N]$.

Given a projection $p \in N' \cap M$, we can consider the compressed inclusion $p \in Np \subset pMp$. Note that Np and pMp are both type II factors by Corollary 4.2.3 and Remark 4.3.9, and since $\frac{1}{\tau_M(p)}\tau_M$ defines a trace on pMp we see that they are in fact II_1 factors. Thus we can consider the index $[pMp \colon Np]$. Using Remark 6.1.3 and a few facts about the coupling constant, one can show

$$[pMp : Np] = [M : N]\tau_M(p)\tau_{N'}(p). \tag{6.2}$$

We can use this fact to derive some nice consequences for certain values of the index.

Proposition 6.1.4. *If $[M : N] < \infty$, then $N' \cap M$ is finite dimensional.*

Proof. Let $p_1, \ldots, p_n \in \mathcal{P}(N' \cap M)$ be non-zero pairwise orthogonal projections. Then since the index is always greater than or equal to one, (6.2) implies

$$[M : N] \geq [M : N] \sum_{i=1}^{n} \tau_M(p_i) = \sum_{i=1}^{n} \frac{1}{\tau_{N'}(p_i)} [p_i M p_i : N p_i] \geq \sum_{i=1}^{n} \frac{1}{\tau_{N'}(p_i)}.$$

Note that the condition $\sum_{i=1}^{n} \tau_{N'}(p_i) \leq 1$ implies $\tau_{N'}(p_i) \leq \frac{1}{n}$ for some $i = 1, \ldots, n$. Consequently, $[M : N] \geq n$, and so for any family of non-zero pairwise orthogonal projections $\mathcal{P} \subset \mathcal{P}(N' \cap M)$ we must have $|\mathcal{P}| \leq [M : N] < \infty$. Suppose $\mathcal{P}$ is a maximal family of pairwise orthogonal projections. We must have

$$\sum_{p \in \mathcal{P}} p = 1,$$

since otherwise $\{1 - \sum_p p\} \cup \mathcal{P}$ contradicts the maximality of $\mathcal{P}$. Also, each $p \in \mathcal{P}$ must be minimal in $N' \cap M$ because otherwise for $0 < q < p$ the maximality of $\mathcal{P}$ is contradicted by $\{q, p - q\} \cup \mathcal{P} \setminus \{p\}$. Now, as minimal projections, $p, q \in \mathcal{P}$ are either centrally orthogonal or equivalent in $N' \cap M$ by Proposition 4.1.9. If they are centrally orthogonal, then the same proposition implies $pxq = 0$ for all $x \in N' \cap M$. If they are equivalent, say by $vv^* = p$ and $v^*v = q$, then for $x \in N' \cap M$ we have

$$pxq = pxqq = px(v^*v)(v^*v) = pxv^*(vv^*)v = pxv^*pv = cpv = cv$$

for some $c \in \mathbb{C}$. Denote $v := v_{p,q}$, and if p and q are centrally orthogonal set $v_{p,q} := 0$. Thus for any $x \in N' \cap M$, we have

$$x = (\sum_{p \in \mathcal{P}} p)x(\sum_{q \in \mathcal{P}} q) = \sum_{p,q \in \mathcal{P}} pxq = \sum_{p,q \in \mathcal{P}} c_{p,q} v_{p,q},$$

for $c_{p,q} \in \mathbb{C}$. Hence $N' \cap M = \operatorname{span}\{v_{p,q} \colon p, q \in \mathcal{P}\}$, and since $\mathcal{P}$ is a finite set we see that $N' \cap M$ is finite dimensional. □

Proposition 6.1.5. *If $[M \colon N] < 4$, then $N' \cap M = \mathbb{C}$.*

Proof. Suppose, towards a contradiction that $p, q \in \mathcal{P}(N' \cap M)$ are orthogonal and non-zero. Then (6.2) implies (by the same argument as in the proof of the previous proposition)

$$[M : N] \geq \frac{1}{\tau_{N'}(p)} + \frac{1}{\tau_{N'}(q)} \geq \frac{1}{\tau_{N'}(p)} + \frac{1}{1 - \tau_{N'}(p)}.$$

This last expression is minimized at $\tau_{N'}(p) = \frac{1}{2}$, and hence we obtain the contradiction $[M : N] \geq 4$. □

We present the next result without proof, but we direct the interested reader to Jones' original paper.

Theorem 6.1.6 (Jones, 1983)**.** *Let $1_M \in N \subset M$ be an inclusion of II_1 factors. Then*

$$[M : N] \in \{4 \cos^2(\pi/n) \colon n \geq 3\} \cup [4, \infty].$$

Moreover, every value in the set above occurs as the index of some unital inclusion of II_1 factors.

This result is part of the work that would ultimately earn Vaughan Jones the Fields Medal. That the index has a discrete component to its range was a remarkable revelation at the time[1].

[1] Masamichi Takesaki says he first heard about the result when picking Vaughan Jones up from the airport for a visit to UCLA, and was so startled by it that he nearly crashed the car.

Exercises

6.1.1. Let $N \subset P \subset M$ be inclusions of II_1 factors. Show that $[M : N] = [M : P][P : N]$. [**Hint:** use (6.1).]

6.1.2. Let $N \subset B(\mathcal{H})$ be a II_1 factor. For $d \in \mathbb{N}$, embed $N \hookrightarrow M_d(N)$ by

$$x \mapsto \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \qquad x \in N.$$

In this exercise, you will compute $[M_d(N) : N]$.

(a) Show that $B(L^2(M_d(N))) = M_{d^2}(B(L^2(N)))$, where the entries in the latter space are indexed by pairs of pairs: $((i, j), (k, \ell))$ for $i, j, k, \ell = 1, \ldots, d$.

[**Hint:** first show that $L^2(M_d(N)) \cong L^2(N)^{\oplus d^2}$.]

(b) Show that $N' \cap B(L^2(M_d(N))) = M_{d^2}(N' \cap L^2(N))$.

(c) For $X = (x_{i,j})_{i,j=1}^d \in M_d(N)$, show that

$$e_N X = \begin{pmatrix} \frac{1}{d}\sum_{i=1}^d x_{i,i} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{d}\sum_{i=1}^d x_{i,i} \end{pmatrix}.$$

as vectors in $L^2(M_d(N))$.

(d) Viewing $e_N \in M_{d^2}(N' \cap L^2(N))$, show that the $((i, j), (k, \ell))$-entry of e_N is $\frac{1}{d}\delta_{i=j}\delta_{k=\ell}$.

(e) Compute $\tau_{M_d(N)}(e_N)$ and $[M_d(N) : N]$.

6.1.3. Let $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ be a free ergodic p.m.p action of a countably infinite discrete group on a probability space (X, μ). Let $\Lambda < \Gamma$ be a finite index subgroup with

$$\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n.$$

for some $g_2, \ldots, g_n \in \Gamma \setminus \Lambda$. Assume $\alpha|_\Lambda$ is ergodic and set

$$M := L^\infty(X, \mu) \rtimes_\alpha \Gamma$$
$$N := L^\infty(X, \mu) \rtimes_{\alpha|_\Lambda} \Lambda.$$

Recall that $L^2(M) = \ell^2(\Gamma) \otimes L^2(X, \mu)$ and $L^2(N) = \ell^2(\Lambda) \otimes L^2(X, \mu)$.

(a) For each $i = 2, \ldots, n$, show that $\ell^2(\Lambda g_i) \otimes L^2(X, \mu)$ is reducing for N.

(b) Let J be the canonical conjugation operator on $L^2(M)$: $J\hat{x} = \widehat{x^*}$. Show that

$$J(\delta_g \otimes f) = \delta_{g^{-1}} \otimes \alpha_{g^{-1}}(\bar{f})$$

for $g \in \Gamma$ and $f \in L^\infty(X, \mu)$.

(c) For each $i = 2, \ldots, n$, show that $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J \in N'$ and that this is the projection onto the subspace $\ell^2(\Lambda g_i) \otimes L^2(X, \mu)$.

(d) For each $i = 2, \ldots, n$, show that e_N is equivalent to $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J$ in N'.

(e) Compute $\tau_{N'}(e_N)$ and $[M : N]$.

6.1.4. Let Γ be an i.c.c. group, let $\Lambda < \Gamma$ be a finite index subgroup, and set $M := L(\Gamma)$ and $N := L(\Lambda)$.

(a) Show that Λ is i.c.c.

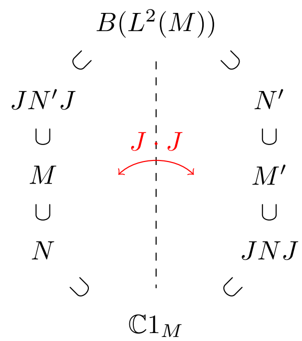
(b) Suppose $\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$ for $g_2, \ldots, g_n \in \Lambda \setminus \Lambda$. For each $i = 2, \ldots, n$, show that $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J \in N'$ and that this is the projection onto $\ell^2(\Lambda g_i)$.

(c) For each $i = 2, \ldots, n$, show that e_N is equivalent to $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J$ in N'.

(d) Compute $\tau_{N'}(e_N)$ and $[M : N]$.

6.2 The Basic Construction

Once more we let $1_M \in N \subset M \subset B(L^2(M))$ be an inclusion of II_1 factors with unique traces τ_N and τ_M, respectively. Let $e_N \in B(L^2(M))$ be the projection onto the subspace $L^2(N) \subset L^2(M)$, so that $e_N \in N'$. Recall from Theorem 5.2.5, that if J is the canonical commutation on $L^2(M)$ then $JMJ = M'$. Consequently, $JNJ \subset M'$. However, $N \subset M$ implies $N' \subset M'$. Thus we cannot have $JNJ = N'$ unless $N' = M'$, in which case $N = M$. Since this case corresponds to $[M : N] = 1$, we see that whenever $[M : N] > 1$ we have M' is a strict subset of N', and $JN'J \supset JM'J = M$. We summarize these various relations in the diagram below.



Horizontal reflection in the above diagram corresponds to conjugating by J. There is another important symmetry: reflecting through the center of the diagram corresponds to taking the commutant. This is clear for the pairs $(\mathbb{C}1_M, B(L^2(M)))$, (M, M'), and (N, N'), but it also holds for $(JN'J, JNJ)$. That is, $(JN'J)' = JNJ$. Indeed, $x \in (JN'J)'$ if and only if $x(JyJ) = (JyJ)x$ for all $y \in N'$, and conjugating the equation by J shows this is equivalent to $(JxJ)y = y(JxJ)$ for all $y \in N'$. Consequently, $x \in (JN'J)'$ if and only if $JxJ \in N'' = N$, and thus the claimed equality holds. In particular, this implies $JN'J$ is a factor:

$$\mathcal{Z}(JN'J) = (JN'J) \cap (JN'J)' = (JN'J) \cap JNJ = J(N' \cap N)J = \mathbb{C},$$

since N is a factor. Thus using only conjugation by J and taking commutants, we have produced a new factor extending our original inclusion: $N \subset M \subset JN'J$. We will study this new factor further, but first we require a lemma.

Recall from Theorem 5.2.7 that there is a faithful normal trace-preserving conditional expectation $E_N : M \to N$. This map is positive, restricts to the identity on N, and satisfies $E_N(axb) = aE_N(x)b$ for all $a, b \in N$ and $x \in M$. Also recall that for each $x \in M$, $E_N(x)$ is uniquely determined by $E_N(x)\hat{1} = e_N\hat{x}$.

Lemma 6.2.1.

(i) *For $x \in M$, $e_N x e_N = E_N(x) e_N$.*

(ii) *$N = \{e_N\}' \cap M$.*

(iii) *$N' = \{M' \cup \{e_N\}\}''$.*

(iv) *$Je_N = e_N J$.*

Proof.

(i): For $y \in M$, we have

$$e_N x e_N \hat{y} = e_N x E_N(y)\hat{1} = e_N \widehat{xE_N(y)} = E_N(xE_N(y))\hat{1} = E_N(x)E_N(y)\hat{1} = E_N(x)e_N\hat{y}.$$

Since $\widehat{M}$ is dense in $L^2(M)$, we have $e_N x e_N = E_N(x)e_N$.

(ii): Since $e_N \in N'$, we have $N \subset \{e_N\}' \cap M$. On the other hand, for $x \in \{e_N\}' \cap M$ we have

$$E_N(x)\hat{1} = e_N\hat{x} = e_N x\hat{1} = xe_N\hat{1} = x\hat{1}.$$

Since $\hat{1}$ is separating for M, we must have $x = E_N(x) \in N$.

(iii): The Bicommutant Theorem implies it suffices to show $\{M' \cup \{e_N\}\}' = N$. Note that $\{M' \cup \{e_N\}\}' \subset M'' \cap \{e_N\}' = M \cap \{e_N\}' = N$ by the previous part. The reverse inclusion follows from $N \subset M$ and the previous part.

(iv): For $x \in M$ we have

$$Je_N\hat{x} = JE_N(x)\hat{1} = E_N(x)^*\hat{1} = E_N(x^*)\hat{1} = e_N\widehat{x^*} = e_N J\hat{x}.$$

Thus the density of $\widehat{M}$ in $L^2(M)$ yields $Je_N = e_N J$. □

Proposition 6.2.2. *Let $1_M \in N \subset M \subset B(L^2(M))$ be an inclusion of II_1 factors. If $e_N \in B(L^2(M))$ is the projection onto $L^2(N)$, then the factor $JN'J$ is generated by $M \cup \{e_N\}$. In fact, $JN'J$ is generated by the $*$-algebra span$(M \cup Me_NM)$.*

Proof. Recall that we have already seen that $JN'J$ is a factor in the discussion at the beginning of the section. From Lemma 6.2.1.(iii), we see that N' is the von Neumann algebra generated by $M' \cup \{e_N\}$. Note the unital *-algebra generated by $M' \cup \{e_N\}$ is spanned by elements of the form $y_1e_Ny_2e_N \cdots e_Ny_d$ for $d \geq 1$ and $y_1, \ldots, y_d \in M'$. Using Lemma 6.2.1.(iv) to assert $e_N = Je_NJ$ we have

$$J(y_1e_Ny_2e_N \cdots e_Ny_d)J = (Jy_1J)e_N(Jy_2J)e_N \cdots e_N(Jy_dJ).$$

Since $JM'J = M$ by Theorem 5.2.5, the above element is in the *-algebra generated by $M \cup \{e_N\}$. Consequently, $JN'J$ is the von Neumann algebra generated by $M \cup \{e_N\}$.

The $*$-algebra generated by $M \cup \{e_N\}$ is $\operatorname{span}\{x_1e_Nx_2e_N \cdots e_Nx_d \colon d \geq 1,\ x_1, \ldots, x_d \in M\}$. But Lemma 6.2.1.(i),(iii) imply for $d \geq 3$

$$x_1e_Nx_2e_Nx_3e_N \cdots e_Nx_d = x_1E_N(x_2)e_NE_N(x_3)e_N \cdots e_Nx_d = x_1E_N(x_2)E_N(x_3)\cdots E_N(x_{n-1})e_Nx_d.$$

So span$(M \cup Me_NM)$ is a $*$-algebra generating $JN'J$.

□

In light of the above proposition, we make the following definition.

Definition 6.2.3. The **basic construction** for $N \subset M$ is $\langle M, e_N \rangle := \{M \cup \{e_N\}\}'' \subset B(L^2(M))$.

By the discussion of at the beginning of the section, we know the commutant of $\langle M, e_N \rangle = JN'J$ is JNJ, which is a II_1 factor since N is a II_1 factor. So by Remark 4.3.9 we know $\langle M, e_N \rangle$ is a type II factor, but it could be either type II_1 or type II_∞. As we will see in the next theorem, the former case happens precisely when the index $[M : N]$ is finite.

Theorem 6.2.4. *Let $1_M \in N \subset M \subset B(L^2(M))$ be an inclusion of II_1 factors, and let $\langle M, e_N \rangle$ be its basic construction. Then $\langle M, e_N \rangle$ is a II_1 factor if and only if $[M : N] < \infty$. In this case, we have*

$$[\langle M, e_N \rangle : M] = [M : N].$$

If $\tau_{\langle M, e_N \rangle}$ is the unique trace on $\langle M, e_N \rangle$, then

$$\tau_{\langle M, e_N \rangle}(xe_N) = \frac{1}{[M : N]}\tau_M(x) \qquad \forall x \in M,$$

and in particular $\tau_{\langle M, e_N \rangle}(e_N) = [M : N]^{-1}$.

Proof. By the discussion preceding the theorem we know that $\langle M, e_N \rangle$ is a type II factor, and so it suffices to show $\langle M, e_N \rangle$ is finite if and only if $[M : N] < \infty$. Recall that $[M : N] < \infty$ if and only if N' is a finite by definition of the index. Thus it further suffices to show $\langle M, e_N \rangle$ is finite if and only if N' is finite, and by Theorem 5.1.5 it yet further suffices to show $\langle M, e_N \rangle$ has a trace if and only if N' has a trace. But this follows from $\langle M, e_N \rangle = JN'J$ because a trace on one algebra can be used to define a trace on the other:

$$\begin{aligned} \tau_{\langle M, e_N \rangle}(x) &:= \tau_{N'}(JxJ) \qquad && x \in \langle M, e_N \rangle . \\ \tau_{N'}(y) &:= \tau_{\langle M, e_N \rangle}(JyJ) \qquad && y \in JN'J. \end{aligned}$$

Thus $\langle M, e_N\rangle$ is II_1 factor if and only if $[M : N] < \infty$.

Let $\tau_{\langle M,e_N\rangle}$ be the unique trace on $\langle M, e_N\rangle$. By the above, we have $\tau_{\langle M,e_N\rangle}(x) = \tau_{N'}(JxJ)$ for all $x \in \langle M, e_N\rangle$, and in particular

$$\tau_{\langle M,e_N\rangle}(e_N) = \tau_{N'}(Je_NJ) = \tau_{N'}(e_N) = \frac{1}{[M : N]},$$

where in the second equality we have used $e_N = Je_NJ$ from Lemma 6.2.1.(iv). Now, by Lemma 6.2.1.(ii) we see that $N \ni x \mapsto \tau_{\langle M,e_N\rangle}(xe_N)$ defines a tracial positive linear functional on N, and so must equal $c\tau_N$ for some $c \in \mathbb{C}$ by the uniquness of τ_N. Setting $x = 1$ reveals

$$c = c\tau_N(1) = \tau_{\langle M,e_N\rangle}(1e_N) = \tau_{\langle M,e_N\rangle}(e_N) = \frac{1}{[M : N]}.$$

Thus $\tau_{\langle M,e_N\rangle}(xe_N) = \frac{1}{[M:N]}\tau_N(x)$ for $x \in N$. Using Lemma 6.2.1.(i), we can show this also holds for $x \in M$:

$$\begin{aligned}\tau_{\langle M,e_N\rangle}(xe_N) &= \tau_{\langle M,e_N\rangle}(e_Nxe_N) = \tau_{\langle M,e_N\rangle}(E_N(x)e_N)\\ &= \frac{1}{[M : N]}\tau_N(E_N(x)) = \frac{1}{[M : N]}\tau_M(E_N(x)) = \frac{1}{[M : N]}\tau_M(x),\end{aligned}$$

where the last equality uses the fact that E_N is trace-preserving.

Finally, we compute the index $[\langle M, e_N\rangle : M]$ using (6.1). We take $\mathcal{H} = L^2(M)$ and $\xi = \hat{1}$. Note that $\hat{1}$ is cyclic for $\langle M, e_N\rangle$ since it is cyclic for M, and it is cyclic for M' since it is separating for M. Consequenlty, $[\langle M, e_N\rangle\, \hat{1}] = [M'\hat{1}] = [M\hat{1}] = 1$. Thus

$$[\langle M, e_N\rangle : M] = \frac{\tau_M([M'\hat{1}])}{\tau_{M'}([M\hat{1}])}\frac{\tau_{\langle M,e_N\rangle'}([\langle M, e_N\rangle]\hat{1})}{\tau_{\langle M,e_N\rangle}([\langle M, e_N\rangle'\,\hat{1}])} = \frac{\tau_M(1)}{\tau_{M'}(1)}\frac{\tau_{\langle M,e_N\rangle'}(1)}{\tau_{\langle M,e_N\rangle}([\langle M, e_N\rangle'\,\hat{1}])} = \frac{1}{\tau_{\langle M,e_N\rangle}([\langle M, e_N\rangle'\,\hat{1}])}$$

Now, as we saw above $\langle M, e_N\rangle' = JNJ$ and so $[JNJ\hat{1}] = [JN\hat{1}] = [N\hat{1}] = e_N$. Since $\tau_{\langle M,e_N\rangle}(e_N) = [M : N]^{-1}$, the above computation yields $[\langle M, e_N\rangle : M] = [M : N]$. □

We now see that a finite index inclusion of II_1 factors $N \subset M$ begets another finite index inclusion of II_1 factors: $M \subset \langle M, e_N\rangle$. Moreover, the index of this new inclusion equals the original index and is therefore finite. Consequently, we can iterate this process and generate a tower of II_1 factors:

$$N \subset M \subset \langle M, e_N\rangle \subset \langle\langle M, e_N\rangle, e_M\rangle \subset \cdots$$

If we relabel these von Neumann algebras by $M_0 := N$, $M_1 := M$, $M_2 := \langle M, e_N\rangle$, etc. then we have

$$M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots,$$

and $[M_i : M_{i-1}] = [M : N]$ for all $i \geq 1$. Moreover, by Exercise 6.1.1 for any $i > j \geq 0$ we have

$$[M_i : M_j] = \prod_{k=j+1}^{i}[M_k : M_{k-1}] = [M : N]^{i-j}.$$

In particular, $[M_i : M_0] = [M : N]^i < \infty$ and $[M_i : M_1] = [M : N]^{i-1} < \infty$, and so $M_0' \cap M_i$ and $M_1' \cap M_i$ are finite dimensional by Proposition 6.1.4.

Definition 6.2.5. The **Jones tower** for a finite index inclusion of II_1 factors $N \subset M$ is series of inclusions constructed above:

$$\begin{array}{ccccccccc} M_0 & \subset & M_1 & \subset & M_2 & \subset & M_3 & \subset & \cdots \\ \shortparallel & & \shortparallel & & \shortparallel & & & & \\ N & & M & & \langle M, e_N\rangle & & & & \end{array}$$

The **standard invariant** of $N \subset M$ is the collection of finite dimensional relative commutants $\{M_0' \cap M_i\}_{i\geq 0} \cup \{M_1' \cap M_i\}_{i\geq 1}$:

$$\begin{array}{ccccccccc} \mathbb{C} = M_0' \cap M_0 & \subset & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & M_0' \cap M_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} = M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & M_1' \cap M_2 & \subset & \cdots \end{array}$$

While the standard invariant may seem to be a dizzying array of von Neumann algebras, remember that each $M_j' \cap M_i$ is finite-dimensional and consequently is isomorphic to a direct sum of matrix algebras. Moreover, one can diagrammatically encode the data of these relative commutants and their various inclusions using planar algebras. These objects also provide a bridge between subfactors and category theory, and although they are worthy of their own entire course we will not go into further detail on them here.

We conclude with an example where the basic construction can be explicitly described. The resulting von Neumann algebra is the generalization of the crossed product construction from Example 4.3.16, where $L^\infty(X,\mu)$ (i.e. a commutative von Neumann algebra) has been replaced with II_1 factor.

Example 6.2.6. Consider a II_1 factor $M \subset B(L^2(M))$. Let $\mathcal{U}(L^2(M))$ denote the group of unitary operators on $L^2(M)$, and suppose $U < \mathcal{U}(L^2(M))$ is a finite subgroup satisfying $U \cap M = \{1\}$, $uMu^* = M$ for all $u \in U$, and $u\hat{1} = \hat{1}$ for all $u \in U$. Denote

$$M^U := \{x \in M : uxu^* = x \ \forall u \in U\}.$$

The hypotheses on U imply that this is a factor. This is not obvious but we will assume it as a fact. Then

$$p := \frac{1}{|U|}\sum_{u\in U} u \in (M^U)',$$

and p is a projection (Exercise 6.2.3). Observe for $x \in M$ that

$$p\hat{x} = \frac{1}{|U|}\sum_{u\in U} ux\hat{1} = \frac{1}{|U|}\sum_{u\in U} uxu^*u\hat{1} = \frac{1}{|U|}\sum_{u\in U} uxu^*\hat{1},$$

and $\frac{1}{|U|}\sum_u uxu^* \in M^U$. Thus $p = e_{M^U}$. We claim that

$$\langle M, e_{M^U}\rangle = \left\{\sum_{u\in U} x_u u : x_u \in M\right\}''.$$

Denote the set on the right by B. For $x, y \in M$ we have

$$xe_{M^U}y = \frac{1}{|U|}\sum_{u\in U} xuy = \frac{1}{|U|}\sum_{u\in U} x(uyu^*)u \in B.$$

Since the identity of the group U is 1, we have $x = x1 \in B$ for $x \in M$. Thus $\mathrm{span}(M \cup Me_N M) \subset B$, and the former is a $*$-algebra generating $\langle M, e_{M^U}\rangle$ by Proposition 6.2.2. Thus to prove the claim it suffices to show $B \subset \langle M, e_{M^U}\rangle$, and this will follow if $U \subset \langle M, e_{M^U}\rangle$. For $x \in M$ we have

$$JuJ\hat{x} = Jux^*\hat{1} = Jux^*u^*u\hat{1} = Jux^*u^*\hat{1} = \widehat{uxu^*} = uxu^*\hat{1} = ux\hat{1} = u\hat{x}.$$

So $JuJ = u$ by the density of $\widehat{M} \subset L^2(M)$. Since $U \subset (M^U)'$, this shows $U = JUJ \in J(M^U)'J = \langle M, e_{M^U}\rangle$, and so the claim holds. The trace on $\langle M, e_{M^U}\rangle$ is given by

$$\tau_{\langle M, e_{M^U}\rangle}\left(\sum_{u\in U} x_u u\right) = \tau_M(x_1)$$

(see Exercise 6.2.4). In particular,

$$\tau_{\langle M, e_{M^U}\rangle}(e_{M^U}) = \tau_{\langle M, e_{M^U}\rangle}\left(\frac{1}{|U|}\sum_{u\in U} u\right) = \frac{1}{|U|}\tau_M(1) = \frac{1}{|U|}.$$

So by Theorem 6.2.3 we have $[M : M^U] = |U|$.

Exercises

6.2.1. Show that the basic construction for $N \subset M_d(N)$ is $M_{d^2}(N)$. [**Hint:** use Proposition 6.2.2 and the computation of e_N in Exercise 6.1.2.]

6.2.2. Let Γ be an i.c.c. group, let $\Lambda < \Gamma$ be a finite index subgroup, and set $M := L(\Gamma)$ and $N := L(\Lambda)$. Suppose

$$\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$$

for $g_2, \ldots, g_n \in \Gamma \setminus \Lambda$. Set $p_1 := e_N$ and $p_i = \lambda(g_i) e_N \lambda(g_i^{-1})$ for $i = 2, \ldots, n$.

(a) Show that $p_i M p_i$ is spatially isomorphic to $N e_N$ for each $i = 2, \ldots, n$.

(b) Show that $\langle M, e_N \rangle$ is isomorphic to $M_n(N)$. What is the image of M under this isomorphism?

6.2.3. Let $\mathcal{U}(\mathcal{H})$ be the group of unitaries on a Hilbert space $\mathcal{H}$. For a finite subgroup $U < \mathcal{U}(\mathcal{H})$, show that

$$\frac{1}{|U|} \sum_{u \in U} u$$

is a projection.

6.2.4. Let $M \subset B(L^2(M))$ be a II_1 factor and let $U < \mathcal{U}(L^2(M))$ be a finite subgroup satisfying $U \cap M = \{1\}$ and $uMu^* = M$ for all $u \in M$.

(a) Show that $\tau_M(uxu^*) = \tau_M(x)$ for all $u \in U$ and $x \in M$. [**Warning:** since $u \notin M$ when u is non-trivial, this is **not** simply a consequence of the tracial property of τ_M.]

(b) Show that

$$\tau\left(\sum_{u \in U} x_u u\right) = \tau_M(x_1)$$

defines a faithful trace on the $*$-algebra $\{\sum_u x_u u \colon x_u \in M\}$.