Problem Session

July 14th, 2020

Mark Tomforde *K-theory: An Elementary Introduction*

Let A be a C^*-algebra, and let I be an ideal of A. Prove that if $K_0(I) \cong K_1(I) \cong \{0\}$, then $K_0(A) \cong K_0(A/I)$ and $K_1(A) \cong K_1(A/I)$.

If you have extra time, consider the following exercises:

1. Suppose A is a unital C^*-algebra that is Morita equivalent to a crossed product of an AF-algebra by $\mathbb{Z}$; that is, there exists an AF-algebra B and an automorphism $\alpha\colon B \to B$ such that A is Morita equivalent to the crossed product $B \rtimes_\alpha \mathbb{Z}$. Prove that

   $$K_0(A) \cong \operatorname{coker}(id - \alpha_0) \qquad \text{and} \qquad K_1(A) \cong \ker(id - \alpha_0)$$

   where $(id - \alpha_0)\colon K_0(B) \to K_0(B)$. Also show that $K_1(A)$ is torsion-free abelian group.

   (Recall: if $h\colon G \to H$ is a homomorphism between abelian groups, then the *cokernel* of h is defined $\operatorname{coker}(h) := H/\operatorname{im}(h)$.)

   [**Hint:** use the Pimsner–Voiculescu (PV) sequence.]

2. Prove that K_0 and K_1 distribute over a direct sum; that is, for any C^*-algebras A and B prove that

   $$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B) \qquad \text{and} \qquad K_1(A \oplus B) \cong K_1(A) \oplus K_1(B).$$

   There are several ways to do this problem. The hints below outline one possible approach.

   [**Hint 1:** use the fact that K_0 and K_1 each take split exact sequences to split exact sequences.]

   [**Hint 2:** obtain the following commutative diagram

   $$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus K_0(B) & \longrightarrow & K_0(B) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A \oplus B) & \longrightarrow & K_0(B) & \longrightarrow & 0 \end{array}$$

   and apply the three-lemma (i.e. a special case of the five-lemma). Similarly for K_1.]

Ian Charlesworth *Free Probability*

Let G and H be countable discrete groups and let $G * H$ denote their free product. View $L(G)$ and $L(H)$ as subalgebras of $L(G * H)$, whose trace we denote by τ.

(a) For $g_1, \ldots, g_n \in G \setminus \{e\}$ and $h_1, h_2, \ldots, h_n \in H \setminus \{e\}$, show that

$$\tau(\lambda(g_1)\lambda(h_1) \cdots \lambda(g_n)\lambda(h_n)) = 0.$$

(b) For $x \in \mathbb{C}[\lambda(G)]$, characterize when $\tau(x) = 0$. Similarly for $y \in \mathbb{C}[\lambda(H)]$.

(c) For $x_1, \ldots, x_n \in \mathbb{C}[\lambda(G)]$ and $y_1, \ldots, y_n \in \mathbb{C}[\lambda(H)]$ assume $\tau(x_i) = \tau(y_i) = 0$ for $i = 1, \ldots, n$. Show that

$$\tau(x_1 y_1 \cdots x_n y_n) = 0.$$

(d) Show that the previous part holds for $x_i \in L(G)$ and $y_i \in L(H)$.