Problem Session July 13th, 2020

Michael Brannan: *Quantum Groups: what are they and what are they good for?*

Recall that the *Brown Algebra* B_n is the unital C*-algebra satisfying the following universal property: B_n is generated by the elements $u_{ij}, 1 \leq i, j \leq n$ satisfying the property that $[u_{ij}]_{ij}$ is a unitary in $M_n(B_n)$, and if A is another unital C*-algebra generated by elements v_{ij} satisfying the same relations then there exists a unique unital $*$-homomorphism $\pi : B_n \to A$ such that $u_{ij} \mapsto v_{ij}$ for all i, j.

Exercise: Prove that Brown's universal unitary algebras B_n, equipped with their canonical co-products, do **not** define compact quantum groups.

If you have extra time, consider the following exercise:

Let $G = (A, \Delta)$ be a compact quantum group with comultiplication $\Delta : A \to A \otimes_{\min} A$. Define $\Delta^{\text{opp}} := {}^t(\Delta) := t \circ \Delta$, where $t : A \otimes_{\min} A \to A \otimes_{\min} A$ denotes the flip map, i.e., $a \otimes b \mapsto b \otimes a$. Show that $G^{\text{opp}} = (A, \Delta^{\text{opp}})$ is a compact quantum group.

Dawn Archey: *A Crash Course in Crossed Product C*-Algebras*

We say that a C^*-algebra A has **real rank zero** if the invertible elements in $A_{s.a.}$ are dense in $A_{s.a.}$.

Exercise: Show that $C([0,1])$ does not have real rank zero, by finding a function $f \in C([0,1])_{s.a.}$ which cannot be approximated within $\epsilon = 1/4$ by an invertible self-adjoint element.

If you have extra time, consider the following exercises:

1. Let G be a finite group. Let A be a unital $C*$-algebra. Let $\alpha \colon G \to \text{Aut}(A)$ be a homomorphism. As short hand, write α_t instead of $\alpha(t)$. Consider the algebra AG of all sums $\sum_{t \in G} a_t t$.
   (a) We will define multiplication on AG by the formal rule $tat^{-1} = \alpha_t(a)$.
   Work out an explcit formula for the product fg where $f = \sum_{t \in G} a_t t$ and $g = \sum_{s \in G} b_s s$. Your final answer should be in the same format (a sum of things of the form: algebra element times group element).
   (b) Later we will complete this to create a C^*-algebra. So we will need an adjoint. The adjoint is determined by $s^* = s^{-1}$. Use this to determine a formula for the adjoint of f as defined in the previous part of the problem.
2. Let $h : X \to X$ be a homeomorphism. We say (H, h) is a **minimal dynamical system** if X has no proper closed h invariant subsets. Let $X = S_1$. Let $h(z) = e^{-2\pi\theta z}$. If $\theta \in \mathbb{Q} \setminus \{0\}$ then show h is not minimal.

Lecture Exercises

C*.1 Finish the proof of the following proposition from the lecture notes:

Proposition 0.1. *For* C*-*algebras* A_1 *and* A_2*, and* $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$,

$$\|x\|_{\min} = \sup\{\|\sum_{j=1}^n \pi_1(a_j) \otimes \pi_2(b_j)\| \ : \ \pi_i : A_i \to B(\mathcal{H}_i) \text{ (nondegenerate) representations}\}.$$

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be representations and $\sigma_i : A_i \to B(\mathcal{H}_i')$ be faithful representations. Then by Exercise 4.16, $\pi_i \oplus \sigma_i : A_i \to B(\mathcal{H}_i \oplus \mathcal{H}_i')$ is a faithful representation. Let $P_i \in B(\mathcal{H}_i \oplus \mathcal{H}_i')$ be the compression to $\mathcal{H}_i$ for each $i = 1, 2...$ □

(This is an example of a technique where one can *dilate* a map to one with a desired property (e.g. faithfulness) and then *cut down* to the original map to draw the desired conclusion.)

W*.1 For each $N \subset M$ below, compute the conditional expectation $E_N : M \to N$. Recall that the conditional expectation is determined by the formula

$$\langle E_N(x), y\rangle_2 = \langle x, y\rangle_2 \qquad x \in M,\ y \in N$$

where $\langle a, b\rangle_2 = \tau(b^* a)$ for $a, b \in M$.

(a) For $d \in \mathbb{N}$, let $M := M_d(\mathbb{C})$ and let N be the subalgebra of diagonal matrices.

(b) Let M be an arbitrary finite factor and let $N := \mathbb{C}$.

(c) Let Γ be a discrete i.c.c. group. Let $\Lambda < \Gamma$ be a subgroup. Take $M := L(\Gamma)$ and $N := L(\Lambda)$.