Problem Session

July 15th, 2020

Lauren Ruth *Operator Algebras and Equivalences between Groups*

Exercise: Show that measure equivalence of groups is an equivalence relation.

If you have extra time, consider the following exercises:

1. Give an example of a space measure (X, μ) and a measure-preserving action of $\mathbb{Z}$ on X along with a fundamental domain.
2. Let $\Gamma = SL_2(\mathbb{Z})$ act on the upper-half plane $H \subseteq \mathbb{C}$ by fractional linear transformations:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$
   (a) Show that if $\mu(E) := \int_E \frac{dx\,dy}{y^2}$ then
$$\mu(E) = \mu(g \cdot E)$$
   for every $g \in \Gamma$ and $E \subset H$ measurable.

   (b) Show that the set $\mathcal{F} = \{z \in H : -\frac{1}{2} \leq \text{Re }(z) < \frac{1}{2}, |z| \geq 1\} \cup \{z \in H : |z| = 1, \text{Re }(z) \leq 0\}$ is a fundamental domain for Γ.
   [**Hint:** Use the fact that Γ is generated by the elements
$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
   How do the elements a and b act on a point $z \in H$?]

Nate Brown *Duality as the bridge between C*- and W*-algebras*

We say a von Neumann algebra $M \subset B(\mathcal{H})$ is **injective** if there is a contractive linear map $\Phi\colon B(\mathcal{H}) \to M$ such that $\Phi(x) = x$ for all $x \in M$.

Exercise: For a direct sum of von Neumann algebras $M := M_1 \oplus M_2$, show that M is injective if and only if M_1 and M_2 are injective.

Exercise: Suppose M is injective and $I \subset M$ is a σ-WOT closed ideal. Show that I and M/I are injective.

Lecture Exercises

W*.1 Let Γ be an i.c.c. group, let $\Lambda < \Gamma$ be a finite index subgroup, and set $M := L(\Gamma)$ and $N := L(\Lambda)$.

(a) Show that Λ is i.c.c.

(b) Suppose $\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$ for $g_2, \ldots, g_n \in \Lambda \setminus \Lambda$. For each $i = 2, \ldots, n$, show that $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J \in N'$ and that this is the projection onto $\ell^2(\Lambda g_i)$.

(c) For each $i = 2, \ldots, n$, show that e_N is equivalent to $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J$ in N'.

(d) Compute $\tau_{N'}(e_N)$ and $[M : N]$.

(e) Show that $\langle M, e_N \rangle$ is isomorphic to $M_n(N)$. What is the image of M under this isomorphism?

C*.1 Prove the following fact used in the proof of Theorem 12.7: if $f \in \ell^\infty(G)$, $f = \sum_{g \in G} a_g u_g$, then $\lambda_s(f) = u_s f u_s^*$ as operators on $\ell^2(G)$. In other words, left translation is spatially implemented.