12. Amenability

Preview of Lecture: In lecture, we'll discuss the paradoxical decomposition of $\mathbb{F}_2$ (Example 12.3), but probably not the proof of Proposition 12.4 or Proposition 12.5. My goal in lecture will be to discuss the proof of Theorem 12.13; this will require also discussing Følner sets, but we won't get into the proof of Proposition 12.10 or Proposition 12.12.

The concept of amenability for groups was introduced by John von Neumann in 1929, in response to the Banach-Tarski paradox. For modern operator algebraists, amenable groups are important because these are precisely the groups G for which $C^*(G) \cong C^*_r(G)$. Another C*-algebraic characterization of amenability is that G is amenable iff $C^*_r(G)$ is nuclear – indeed, this is what underlies the use of the word "amenable" instead of "nuclear" for more general C*-algebras. More generally, if a C*-algebra A is nuclear and $\alpha : G \to \mathrm{Aut}(A)$ is an action of an amenable group on A, then the crossed product C*-algebra $C^*(G, A, \alpha)$ will be nuclear. (In particular, this is true for all of the crossed products Dawn Archey mentioned yesterday in her talk.)

There are many (many) equivalent characterizations of amenability (and they all have analogues for locally compact groups, although in these notes we'll just treat the discrete case). If you want to know more than what's presented here, [3, Section 2.6] is a good place to start. For a more exhaustive account, check out [8].

Definition 12.1. A discrete group G is *amenable* if it admits a left-invariant mean: that is, there is a state[20] μ on $\ell^\infty(G)$ such that

$$\mu(f) = \mu(g \mapsto f(s^{-1}g))$$

for all $f \in \ell^\infty$ and $s \in G$.

Example 12.2. Any finite group G is amenable. We define $\mu(\delta_g) = \frac{1}{|G|}$ for each $g \in G$. It is easy to check that if we extend μ to $\ell^\infty(G)$ by requiring it to be linear, the result is a state.

Example 12.3. The free group $\mathbb{F}_2$ is not amenable.

Recall that $\mathbb{F}_2 = \langle a, b\rangle$ is the set of all words in two noncommuting generators (here called a, b) and their inverses. We will assume that the words are *reduced* in the sense that a variable is never immediately followed by its inverse. Let A_+ denote the set of words in $\mathbb{F}_2$ whose first letter is a, and A_- denote the set of words whose first letter is a^{-1}, and note that

$$\mathbb{F}_2 = A_+ \sqcup aA_-;$$

if a reduced word w doesn't start with a, then $a^{-1}w \in \mathbb{F}_2$ lies in A_-, and so $w \in aA_-$.

Similarly, define B_+ (resp. B_-) to be the words whose first letter is b (resp. b^{-1}). So if $C = \{b^n : n \geq 0\}$, then we can also write

$$\mathbb{F}_2 = A_+ \sqcup A_- \sqcup (B_+\backslash C) \sqcup (B_- \cup C).$$

Finally, I claim that $\mathbb{F}_2 = b^{-1}(B_+\backslash C) \sqcup (B_- \cup C)$. Why? Notice that $(B_+\backslash C)$ is the set of words whose first letter is b (so the second letter can't be b^{-1}) but which contain other letters, so $b^{-1}(B_+\backslash C)$ consists of words whose first letter is not b^{-1}, and which contain some letter that's not b. On the other hand, $(B_- \cup C)$ is the set of words which either have b^{-1} as the first letter, or contain only nonnegative powers of b.

Now that we have these three decompositions of $\mathbb{F}_2$, suppose that we did in fact have a left-invariant mean μ on $\ell^\infty(\mathbb{F}_2)$. Observe that $\chi_{tS} = \chi_S(t^{-1}\cdot)$, for any $t \in \mathbb{F}_2$. In other words (abusing notation and writing $\mu(S)$ rather than $\mu(\chi_S)$ for $S \subseteq \mathbb{F}_2$) we have $\mu(tS) = \mu(S)$ for any $S \subseteq \mathbb{F}$ and any $t \in \mathbb{F}$. It follows that

$$1 = \mu(\mathbb{F}_2) = \mu(A_+ \sqcup aA_-) = \mu(A_+) + \mu(A_-).$$

On the other hand, $\mu(\mathbb{F}_2) = \mu(A_+) + \mu(A_-) + \mu(B_+\backslash C) + \mu(B_- \cup C)$, so we must have $\mu(B_+\backslash C) = \mu(B_- \cup C) = 0$. However, this contradicts the fact that (by our third decomposition)

$$1 = \mu(\mathbb{F}_2) = \mu(B_+\backslash C) + \mu(B_- \cup C).$$

Notice that our decomposition $\mathbb{F}_2 = A_+ \sqcup A_- \sqcup (B_+\backslash C) \sqcup (B_- \cup C)$ thus writes $\mathbb{F}_2$ as the disjoint union of two subsets, namely $A_+ \sqcup A_-$ and $(B_+\backslash C) \sqcup (B_- \cup C)$, which both end up having the same measure as $\mathbb{F}_2$ under any translation-invariant measure (thanks to our first and last decompositions of $\mathbb{F}_2$). This is often called a *paradoxical decomposition* of $\mathbb{F}_2$, and is what underlies the Banach-Tarski paradox.

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[20] We've only defined states on C*-algebras so far, but the definition in this context is the same: a linear functional of norm 1 which assigns a nonnegative real number to any nonnegative function.

Proposition 12.4. *If G is abelian then G is amenable.*

The proof uses the *Markov-Kakutani fixed point theorem:* [5, Theorem VII.2.1] if X is a topological vector space, $K \subseteq X$ is compact and convex, and T is a collection of continuous, linear, pairwise commuting maps $t : X \to X$ is such that every $t \in T$ satisfies $tK \subseteq K$, then there is a point in K which is fixed by all $t \in T$.

Proof of Proposition 12.4. The compact convex set K of interest here is the set $S(\ell^\infty(G))$ of states on $\ell^\infty(G)$; take $T = \{\lambda_s^* : s \in G\}$, where

$$\lambda_s^*(\phi)(f) = \phi(\lambda_s f) = \phi(g \mapsto f(s^{-1}g)).$$

Then one checks that every element of T is continuous, in the sense that if a net $(\phi_i)_i \in \ell^\infty(G)^*$ satisfies $\phi_i \to \phi$ in the weak-* topology, then $\lambda_s^*(\phi_i) \to \lambda_s^*(\phi)$ for all $s \in G$. The fact that G is abelian implies that T is a set of pairwise commuting maps, and one can check that T preserves $S(\ell^\infty(G))$. So, the Markov-Kakutani fixed point theorem gives us $\mu \in S(\ell^\infty(G))$ such that $\lambda_s^*(\mu) = \mu$ for all s. By construction, μ is a left-invariant mean on $\ell^\infty(G)$. $\square$

Proposition 12.5. *The class of amenable groups is closed under taking subgroups, quotients, extensions, and inductive limits.*

Proof. We will prove that the class of amenable groups is closed under extensions, and leave the rest as an exercise. So, suppose that N, H are amenable, with left invariant means μ_N, μ_H respectively, and $1 \to N \to G \to H \to 1$ is a short exact sequence of groups (so that N is normal in G and $H \cong G/N$). We define a functional μ on $\ell^\infty(G)$ by

$$\mu(f) = \mu_H(sN \mapsto \mu_N(g \mapsto f(sg))).$$

Notice that the function $sN \mapsto \mu_N(g \mapsto f(sg))$ is well defined by our hypothesis that μ_N is left invariant; we have

$$\mu_N(g \mapsto f(sg)) = \mu_N(g \mapsto f(sng)).$$

Moreover, if f is positive, then the fact that μ_H, μ_N are positive linear functionals implies that μ is also a positive linear functional. To see that μ is indeed a left invariant mean, then, it merely remains to check left invariance. If $\tilde{g} \in G$, then

$$\mu(\lambda_{\tilde{g}} f) = \mu_H(sN \mapsto \mu_N(g \mapsto (\lambda_{\tilde{g}}(sg))) = \mu_H(sN \mapsto \mu_N(g \mapsto f(\tilde{g}^{-1}sg))) = \mu_H(\tilde{g}^{-1}sN \mapsto \mu_N(g \mapsto f(\tilde{g}^{-1}sg)))$$

by the left invariance of μ_H. However, replacing the variable $s \in G$ with $\tilde{g}s$ reveals that this latter is precisely $\mu(f)$, as desired. $\square$

Exercise 12.6. Complete the proof of Proposition 12.5. Some hints:

- If $H \leq G$ is a subgroup of an amenable group, pick a set S of left coset representatives of $H \leq G$, so that you can write any $g \in G$ uniquely as $g = sh$ for $s \in S, h \in H$. Use this to embed $\ell^\infty(H)$ into $\ell^\infty(G)$.
- To show that $G = \varinjlim G_n$ is amenable whenever all the groups G_n are, you'll need to take a weak-$*$ cluster point of the left invariant means witnessing amenability of the G_ns.

In particular, Proposition 12.5 implies that $\mathbb{F}_n$ is not amenable for any $n \geq 2$: Each such $\mathbb{F}_n$ contains $\mathbb{F}_2$ as a subgroup.

Theorem 12.7. *G is amenable iff $\mathrm{C}^*_r(G) \cong \mathrm{C}^*(G)$.*

Proof. We will prove the backwards direction; the forwards direction (cf. [5, Theorem VII.2.8] or [3, Theorem 2.6.8]) uses a lot of machinery that we don't have time to introduce.

Suppose $\mathrm{C}^*_r(G) \cong \mathrm{C}^*(G)$. Note that the universal property of $\mathrm{C}^*(G)$ means that it always admits a one-dimensional representation χ, arising from the unitary representation $\pi(u_g) = 1$ for all $g \in G$. Then, since we assumed that the canonical surjection $\pi_\lambda : \mathrm{C}^*(G) \to \mathrm{C}^*_r(G)$ is an isomorphism, χ becomes a 1-dimensional representation on $\mathrm{C}^*_r(G)$.

By the Hahn-Banach Theorem, extend χ to a norm-1 bounded linear functional (also called χ) on $B(\ell^2(G))$, and then restrict it to a bounded linear functional on $\ell^\infty(G)$ (viewed as a subalgebra of $B(\ell^2(G))$, acting by left multiplication). If $f \in \ell^\infty(G)$ is positive, $f = \sup\{f|_F : F \subseteq G \text{ finite}\}$, and as each $f|_F$ is positive in $\mathrm{C}^*_r(G)$, the fact that $\chi|_{\mathrm{C}^*_r(G)}$ is a $*$-homomorphism (and hence positive) implies $\chi(f) \geq 0$ for all $f \geq 0$ in $\ell^\infty(G)$.

It's straightforward to check **Exercise: do it!** that if $f \in \ell^\infty(G)$, $f = \sum_{g\in G} a_g u_g$, then $\lambda_s(f) = u_s f u_s^*$ as operators on $\ell^2(G)$. Moreover, as χ is a $*$-homomorphism on $C^*_r(G) \ni u_g$, these elements are in the multiplicative domain of χ (see Proposition 9.26). Therefore, $\chi(\lambda_s f) = \chi(f)$ for any $f \in \ell^\infty(G)$, so χ is our left-invariant mean. $\square$

We would also like to prove that G is amenable iff $C^*_r(G)$ is nuclear. To do this, it will be easier to work with a different characterization of amenability. To introduce it, recall that if S, T are sets, then $S\Delta T = (S\cup T)\backslash(S\cap T)$ is the set of elements which are in precisely one of S, T.

Definition 12.8. A discrete group G satisfies the *Følner condition* if for any finite subset $E \subseteq G$ and any $\epsilon > 0$, there is a finite subset $F \subseteq G$ such that

$$\frac{|sF\Delta F|}{|F|} < \epsilon \text{ for all } s \in E.$$

It is a fact (although not one we'll prove here) that G satisfies the Følner condition iff G is amenable. However, we can prove that satisfying the Følner condition is equivalent to the following property, which is hopefully sufficiently reminiscent of the definition of amenability that you're willing to believe said fact. If you recall that $\ell^1(G)$ is the predual of $\ell^\infty(G)$ and hence is dense in $\ell^\infty(G)^*$, you may be even more credulous.

Definition 12.9. A discrete group G admits an *approximate invariant mean* if, for any finite subset $E \subseteq G$ and any $\epsilon > 0$, there is a positive function $m = m(E, \epsilon) \in \ell^1(G)$ with $\sum_{s\in G} m(s) = 1$ and such that

$$\sup_{s\in E} \sum_{t\in G} |m(s^{-1}t) - m(t)| < \epsilon.$$

Proposition 12.10. *G satisfies the Følner condition iff G admits an approximate invariant mean.*

Proof. Suppose G satisfies the Følner condition. Given a finite set E and $\epsilon > 0$, let $F \subseteq G$ be the finite set guaranteed by the Følner condition and let $m = \frac{1}{|F|}\chi_F$. Note that

$$\chi_F(s^{-1}t) = 1 \Leftrightarrow s^{-1}t \in F \Leftrightarrow t \in sF,$$

so $\sum_{t\in G} |m(s^{-1}t) - m(t)| = \frac{|sF\Delta F|}{|F|} < \epsilon$ for all $s \in E$.

On the other hand, suppose that G admits an approximate invariant mean. We first make a helpful technical observation. Given a positive function $f \in \ell^1(G)$ and $r \geq 0$, set $F(f, r) = \{t : f(t) > r\}$. Notice first that $F(f, r)$ must be finite for each fixed r, in order to have $f \in \ell^1(G)$. We now observe that if f, h are two such functions, both bounded above by 1, then

$$|f(t) - h(t)| = \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr.$$

To see this, suppose without loss of generality that $f(t) = x, h(t) = y$ with $x \leq y$. Then $\chi_{F(f,r)}(t) = 1$ iff $r < x$ and $\chi_{F(h,r)}(t) = 1$ iff $r < y$, so the integrand is 1 precisely on the interval $[x, y)$.

Now, supposing G admits an approximate invariant mean, fix a finite subset $E \subseteq G$ and $\delta > 0$; write $\epsilon = \delta/|E|$, and let $m \in \ell^1(G)$ be a norm-1 positive function such that $\sum_{t\in G} |m(t) - m(s^{-1}t)| < \epsilon$ for all $s \in E$. Applying our above observation to the functions $f = m, h = (t \mapsto m(s^{-1}t))$, we have

$$\sum_{t\in G} |m(t) - m(s^{-1}t)| = \sum_{t\in G} \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr = \int_0^1 \sum_{t\in G} |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr$$

(as the integrand is positive we can exchange the integral and the sum). Moreover, we have $t \in F(h, r)$ precisely if $m(s^{-1}t) > r$, that is, if $t \in sF(m, r)$. It follows that

$$\sum_{t\in G} |m(t) - m(s^{-1}t)| = \int_0^1 |F(m, r)\,\Delta\, sF(m, r)|\, dr < \epsilon$$

for all $s \in G$. Furthermore, as m is positive, $1 = \sum_{t\in G} m(t) = \int_0^1 |F(m, r)|\, dr$. It follows that

$$\sum_{s\in E} \int_0^1 |sF(m, r)\,\Delta\, F(m, r)|\, dr < \int_0^1 |E|\epsilon|F(m, r)|\, dr,$$

and so we must have

$$\sum_{s\in E} |sF(m,r)\,\Delta\,F(m,r)| < |E|\epsilon|F(m,r)|$$

for some r. Then, in particular, for each $s \in E$ we have

$$\frac{|sF(m,r)\,\Delta\,F(m,r)|}{|F(m,r)|} < |E|\epsilon = \delta,$$

so $F(m,r)$ satisfies the Følner condition for the given E and $\delta > 0$. □

The proof of the following Proposition can be found in [3, Theorem 2.6.8] (see also [5, Theorem VII.2.8]). It uses a lot more Banach space theory than one might expect.

Proposition 12.11. *G is amenable iff G admits an approximate invariant mean (iff G satisfies the Følner condition).*

Before proving our next theorem, we need the following useful fact about completely positive maps.

Proposition 12.12. *A map $\phi : M_n(\mathbb{C}) \to A$ is completely positive iff $[\phi(E_{ij})] \in M_n(A)$ is positive.*

Proof. We prove the backwards direction and leave the forwards direction as an easy **exercise** to the reader. So, suppose $a = [\phi(E_{ij}] \in M_n(A)$ is positive; write $[b_{ij}] := a^{1/2}$, so that

$$a_{ij} = \phi(E_{ij}) = (b^*b)_{ij} = \sum_{k=1}^{n} b_{ki}^* b_{kj}.$$

Without loss of generality, assume $A \subseteq B(\mathcal{H})$, so that each entry b_{ij} of $b \in M_n(A)$ lies in $B(\mathcal{H})$. Define $V : \mathcal{H} \to \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H}$ by

$$V(\xi) = \sum_{j,k=1}^{n} e_j \otimes e_k \otimes b_{k,j}\xi.$$

Then we compute that if $T = [t_{ij}] \in M_n(\mathbb{C})$,

$$\begin{aligned}
\langle V^*(T \otimes 1 \otimes 1)V\eta, \xi\rangle &= \langle (T \otimes 1 \otimes 1)(V\eta), V\xi\rangle \\
&= \langle \sum_{i,j,k=1}^{n} t_{ij} e_i \otimes e_k \otimes b_{k,j}\eta, \sum_{\ell,m=1}^{n} e_\ell \otimes e_m \otimes b_{m,\ell}\xi\rangle \\
&= \sum_{i,j,k=1}^{n} t_{ij}\langle b_{k,j}\eta, b_{k,i}\xi\rangle = \sum_{i,j,k=1}^{n} \langle b_{k,i}^* b_{k,j}\eta, \xi\rangle \\
&= \langle \phi([t_{ij}])\eta, \xi\rangle.
\end{aligned}$$

In other words, $\phi(T) = V^*(T\otimes 1\otimes 1)V$ is a compression of the $*$-homomorphism $\psi : M_n(\mathbb{C}) \to B(\mathbb{C}^n\otimes\mathbb{C}^n\otimes\mathcal{H})$ given by $\psi(T) = T \otimes 1 \otimes 1$, so ϕ is cp. □

Finally, we can prove our second marquee theorem.

Theorem 12.13. *G is amenable iff $\mathrm{C}_r^*(G)$ is nuclear.*

Proof. Suppose G is amenable (and, for simplicity, countable, so that we can enumerate the elements of G). By Proposition 12.11, we can assume that G satisfies the Følner condition. Choose, then, a sequence of finite sets F_n such that F_n satisfies the Følner condition for $\epsilon = 1/n$ and the finite set consisting of the first n elements of G. Let $P_n \in B(\ell^2(G))$ be the projection onto the subspace spanned by $\{\delta_g : g \in F_n\}$, so that we can identify $P_n B(\ell^2(G))P_n$ with $M_{F_n}(\mathbb{C})$. Define $\phi_n : \mathrm{C}_r^*(G) \to M_{F_n}(\mathbb{C})$ by $\phi_n(x) = P_n x P_n$. Example 9.9 shows that ϕ_n is ccp.

To define $\psi_n : M_{F_n}(\mathbb{C}) \to \mathrm{C}_r^*(G)$, write E_{pq} for the matrix unit in $M_{F_n}(\mathbb{C})$ such that $E_{pq}(\delta_q) = \delta_p$. Then define

$$\psi_n(E_{pq}) = \frac{1}{|F_n|} u_p u_q^*,$$

and extend ψ_n to be a linear map on $M_{F_n}(\mathbb{C})$. If we enumerate the elements of F_n as $p_1, \ldots, p_{|F_n|}$, then $[\psi_n(E_{pq})]$ satisfies

$$[\psi_n(E_{pq})] = \frac{1}{|F_n|^2} \begin{bmatrix} u_{p_1} & 0 & \cdots & 0 \\ u_{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ u_{p_{|F_n|}} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_{p_1} & 0 & \cdots & 0 \\ u_{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ u_{p_{|F_n|}} & 0 & \cdots & 0 \end{bmatrix}^* \geq 0,$$

so Proposition 12.12 tells us that ψ_n is also cp. In fact, ψ is ucp: our choice of scaling factor and the fact that each u_p is a unitary means that

$$\psi_n(1) = \sum_{p \in F_n} \psi_n(E_{pp}) = 1.$$

To complete the proof that $\mathrm{C}^*_r(G)$ is nuclear when G is amenable, it remains to show that for any $a \in \mathrm{C}^*_r(G)$ we have $\lim_{n\to\infty} \|a - \psi_n(\phi_n(a))\| = 0$. In fact, since the generators u_s densely span $\mathrm{C}^*_r(G)$, it suffices to show that $\lim_{n\to\infty} \|u_s - \psi_n(\phi_n(u_s))\| = 0$ for all $s \in G$.

One quickly computes that $\phi_n(u_s) = \sum_{p : p, s^{-1}p \in F_n} E_{p, s^{-1}p}$, and therefore

$$\psi_n(\phi_n(u_s)) = \frac{1}{|F_n|} \sum_{p : p, s^{-1}p \in F_n} u_p u^*_{s^{-1}p} = \frac{1}{|F_n|} \sum_{p : p, s^{-1}p \in F_n} u_s = u_s \frac{|F_n \cap sF_n|}{|F_n|}.$$

As $|F_n \,\Delta\, sF_n| = 2|F_n| - 2|F_n \cap sF_n|$, our choice of the sets F_n implies that

$$0 = \lim_{n\to\infty} \frac{|F_n \,\Delta\, sF_n|}{|F_n|} = \lim_{n\to\infty} 1 - \frac{|F_n \cap sF_n|}{|F_n|}$$

for any $s \in G$. In particular,

$$\lim_{n\to\infty} \|u_s - \psi_n(\phi_n(u_s))\| = \lim_{n\to\infty} 1 - \frac{|F_n \cap sF_n|}{|F_n|} = 0,$$

as desired.

Now, for the converse. Assume $\mathrm{C}^*_r(G)$ is nuclear, so that we have cpc maps $\phi_n : \mathrm{C}^*_r(G) \to M_{k(n)}$ and $\psi_n : M_{k(n)} \to \mathrm{C}^*_r(G)$. By Arveson's Extension Theorem, we might as well assume that ϕ_n is defined on all of $B(\ell^2(G))$, so that the composition $\Phi_n = \psi_n \circ \phi_n$ is a cpc map from $B(\ell^2(G))$ to $\mathrm{C}^*_r(G)$, such that $\Phi_n(x) \to x$ for all $x \in \mathrm{C}^*_r(G)$. Take a point-ultraweak limit of the maps Φ_n (ask Brent and Rolando), and we end up with a cpc map $\Phi : B(\ell^2(G)) \to L(G)$ which restricts to the identity on $\mathrm{C}^*_r(G)$.

Recall from your von Neumann algebra lectures that there is a canonical trace τ on $L(G)$, given by $\tau(x) = \langle x\delta_e, \delta_e \rangle$. Define $\mu = \tau \circ \Phi$; we claim that μ is a left invariant mean. To see this, we again use that the left translation action λ_s on functions in $\ell^\infty(G) \subseteq B(\ell^2(G))$ is given by $\lambda_s(f) = u_s f u_s^*$. Since $\Phi|_{\mathrm{C}^*_r(G)} = \mathrm{id}$, we have u_g in the multiplicative domain of Φ for all g. Consequently, for any $f \in \ell^\infty(G)$,

$$\mu(\lambda_s(f)) = \tau(\Phi(u_s f u_s^*)) = \tau(u_s \Phi(f) u_s^*) = \tau(\Phi(f)),$$

since τ is a trace and u_s is a unitary. $\square$

References

[1] W. Arveson, *An Invitation to* C*-*algebras*. Graduate Texts in Mathematics, Springer-Verlag New York Inc., 1976.

[2] B. Blackadar, *Operator Algebras*. Encyclopaedia of Mathematical Sciences, vol. 122, Springer, 2006.

[3] N. Brown and N. Ozawa, *C^*-Algebras and Finite-Dimensional Approximations*. Graduate Studies in Mathematics. Volume 88 AMS. (2008).

[4] J. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics **96** 2nd Edition, Springer.

[5] K. Davidson, *C^*-algebras by Example*. Fields Institute Monograph 6, AMS, 1996.

[6] E.C. Lance, *Hilbert C^*-Modules. A Toolkit for Operator Algebraists.* London Mathematical Society Lecture Note Series, **210**. Cambridge University Press, Cambridge, 1995.

[7] G. J. Murphy, C*-*algebras and Operator Theory*. Academic Press, 1990.

[8] A.L.T. Paterson, *Amenability*. Mathematical Surveys and Monographs, vol. 29. American Mathematical Society, Providence, RI, 1988.

[9] V. Paulsen, *Completely Bounded Maps and Operator Algebras*. Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.

[10] G.K. Pedersen, *Analysis Now*. Graduate Texts in Mathematics, Springer-Verlag New York Inc., 1989.

[11] G.K. Pedersen, *C^*-algebras and Their Automorphism Groups, 2nd Edition.* Edited by Søren Eilers and Dorte Olesen. London Math. Soc. Monographs 14. Academic Press, 2018.

[12] I. Raeburn, *Graph Algebras*. CBMS Regional Conference Series in Mathematics, vol. 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005.

[13] M. Rørdam, F. Larsen, N.J. Laustsen. *An Introduction to K-Theory for C^*-Algebras.* London Mathematical Society Student Texts, **49**. Cambridge University Press, Cambridge, 2000.