11. Tensor Products of C*-algebras

Overall, sections 11.1 and 11.2 will be treated as preliminary material in the lecture, which will focus more on sections 11.3 and 11.7. Section 11.4 goes into much more difficult problems concerning injectivity and exactness for tensor products. The point there is to just give a feel for the the questions and obstacles in both settings. With time, we will touch on the topics in lecture. Section 11.5 gives a tensor product characterization of nuclearity (Theorem 11.48) and highlights some important examples (Remark 11.51). We will mention these in lecture but without much discussion. Section 11.6 establishes an important class of examples (Theorem 11.55), which we will be sure to mention in lecture, but without much word on the proof.

Section 11.3 defines the two primarily studied C*-norms on tensor products. These are quite analogous to the universal and reduced norms for discrete groups, and we will explore several tensor product analogies to results we saw for groups, e.g. Corollary 11.28, Proposition 11.33, and Proposition 11.32. Section 11.7 justifies our use of *completely* positive maps. We will cover Example 11.58 and mention how Stinespring's Dilation theorem is used in the proof of Theorem 11.59.

The way you read these notes will depend on your background and comfort level. If algebraic tensor products are new to you, spend more time in section 11.1. Regardless of your comfort level with algebraic tensors, be sure you've digested Exercise 11.11, which is quite foundational to the later sections. If you are still shaky on Hilbert space operators, linger in section 11.2. If you feel comfortable with (assuming) the material in these sections, but still want some more fundamental examples and arguments under your belt, check out sections 11.6 and 11.5.

One of the most important constructions in C*-algebras is the tensor product. Given two C*-algebras A and B, we form a C*-tensor product $A \otimes_\alpha B$ by taking the $*$-algebraic tensor product $A \odot B$ and completing with some C*-norm. In this section, we consider the two most prominent ones. This section is taken heavily from the first half of [3, Chapter 3].

One word on notation. Because there is so much significance to the norm on a given tensor product, we will denote algebraic tensor products by $\odot$ and tensor products that are also complete with respect to a norm by $\otimes$ (possibly with decoration to denote which norm). Sometimes $\otimes$ is used in the literature to denote an algebraic tensor product, and sometimes it is used to indicate the normed tensor product space with the spatial tensor product norm Definition 11.21. Usually authors are good about warning you of this.

11.1. Facts about algebraic tensor products.

In this section we list some relevant facts about algebraic tensor products that we will take for granted in the lecture. Many of these are proved in [3, Section 3.1-3.2].

We give a non-constructive definition since it highlights the key properties: Let A and B be $\mathbb{C}$-vector spaces. Their tensor product is the vector space $A \odot B$, together with a bilinear map $\odot : A \times B \to A \odot B$, such that $A \odot B$ is universal in the following sense:
For any $\mathbb{C}$-vector space C and any bilinear map $\phi : A \times B \to C$, there exists a unique bilinear map $\tilde{\phi} : A \odot B \to C$ so that $\tilde{\phi}(a \odot b) = \phi(a, b)$ for all $a \in A$ and $b \in B$. The bilinearity of the map $\odot : A \times B \to A \odot B$ means that we have the following algebraic relations in $A \odot B$:

(1) $(a_1 + a_2) \odot b = (a_1 \odot b) + (a_2 \odot b)$ and
$a \odot (b_1 + b_2) = (a \odot b_1) + (a \odot b_2)$ for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$; and
(2) $\lambda(a \odot b) = (\lambda a) \odot b = a \odot (\lambda b)$ for all $a \in A$, $b \in B$, and $\lambda \in \mathbb{C}$.

Elements of the form $a \odot b$ for $a \in A$ and $b \in B$ are called *simple tensors*. Note that if $a = 0$ or $b = 0$, then $a \odot b = 0$.

Remark 11.1. $A \odot B$ is spanned by its simple tensors, but consists of many more elements. For example, in general the element $(a_1 \odot b_1) + (a_2 \odot b_2)$ cannot be written as a simple tensor $a \odot b$.

As a vector space, the notion of linear independence in an algebraic tensor product is a little technical but also technically very useful. We lay out the following propositions for later use.

As far as linear independence goes, the following propositions can be useful:

Proposition 11.2. *Suppose $\{a_1, ..., a_n\} \subset A$ are linearly independent and $\{b_1, ..., b_n\} \subset B$. Then*

$$\sum_{1}^{n} a_i \odot b_i = 0 \Rightarrow b_i = 0, \text{ for } 1 \leq i \leq n.$$

Proposition 11.3. *If $\{e_i\}_{i\in I}$ is a basis for A and $\{e'_j\}_{j\in J}$ is a basis for B, then $\{e_i \odot e'_j\}_{(i,j)\in I\times J}$ is a basis for $A \odot B$.*

Proposition 11.4. *If $\{e_i\}_{i\in I}$ is a basis for A and $x \in A \odot B$, then there exists a unique finite set $I_0 \subset I$ and $\{b_i\}_{i\in I_0}$ so that $x = \sum_{i\in I_0} e_i \odot b_i$.*

Just as we take tensor products of linear spaces, we can take tensor products of linear maps.[10] The following is more of a proposition/ definition; existence and uniqueness of these maps come from the above universal property.

Definition 11.5. Suppose $A_1 A_2, B_1, B_2$ are $\mathbb{C}$-vector spaces and $\phi_i : A_i \to B_i$, $i = 1, 2$ are linear maps. Then there is a unique linear map

$$\phi_1 \odot \phi_2 : A_1 \odot B_1 \to A_2 \odot B_2$$

so that $\phi_1 \odot \phi_2(a \odot b) = \phi_1(a) \odot \phi_2(b)$ for all $a \in A_1$, $b \in A_2$. This is called the *tensor product* of the maps ϕ_1 and ϕ_2.

The tensor product of linear maps preserves both injectivity and exact sequences:

Proposition 11.6. *Suppose A_1, A_2, B_1, B_2 are $\mathbb{C}$-vector spaces and $\phi_i : A_i \to B_i$, $i = 1, 2$ are injective linear maps. Then $\phi_1 \odot \phi_2$ is also injective.*

Proposition 11.7. *Suppose J, A, B, C are $\mathbb{C}$-vector spaces. If $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ is a short exact sequence (i.e. ι is injective, π is surjective, and $\ker(\pi) = \iota(J)$), then so is*

$$0 \to J \odot C \xrightarrow{\iota\odot id_C} A \odot C \xrightarrow{\pi\odot id_C} B \odot C \to 0.$$

We highlight a special case of this tensor product map when $B_1 = B_2$ is an algebra.

Definition 11.8. Suppose A_1, A_2 are $\mathbb{C}$-vector spaces, B a $\mathbb{C}$-algebra, and $\psi_i : A_i \to B$ are linear maps. Then there exists a unique linear map

$$\psi_1 \times \psi_2 : A_1 \odot A_2 \to B$$

so that $\psi_1 \times \psi_2(a \odot b) = \psi_1(a)\psi_2(b)$ for all $a \in A_1$, $b \in A_2$. This is called the *product* of the maps ψ_1 and ψ_2.

Exercise 11.9. Explain what is meant by $\psi_1 \times \psi_2$ is a "special case" of a tensor product of maps. (Think of the universal property and the bilinear map $B \odot B \to B$ given on simple tensors by $b_1 \odot b_2 \mapsto b_1 b_2$.)

We are interested in particular in tensor products of C*-algebras. When A and B are C*-algebras, then the algebraic tensor product is a $*$-algebra with multiplication and involution defined on simple tensors as

$$(a \odot b)^* = a^* \odot b^* \quad \text{ and } \quad (a_1 \odot b_1)(a_2 \odot b_2) = a_1 a_2 \odot b_1 b_2,$$

and extended linearly to all of $A \odot B$.

When we take the product of two $*$-homomorphisms $\psi_1 : A_1 \to B$ and $\psi_2 : A_2 \to B$, we are forced to impose an extra condition to guarantee that the product $\psi_1 \times \psi_2$ is again a $*$-homomorphism: the images must commute, i.e. for each $a_1 \in A_1$ and $a_2 \in A_2$, $\psi_1(a_1)\psi_2(a_2) = \psi_2(a_2)\psi_1(a_1)$.

Exercise 11.10. Justify the claim above, i.e. the product $\psi_1 \times \psi_2$ of two $*$-homomorphisms $\psi_1 : A_1 \to B$ and $\psi_2 : A_2 \to B$ is a $*$-homomorphism provided that the ranges $\psi_1(A_1)$ and $\psi_2(A_2)$ commute in B.

Recall from Section 9 where we defined a natural C*-norm on

$$\mathrm{M}_n(A) := \{[a_{ij}] \ : \ a_{i,j} \in A, 1 \leq i, j \leq n\}. \tag{11.1}$$

Exercise 11.11. Let A be any C*-algebra, $1 \leq n < \infty$, and let $E_{i,j}$ denote the matrix units on $\mathrm{M}_n(\mathbb{C})$ (i.e. the matrices with 1 in the i, j coordinate and 0 elsewhere). Define a map $\pi : \mathrm{M}_n(A) \to \mathrm{M}_n \odot A$ by $\pi([a_{i,j}]) = \sum_{i,j=1}^{n} E_{i,j} \odot a_{ij}$. Show that this is an algebraic $*$-isomorphism.

[10] For those categorically inclined, tensors play well with linear categories and act like "multiplication" for objects/ morphisms. Ask Corey Jones after his expository talk.

11.2. **Tensor Products of Hilbert Space Operators.** We saw in the prereqs how to define a tensor product of two Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$. (Recall that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of the algebraic tensor product $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the norm coming from the inner product which is given on simple tensors by $\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle$.)

Given operators $T_i \in B(\mathcal{H}_i)$ for $i = 1, 2$, we have a natural algebraic tensor product mapping $T_1 \odot T_2 : \mathcal{H}_1 \odot \mathcal{H}_2 \to \mathcal{H}_1 \odot \mathcal{H}_2$ given on simple tensors by

$$(T_1 \odot T_2)(\xi \odot \eta) = T_1\xi \odot T_2\eta.$$

This extends linearly to a linear map $\mathcal{H}_1 \odot \mathcal{H}_2 \to \mathcal{H}_1 \odot \mathcal{H}_2$ defined on sums of simple tensors by

$$T_1 \odot T_2 \sum_1^n c_j(\xi_j \otimes \eta_j) = \sum_1^n c_j(T_1\xi_j \otimes T_2\eta_j).$$

This map extends to an operator $T_1 \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by the following proposition.

Proposition 11.12. *Given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and operators $T_i \in B(\mathcal{H}_i)$, $i = 1, 2$, there is a unique linear operator $T_1 \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that*

$$T_1 \otimes T_2(\xi_1 \otimes \xi_2) = T_1\xi_1 \otimes T_2\xi_2$$

for all $\xi_i \in \mathcal{H}_i$, $i = 1, 2$, and moreover $\|T_1 \otimes T_2\| = \|T_1\|\|T_2\|$.

Proof. First, we want to show that the operator $T_1 \odot T_2$ is bounded on $\mathcal{H}_1 \odot \mathcal{H}_2$, which means we can extend it to a bounded operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Assume for now that $T_2 = 1_{\mathcal{H}_2}$, and write $T = T_1$. Let $\sum_1^n c_j(\xi_j \odot \eta_j) \in \mathcal{H}_1 \odot \mathcal{H}_2$. Using a Gram-Schmidt process, we may assume η_j are orthonormal (check). Then we compute

$$\begin{aligned}
\left\| T \odot 1_{\mathcal{H}_2}(\sum_{j=1}^n c_j(\xi_j \odot \eta_j)) \right\|^2 &= \left\| \sum_{j=1}^n c_j T\xi_j \odot \eta_j \right\|^2 = \left| \langle \sum_{i=1}^n c_i T\xi_i \odot \eta_i, \sum_{j=1}^n c_j T\xi_j \odot \eta_j \rangle \right| \\
&= \left| \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \langle T\xi_i, T\xi_j \rangle \langle \eta_i, \eta_j \rangle \right| = \sum_{j=1}^n |c_j|^2 \|T\xi_j\|^2 \le \|T\|^2 \sum_{j=1}^n |c_j|^2 \|\xi_j\|^2 \\
&= \|T\|^2 \left| \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \langle \xi_i, \xi_j \rangle \langle \eta_i, \eta_j \rangle \right| = \|T\|^2 \left\| \sum_{j=1}^n c_j(\xi_j \odot \eta_j) \right\|^2.
\end{aligned}$$

Then $\|T \odot 1_{\mathcal{H}_2}\| \le \|T\|$ on $\mathcal{H}_1 \odot \mathcal{H}_2$, meaning it extends to an operator in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, denoted by $T \otimes 1_{\mathcal{H}_2}$, with $\|T \otimes 1_{\mathcal{H}_2}\| \le \|T\|$. Similarly, one shows that for any $T_2 \in B(\mathcal{H}_2)$, we have $1_{\mathcal{H}_1} \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Now, for $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$, we compose $(1_{\mathcal{H}_1} \otimes T_2)(T_1 \otimes 1_{\mathcal{H}_2})$ to get $T_1 \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\|T_1 \otimes T_2\| \le \|T_1\|\|T_2\|$ and

$$T_1 \otimes T_2(\xi_1 \otimes \xi_2) = T_1\xi_2 \otimes T_2\xi_2$$

for all $\xi_i \in \mathcal{H}_i$. To show that this norm inequality is an equality, we find, for any $\varepsilon > 0$, unit vectors $\xi_i \in \mathcal{H}_i$ with $|\|T_i\xi_i\| - \|T_i\|| < \varepsilon(2\max_i \|T_i\|)^{-1}$ for $i = 1, 2$. Then, using Exercise 7.49 from Day 1, we have

$$\|(T_1 \otimes T_2)(\xi_1 \otimes \xi_2)\| = \|T_1\xi_1 \otimes T_2\xi_2\| = \|T_1\xi_1\|\|T_2\xi_2\| \sim_\varepsilon \|T_1\|\|T_2\|.$$

(That's shorthand for $\|T_1\xi_1\|\|T_2\xi_2\|$ is within epsilon of $\|T_1\|\|T_2\|$.) □

We will take for granted that taking tensor products of operators is well-behaved with respect to addition, (scalar) multiplication, and adjoints.

Exercise 11.13. For $A = [a_{ij}] \in M_2(\mathbb{C}) = B(\mathbb{C}^2)$ and $B = [b_{i,j}] \in M_3(\mathbb{C}) = B(\mathbb{C}^3)$, write a matrix array for $A \otimes B \in B(\mathbb{C}^2 \otimes \mathbb{C}^3)$. (This is called a Kronecker product.)

In infinite dimensions, we do not have $B(\mathcal{H}_1) \odot B(\mathcal{H}_2) = B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (the former is no longer automatically closed).

Proposition 11.14. *For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we define $*$-homomorphisms $\iota_i : B(\mathcal{H}_i) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by identifying $B(\mathcal{H}_1) \simeq B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \simeq \mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$. These induce a product $*$-homomorphism $\iota_1 \times \iota_2 : B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, which is injective.*

Proof. Since $B(\mathcal{H}_1) \simeq B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \simeq \mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$ (**Exercise:** check) and $B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $\mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$ commute in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (**Exercise:** check), we have from Section 11.1 the product $*$-homomorphism

$$B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2),$$

given by

$$\sum_{j=1}^n S_j \odot T_j \mapsto \sum_{j=1}^n (S_j \otimes 1_{\mathcal{H}_2})(1_{\mathcal{H}_1} \otimes T_i) = \sum_{j=1}^n S_j \otimes T_j.$$

We just need to show that this map is injective, i.e. if the operator $\sum_{j=1}^n S_j \otimes T_j \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is zero, then the sum of elementary tensors $\sum_{j=1}^n S_j \odot T_j \in B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ is also zero. By possibly re-writing the coefficients of the S_j, we may assume that the operators $\{S_j\}$ are linearly independent. If $0 = \sum_{j=1}^n S_j \otimes T_j \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, then for all vectors $\xi_1, \eta_1 \in \mathcal{H}_1$ and $\xi_2, \eta_2 \in \mathcal{H}_2$, we have

$$\begin{aligned} 0 = \langle (\sum_{j=1}^n S_j \otimes T_j)(\xi_1 \otimes \xi_2), (\eta_1 \otimes \eta_2) \rangle &= \sum_{j=1}^n \langle S_j\xi_1 \otimes T_j\xi_2, \eta_1 \otimes \eta_2 \rangle \\ = \sum_{j=1}^n \langle S_j\xi_1, \eta_1 \rangle \langle T_j, \xi_2, \eta_2 \rangle &= \sum_{j=1}^n \langle \big(\langle T_j\xi_2, \eta_2 \rangle\big) S_j\xi_1, \eta_1 \rangle. \end{aligned}$$

Since this holds for all $\xi_1, \eta_1 \in \mathcal{H}_1$ the operator $\sum_{j=1}^n \langle T_j\xi_2, \eta_2 \rangle S_j \in B(\mathcal{H}_1)$ is zero (by Exercise 7.45 from Day 1 Lectures). Since we assumed the $\{S_j\}$ are linearly independent, it follows from Proposition 11.2 that the coefficients $\langle T_j\xi_2, \eta_2 \rangle$ must all be 0. Again, since this holds for all $\xi_2, \eta_2 \in \mathcal{H}_2$, it follows that each $T_j = 0 \in B(\mathcal{H}_2)$, which finishes the proof. □

Corollary 11.15. *Given two representations $\pi_i : A_i \to B(\mathcal{H}_i)$, $i = 1, 2$, there is an induced representation*

$$\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

such that $\pi_1 \odot \pi_2(a_1 \odot a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ for all $a_i \in A_i$, $i = 1, 2$.

We have discussed extending pairs of linear maps to tensor products, but what about restricting maps on tensor products to the tensor factors? Given a $*$-homomorphism on an algebraic tensor product of C*-algebras $\phi : A \odot B \to C$, when can we define restrictions $\phi|_A : A \to C$ and $\phi|_B : B \to C$? In general this is not so easy. In the unital setting, there is a natural way to do this.

Exercise 11.16. Suppose A, B, and C are C*-algebras with A and B unital and $\phi : A \odot B \to C$ a $*$-homomorphism. Then there exist $*$-homomorphisms $\phi_A : A \to C$ and $\phi_B : B \to C$ with commuting ranges such that $\phi = \phi_A \times \phi_B$.

A little harder to prove is the following (without the assumption that A and B are unital). See [3, Theorem 3.6.2].

Theorem 11.17. *Let A and B be C*-algebras and $\pi : A \odot B \to B(\mathcal{H})$ a non-degenerate $*$-homomorphism. Then there exist non-degenerate representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ so that $\pi = \pi_A \times \pi_B$.*

11.3. **C*-norms on tensor products.** For C*-algebras A and B, $A \odot B$ is a $*$-algebra. In order to turn it into a C*-algebra, we need to be able to define a C*-norm $\|\cdot\|$ on $A \odot B$. With this, $(A \odot B, \|\cdot\|)$ will be a *pre-C*-algebra*, i.e. its completion is a C*-algebra. Much like the situation with groups, we are guaranteed the following:

- C*-norms on algebraic tensor products of C*-algebras always exist;
- there can be (very) many different C*-norms on a given algebraic tensor product of two C*-algebras;
- but we know how to describe the largest and smallest;[11] and
- it is extremely interesting to ask when the two coincide (and this is related to the notion of amenability for groups because math is beautiful).

Definition 11.18. For C*-algebras A and B, a *cross norm* on a $A \odot B$ is a norm $\|\cdot\|$ such that $\|a \otimes b\| = \|a\|\|b\|$ for every $a \in A$ and $b \in B$.

[11] The second part of this statement is a deep theorem due to Takesaki.

Example 11.19. We verified in the previous section that for $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$, the norm on $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ inherited from $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a cross norm. In fact as a consequence of Takesaki's theorem[12] (which we will discuss more later in this section) every C*-norm on $A \odot B$ is a cross norm. We will take this as a fact as we proceed.

In Exercise 11.11, we saw that there is an algebraic $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \simeq \mathrm{M}_n(A)$, the latter being a C*-algebra with norm induced by the norm of A. Hence pulling back the norm along this $*$-isomorphism gives a C*-norm on $\mathrm{M}_n(\mathbb{C}) \odot A$ (i.e. $\|[\lambda_{ij}] \odot a\| = \|[\lambda_{ij} a]\|$). Moreover, $\mathrm{M}_n(\mathbb{C}) \odot A$ is already complete with respect to this norm, which means it is a C*-algebra. Hence any other C*-norm we define on $\mathrm{M}_n(A)$ agrees with this norm. (See remarks after Proposition 1.21.) That means we have proved the following proposition.

Proposition 11.20. *Let A be a C*-algebra and $1 \leq n < \infty$. Then there is a unique C*-norm on the algebraic tensor product $\mathrm{M}_n(\mathbb{C}) \odot A$, which comes from the $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \simeq \mathrm{M}_n(A)$. Hence we write $\mathrm{M}_n(\mathbb{C}) \otimes A$.*

This identification also introduces very convenient notation, e.g. for the diagonal matrix in $\mathrm{M}_n(A)$ with $a \in A$ down the diagonal:

$$I_n \otimes a \;\leftrightarrow\; \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a \end{bmatrix}.$$

For general C*-algebras A and B, it should not be taken for granted that a C*-norm exists at all on $A \odot B$. However, it turns out the two most natural candidates both yield C*-norms.

The first is the spatial norm, i.e. the norm inherited as a subspace of bounded operators on a tensor product of Hilbert spaces. Recall that as a consequence of the GNS construction, every C*-algebra has at least one faithful representation on some Hilbert space.

Definition 11.21 (Spatial Norm)**.** Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. The *spatial* norm on $A_1 \odot A_2$ is

$$\left\|\sum a_i \odot b_i\right\|_{\min} = \left\|\sum \pi_1(a_i) \otimes \pi_2(b_i)\right\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}.$$

Remark 11.22. We will explain the $\|\cdot\|_{\min}$ notation later with Takesaki's theorem, which we keep mentioning.

Exercise 11.23. Check that $\|\cdot\|_{\min}$ is a semi-norm satisfying the C*-identity.

Proposition 11.24. *The semi-norm $\|\cdot\|_{\min}$ is a norm, i.e. for each $x \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then $x = 0$.*

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. Then the algebraic tensor product map $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ is injective. By Proposition 11.14, we can view $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ as a $*$-subalgebra of $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, and consequently have $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ injective. Then for any $x = \sum_{i=1}^n a_i \odot b_i \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then

$$0 = \|x\|_{\min} = \|\sum_{i=1}^n \pi_1(a_i) \otimes \pi_2(b_i)\| = \|(\pi_1 \odot \pi_2)(x)\|,$$

which by injectivity means $x = 0$. □

Hence $\|\cdot\|_{\min}$ is a norm, and we can define the C*-algebra

$$A \otimes B := \overline{A \odot B}^{\|\cdot\|_{\min}}.$$

It is sometimes denoted $A \otimes_{\min} B$, but we choose the undecorated notation to match the literature. In most cases this the unofficial "default" norm to take on a tensor product of C*-algebras.[13]

[12]Full disclosure, using this theorem is wayyyy overkill. A functional calculus argument could prove this, but this section is already long enough.

[13]For groups, it's the other way around and the maximal C*-completion of the group algebra is often the undecorated one.

For a sense of perspective, dropping the representation notation, we view $A_1 \subset B(\mathcal{H}_1)$ and $A_2 \subset B(\mathcal{H}_2)$. Then there is a natural way to stick them into a common C*-algebra, i.e. $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, from whence they can inherit the C*-norm, i.e. $A_1 \otimes A_2$ is the closure of the $*$-subalgebra $A_1 \odot A_2 \subset B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

However, the norm was defined with an arbitrary choice of faithful representations. Fortunately, the value of the norm is independent of that choice.

Proposition 11.25. *Given faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$ and $\pi_i' : A_i \to B(\mathcal{H}_i')$, then the minimal tensor norms $\|\cdot\|_{\min}$ and $\|\cdot\|_{\min}'$ defined by each pair of faithful representations agree.*

The proof is nice to see because it highlights two useful techniques. The first, yet again, is approximate identities. The second is the fact that there is only one C*-norm on $\mathrm{M}_n(B)$ for any C*-algebra B.

In our proof, we limit ourselves to the countable setting to avoid the extra notation involved with nets.

Proof. By symmetry, it suffices to prove the case where $\mathcal{H}_1 = \mathcal{H}_1'$ and $\pi_1 = \pi_1'$.

We first consider the case where $A_1 = \mathrm{M}_n(\mathbb{C})$ for some n. Since both $\|\cdot\|_{\min}$ and $\|\cdot\|_{\min}'$ are C*-norms, by Proposition 11.20, for every $x = \sum_{i=1}^m T_i \odot a_i \in \mathrm{M}_n(\mathbb{C}) \odot A_2$,

que min norm}

$$\left\| \sum_{i=1}^n \pi_1(T_i) \otimes \pi_2(a_i) \right\| = \|x\|_{\min} = \|x\|_{\min}' = \left\| \sum_{i=1}^n \pi_1(T_i) \otimes \pi_2'(a_i) \right\|. \tag{11.2}$$

Now, for the general separable case, take an increasing net of finite-rank projections $P_1 \leq P_2 \leq \ldots$ in $B(\mathcal{H}_1)$ where the rank of P_n is n and such that $\|P_n\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}_1$ (i.e. P_n converge in SOT to $1_{\mathcal{H}_1}$). Then for every $T \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})$ converges in $*$-SOT[14] to T, and so we have (check)

$$\|T\| = \sup_n \|(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})\|.$$

That means for any $x = \sum_{i=1}^m a_i \odot b_i \in A_1 \odot A_2$,

$$\|x\|_{\min} = \sup_n \left\| \sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi_2(b_i) \right\|$$

$$\|x\|_{\min}' = \sup_n \left\| \sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi_2'(b_i) \right\|.$$

For $n \geq 1$, define a $*$-isomorphism $\phi : \mathrm{M}_n(\mathbb{C}) \to P_nB(\mathcal{H})P_n$. Since ϕ is a faithful representation of $\mathrm{M}_n(\mathbb{C})$, by (11.2), we have

$$\begin{aligned}
\left\| \sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi_2(b_i) \right\| &= \left\| \sum_{i=1}^m \phi(\phi^{-1}(P_n\pi(a_i)P_n)) \otimes \pi_2(b_i) \right\| \\
&= \left\| \sum_{i=1}^m \phi(\phi^{-1}(P_n\pi(a_i)P_n)) \otimes \pi_2'(b_i) \right\| \\
&= \left\| \sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi_2'(b_i) \right\|.
\end{aligned}$$

It follows that $\|x\|_{\min} = \|x\|_{\min}'$. ◻

So, given C*-algebras A_1 and A_2 and faithful non-degenerate representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we complete $\pi_1 \odot \pi_2$ to a faithful representation

$$\pi_1 \otimes \pi_2 : A_1 \otimes A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

There is another often useful description of the minimal tensor norm.

Proposition 11.26. *For C*-algebras A_1 and A_2, and $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$,*

$$\|x\|_{\min} = \sup\{\|\sum_{j=1}^n \pi_1(a_j) \otimes \pi_2(b_j)\| \ : \ \pi_i : A_i \to B(\mathcal{H}_i) \text{ (nondegenerate) representations}\}.$$

[14] $S_n \to S$ in $*$-SOT if $S_n \to S$ in SOT and $S_n^* \to S^*$ in SOT.

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be representations and $\sigma_i : A_i \to B(\mathcal{H}'_i)$ be faithful representations. Then by Exercise 4.16, $\pi_i \oplus \sigma_i : A_i \to B(\mathcal{H}_i \oplus \mathcal{H}'_i)$ is a faithful representation. Let $P_i \in B(\mathcal{H}_i \oplus \mathcal{H}'_i)$ be the compression to $\mathcal{H}_i$ for each $i = 1, 2...$ □

Exercise 11.27. Finish the proof of Proposition 11.26. This is an example of a technique where one can *dilate* a map to one with a desired property (e.g. faithfulness) and then *cut down* to the original map to draw the desired conclusion.

Corollary 11.28. *For a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism*

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2.$$

Proof. We are charged with showing that $\phi_1 \odot \phi_2$ is continuous with respect to the topologies on $A_1 \odot A_2$ and $B_1 \odot B_2$ induced by their respective $\|\cdot\|_{\min}$ norms. We know that there exist faithful representations $\pi_i^A : A_i \to B(\mathcal{H}_i^A)$ and faithful representations $\pi_i^B : B_i \to B(\mathcal{H}_i^B)$. So if $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$, the fact that $*$-homomorphisms are norm-decreasing means that

$$\|x\|_{A_1\otimes_{\min} A_2} = \|\sum_{j=1}^n \pi_1^A(a_j) \otimes \pi_2^A(b_j)\| \geq \|\sum_{j=1}^n \pi_1^B(\phi_1(a_j)) \otimes \pi_2^B(\phi_2(b_j))\| = \|\phi_1 \odot \phi_2(x)\|_{B_1\otimes_{\min} B_2}.$$

But each $\pi_i^B \phi_i : A_i \to B(\mathcal{H}_i^B)$ is a representation of A_i, so we complete the proof via an appeal to the preceding proposition. □

Just as with groups, there is another natural norm which comes from taking all possible representations.

Definition 11.29 (Maximal Norm)**.** Let A and B be C*-algebras. We define the maximal C*-tensor norm on $A \odot B$ by

$$\|x\|_{\max} = \{\sup \|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}) \text{ a (non-degenerate) rep}\}$$

for all $x \in A \odot B$.

The first question is if this is even finite; it is by Theorem 11.17. Indeed, given $\pi : A \odot B \to B(\mathcal{H})$, with restrictions $\pi|_A$ and $\pi|_B$, then we have for all simple tensors $a \odot b \in A \odot B$,

$$\|\pi(a \odot b)\| = \|\pi|_A(a)\pi|_B(b)\| \leq \|\pi|_A(a)\|\|\pi|_B(b)\| \leq \|a\|\|b\|.$$

Just as we argued for groups (Proposition 5.7), this with the triangle inequality guarantees that $\|x\|_{\max} < \infty$ for all $x \in A \odot B$.

Exercise 11.30. Check that $\|\cdot\|_{\max}$ is a semi-norm satisfying the C*-identity.

For any pair of faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we get a representation $\pi = \pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It follows that for any $x \in A_1 \odot A_2$,

$$\|x\|_{\min} = \|\pi(x)\| \leq \|x\|_{\max}.$$

So, for any $x \in A_1 \odot A_2$,

$$\|x\|_{\max} = 0 \Rightarrow \|x\|_{\min} = 0 \Rightarrow x = 0,$$

which means $\|\cdot\|_{\max}$ is a norm. Hence we define the C*-algebra

$$A_1 \otimes_{\max} A_2 := \overline{A_1 \odot A_2}^{\|\cdot\|_{\max}}.$$

Remark 11.31. Note that by definition, the $*$-algebra $A_1 \odot A_2$ is a dense subalgebra in $A_1 \otimes_{\max} A_2$ and $A_1 \otimes A_2$.

Just as with groups, the maximal tensor product enjoys the following universal property.

Proposition 11.32. *If $\phi : A_1 \odot A_2 \to C$ is a $*$-homomorphism, then there exists a unique $*$-homomorphism $A_1 \otimes_{\max} A_2 \to C$, which extends ϕ. In particular, any pair of $*$-homomorphisms $\phi_i : A_i \to C$ with commuting ranges induces a unique $*$-homomorphism*

$$\phi_1 \times \phi_2 : A \otimes_{\max} B \to C.$$

Note that this is really just a statement about norms, and it is a theme we've seen before (Proposition 5.7). Let's flesh out a more general idea that underlies both.

Suppose B and C are C*-algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. (Notice that, unless $B_0 = B$, this means B_0 is *not* a C*-algebra.) The only obstruction to extending π to a $*$-homorphism on B is if π is not contractive on B_0, i.e. $\|\pi(b)\| > \|b\|$ for some $b \in B_0$. In other words, π extends to B iff π is contractive on B_0. The necessity is easy to see. Indeed, if π does extend to B, then the C*-norm on B forces π to be contractive on all of B, including B_0. On the other hand, if $\pi : B_0 \to C$ is a contractive $*$-homomorphism, then it is in particular bounded, which means it extends to a bounded homomorphism $\pi : B \to C$. Moreover, just as we saw in Proposition 5.7, for any $b \in B$ with $b_n \in B_0$ converging to b, we have $\pi(b_n) \to \pi(b)$ and hence $\pi(b_n)^* \to \pi(b)^*$. Then by uniqueness of limits, $\pi(b^*) = \pi(b)$ since

$$\|\pi(b_n)^* - \pi(b^*)\| = \|\pi(b_n^*) - \pi(b^*)\| \to 0.$$

For the sake of reference, we record this in a proposition:

Proposition 11.33. *Suppose B and C are C*-algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. Then π extends to B iff π is contractive on B_0.*

With that digression, the proof of proposition 11.32 is quite immediate.

Proof of Proposition 11.32. Take a faithful non-degenerate representation $\pi : C \to B(\mathcal{H})$. Then $\pi \circ \phi : A_1 \odot A_2 \to B(\mathcal{H})$ is a contractive $*$-homomorphism (with respect to the $\|\cdot\|_{\max}$ norm) and hence extends to $A \otimes_{\max} A_2$. □

It follows from this that $\|\cdot\|_{\max}$ is the largest possible C*-norm on $A_1 \odot A_2$.

Corollary 11.34. *Given any C*-norm $\|\cdot\|$ on $A_1 \odot A_2$, there is a surjective $*$-homomorphism $A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ extending the identity map on $A_1 \odot A_2$.*

Proof. Suppose $\|\cdot\|$ is another C*-norm on $A_1 \odot A_2$. Then the identity map $A_1 \odot A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ is a $*$-homomorphism, which then extends to a $*$-homorphism

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}.$$

Since it is a $*$-homomorphism, its image is closed and contains the dense subset $A_1 \odot A_2$, and so it is a surjection. As a surjective $*$-homomorphism, it is contractive, and so $\|x\|_{\max} \geq \|x\|$ for all $x \in A_1 \odot A_2$. □

Remark 11.35. Very often in the literature, the closure of $A \odot B$ with respect to an arbitrary tensor norm is denoted by $A \otimes_\alpha B$ where the norm is denoted by $\|\cdot\|_\alpha$.

It turns out that the spatial norm $\|\cdot\|_{\min}$ *is* the minimal C*-norm on $A_1 \odot A_2$. This is an important theorem due to Takesaki whose proof involves some heavy work in extending states to tensor products. For the sake of time, we will have to take this for granted. The proof is worked out in [3, Section 3].

Theorem 11.36 (Takesaki)**.** *The spatial norm $\|\cdot\|_{\min}$ is the minimal C*-norm on $A_1 \odot A_2$. In other words, given any C*-norm $\|\cdot\|$ on $A_1 \odot A_2$, there are surjective $*$-homomorphisms*

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|} \to A_1 \otimes A_2$$

extending the identity map

$$A_1 \odot A_2 \to A_1 \odot A_2 \to A_1 \odot A_2.$$

It follows that if the *natural* surjection $A_1 \otimes_{\max} A_2 \to A_1 \otimes A_2$ is injective, then $A_1 \odot A_2$ has a unique tensor norm. This fact is often indicated by writing

$$A_1 \otimes_{\max} A_2 = A_1 \otimes A_2.$$

Remark 11.37. It is important here that it is this natural surjection that is also injective, i.e. the one that extends the identity map $A_1 \odot A_2$.

We have been avoiding the non-unital elephant in the room. We relegate the proof to [3, Corollary 3.3.12].

Proposition 11.38. *If A and B are C^*-algebras with A non-unital, then any C^*-norm on $A \odot B$ can be extended to a C^*-norm on $\tilde{A} \odot B$ (meaning the norms agree on $A \odot B \subset \tilde{A} \odot B$). Similarly, when both A and B are non-unital, any C^*-norm can be extended to $\tilde{A} \odot \tilde{B}$.*[15]

Exercise 11.39. For C*-algebras A and B, we have canonical[16] isomorphisms $A \otimes B \simeq B \otimes A$ and $A \otimes_{\max} B \simeq B \otimes_{\max} A$.

11.4. **Inclusions and Short Exact Sequences.** This section is dedicated to two properties that held automatically for algebraic tensor products but that can now fail for their C*-completions:

(1) They respect inclusions, i.e. if B and C are C*-algebras and $A \subset B$ a C*-subalgebra, then we have a natural inclusion
$$A \odot C \hookrightarrow B \odot C.$$
(2) They respect exact sequences, i.e. if B and C are C*-algebras and $J \triangleleft B$ an ideal, then the following sequence is exact.
$$0 \to J \odot C \to B \odot C \to B/J \odot C \to 0.$$

Proposition 11.40. *Let B and C be C^*-algebras, $A \subset B$ a C^*-subalgebra, and $J \triangleleft B$ an ideal. Then*

(1) We have a natural inclusion $A \otimes_{\min} C \subseteq B \otimes_{\min} C$.
(2) This can fail for the maximal tensor product.

Exercise 11.41. Check (1). (This is just a statement about norms on sums of simple tensors.)

For (2), that's where things get interesting. Questions about embeddability of maximal tensor products get hard quick. So, it's easiest to explain why it can go wrong. Recall that the maximal tensor product norm was defined as a supremum over all representations. A representation on $B \odot C$ restricts to one on $A \odot C$, but a representation on $A \odot C$ need not extend to $B \odot C$. So, in general the sup taken for the maximal norm on $A \odot C$ is taken over a larger set than the one for $B \odot C$.

Remark 11.42. One fact that will play a role promptly is that this *does* hold when A is an ideal in B. A representation from an ideal $J \triangleleft A$ in a C*-algebra does always extend to a representation on A (see [1, Section 1.3]). So when $J \triangleleft A$ is an ideal, then so is $J \odot C$ for any C*-algebra C, and we have $J \otimes_{\max} C \triangleleft A \otimes_{\max} C$.

Here are some examples of where this can go wrong. Unfortunately, we haven't built up sufficient terminology to explain why.

Example 11.43. Let $A \subset B(\mathcal{H})$ be a separable C*-algebra lacking Lance's Weak Expectation Property ([3, Exercise 2.3.14]), e.g. an exact C*-algebra that is non-nuclear (exactness due to Wasserman), such as $\mathrm{C}^*_r(\mathbb{F}_2)$. Then $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$ does not embed into $B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$.

Using Kirchberg's $\mathcal{O}_2$ embedding theorem (a very difficult and sophisticated result in C*-theory) as well as his groundbreaking work enabling the recent solution to Connes' Embedding Problem (more on that later), we can give another example: $\mathrm{C}^*_r(\mathbb{F}_2)$ embeds into $\mathcal{O}_2$ (because it is exact and separable), but $\mathrm{C}^*_r(\mathbb{F}_2) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$ does not embed into $\mathcal{O}_2 \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$.

Proposition 11.44. *Let B and C be C^*-algebras and $J \triangleleft B$ an ideal. Then*

(1) The sequence
$$0 \to J \otimes_{\max} C \to B \otimes_{\max} C \to B/J \otimes_{\max} C \to 0$$
is exact.
(2) This can fail for the minimal (i.e. spatial) tensor product.

For (1), the proof in full detail is provided in [3, Proposition 3.7.1]. We simply give an idea of what needs to be shown. In either case, $J \otimes_{\max} C \triangleleft B \otimes_{\max} C$ and $J \otimes C \triangleleft B \otimes C$. So we have exact sequences
$$0 \to J \otimes_{\max} C \to B \otimes_{\max} C \to (B \otimes_{\max} C)/(J \otimes_{\max} C) \to 0$$
and
$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to (B \otimes_{\min} C)/(J \otimes_{\min} C) \to 0.$$

[15]In general (i.e. when we don't have $A = \tilde{A}$ or $B = \tilde{B}$, this is a larger algebra than $\widetilde{A \odot B}$.

[16]i.e. This is another way of saying "natural". In this setting, this means the maps extend the usual algebraic maps.

In both cases, from the algebraic identification $B/J \odot C \simeq (B \odot C)/(B \odot J)$ one argues that there is a C*-norm so that

$$(B \otimes_{\max} C)/(J \otimes_{\max} C) \simeq B/J \otimes_\alpha C \quad \textit{and} \quad (B \otimes_{\min} C)/(J \otimes_{\min} C) = B/J \otimes_\beta C.$$

It will follow from the maximality of $\|\cdot\|_{\max}$ that $\otimes_\alpha = \otimes_{\max}$. But for the other quotient, that won't always happen.

Definition 11.45. We say a C*-algebra C is *exact* if the sequence

$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to (B \otimes_{\min} C)/(J \otimes_{\min} C) \to 0$$

is exact for any C*-algebra B and any ideal $J \lhd B$.

Though seemingly unrelated, the two definitions we have given for exactness are indeed equivalent, though the proof of this is not easy.

Theorem 11.46 (Kirchberg)**.** *A* C**-algebra is exact in the sense of Definition 10.17 if and only if the functor* $\otimes_{\min} A$ *is exact, i.e. if the above definition holds.*

The question of when two C*-algebras have a unique C*-tensor norm is very difficult, and resolving this question for certain algebras is equivalent to resolving big open problems.

For example, thanks to deep and groundbreaking work of Kirchberg, we know that a famous recently-resolved problem, Connes' Embedding Problem, is equivalent to answering the question of whether or not $C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2)$. (Ask Brent and Rolando for the the original statement.) Further work (building on Kirchberg's results) connected this to what is known as Tsirelson's problem in quantum information theory, which was what was actually refuted earlier this year.

Another example is A. Thom's example of a hyperlinear group that is not residually finite. (Again, thanks to work of Kirchberg, this is equivalent to the full group C*-algebra of said group not having a unique tensor norm with $B(\mathcal{H})$.)

Another example is Junge and Pisier's proof that $B(\mathcal{H}) \odot B(\mathcal{H})$ does not have a unique C*-tensor norm when $\mathcal{H}$ is infinite dimensional, which was proven by Kirchberg to be equivalent to another collection of open problems.

Remark 11.47. You may have noticed that Kirchberg was very influential in a lot of results pertaining to tensor products of C*-algebras. Yeah.

11.5. **Nuclearity.** On the other end of the spectrum are C*-algebras which always have unique tensor product norms. The term originally used for such C*-algebras was in fact "nuclear." But we've already used this term for C*-algebras satisfying the completely positive approximation property. That these two coincide is a remarkable theorem, independently proved by Choi-Effros and Kirchberg

Theorem 11.48 (Choi-Effros, Kirchberg)**.** *A* C**-algebra* A *satisfies the completely positive approximation property (Definition 10.6) if and only if* $A \odot B$ *has a unique* C**-tensor norm for any* C**-algebra* B.

The proof of this theorem would require us to build up a fair bit of theory first, so we simply point you to Chapters 2 and 3 in [3], where the argument and surrounding theory is laid out quite well.

In general, it's often easier to prove that a C*-algebra has the completely positive approximation property (an internal property) as opposed to always having a unique tensor product norm (an external property). However, it was not so hard to show the latter for one class of C*-algebras.

Example 11.49. From Proposition 11.20, we know that $M_n(\mathbb{C})$ is nuclear for any $n \geq 1$. It turns out that any finite-dimensional C*-algebra is nuclear. (This mostly comes down to Proposition 6.1. See [7, Theorem 6.3.9] for more details.)

We have already seen that $K(\mathcal{H})$, as an AF algebra, is nuclear. Just for fun, here's an argument from the tensor product perspective.

Example 11.50. Let $\mathcal{K}$ denote the compact operators on some Hilbert space $\mathcal{H}$ and A any C*-algebra.

First we claim that $FR(\mathcal{H}) \odot A$ is a dense $*$-subalgebra of $\mathcal{K} \odot A$ with respect to any C*-norm on $\mathcal{K} \odot A$. We know from Day 1 lectures that $FR(\mathcal{H})$ is dense in $\mathcal{K}$. Now, suppose $S \odot a \in \mathcal{K} \odot A$ and $S_j \in FR(\mathcal{H})$

a sequence with $S_j \to S$. Recall that any C*-norm $\|\cdot\|$ on $\mathcal{K} \odot A$ is a cross norm, and so for any norm C*-norm $\|\cdot\|$ on $\mathcal{K} \odot A$, we have

$$\|(S \odot a) - (S_j \odot a)\| = \|(S - S_j) \odot a\| = \|S - S_j\|\|a\| \to 0.$$

Using the triangle inequality, we can extend this to show that any $x = \sum_{j=1}^m T_j \odot a_j \in \mathcal{K} \odot A$ can be approximated in any C*-norm by sums of simple tensors of finite rank operators.

So if we know $\|x\|_{\max} = \|x\|_{\min}$ for any $x \in FR(\mathcal{H}) \odot A$, then it follows that the natural surjection $\mathcal{K} \otimes_{\max} A \to \mathcal{K} \otimes A$ is isometric and $\mathcal{K}$ is nuclear. Fix $x = \sum_{j=1}^m T_j \odot a_j \in FR(\mathcal{H}) \odot A$, and let $\pi : \mathcal{K} \odot A \to B(\mathcal{H}')$ be a representation. Then there exists a projection $P \in B(\mathcal{H})$ of rank $n < \infty$ such that $T_j = PT_jP$ for each j, and $x = \sum_{j=1}^m PT_jP \odot a_j$. Hence $x \in PB(\mathcal{H})P \odot A$. From Exercise 7.41 from Day 1 Lectures, we have a $*$-isomorphism $\phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P$, and hence a representation $\pi' := \pi \circ (\phi \odot id_A) : \mathrm{M}_n(\mathbb{C}) \odot A \to B(\mathcal{H})$.

Since we know $\mathrm{M}_n(\mathbb{C}) \otimes_{\max} A = \mathrm{M}_n(\mathbb{C}) \otimes_{\min} A$, we know that for any faithful representations $\sigma_1 : \mathrm{M}_n(\mathbb{C}) \to B(\mathcal{H}_1)$ and $\sigma_2 : A \to B(\mathcal{H}_2)$,

$$\begin{aligned}
\|\sum_{j=1}^m \sigma_1(\phi^{-1}(PT_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)} &= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\min} \\
&= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\max} \geq \|\pi'(\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j)\| \\
&= \|\pi(\sum_{j=1}^m PT_jP \odot a_j)\| = \|\pi(x)\|.
\end{aligned}$$

In particular, this holds for the faithful representations $\sigma_1 = \mathrm{id}_{\mathcal{K}} \circ \phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P \subset \mathcal{K} \hookrightarrow B(\mathcal{H})$ and any faithful representation σ_2 of A. But then we have

$$\begin{aligned}
\|x\|_{\min} &= \|\sum_{j=1}^m \mathrm{id}_{\mathcal{K}}(S_j) \odot \sigma_2(a_j)\|_{B(\mathcal{H} \otimes \mathcal{H}_2)} \\
&= \|\sum_{j=1}^m \sigma_1(\phi^{-1}(PS_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H} \otimes \mathcal{H}_2)} \\
&\geq \|\pi(x)\|.
\end{aligned}$$

Since $\pi : \mathcal{K} \odot A \to B(\mathcal{H}')$ was arbitrary, it follows that

$$\|x\|_{\min} \geq \|x\|_{\max},$$

which finishes the proof.

Remark 11.51. Consider $\mathcal{K} = \mathcal{K}(\ell^2)$. It follows from Example 11.50 that the completion of $\mathcal{K} \odot \mathcal{K}$ under any tensor norm can be identified with the completion of $\mathcal{K} \odot \mathcal{K}$ with respect to the norm on $B(\ell^2 \odot \ell^2)$ (via the tensor product of faithful representations $id_{\mathcal{K}} \odot id_{\mathcal{K}}$). This will be a closed two-sided ideal in $B(\ell^2 \odot \ell^2)$, which means it must be the compact operators $\mathcal{K}(\ell^2 \odot \ell^2)$. Moreover, after a permutation of the basis elements, we have $\ell^2 \otimes \ell^2 \simeq \ell^2$. With this, one can then argue that $\mathcal{K} \otimes \mathcal{K} \simeq \mathcal{K}$. More generally, we say a C*-algebra is *stable* if $A \otimes \mathcal{K} \simeq A$. (Because of nuclearity, it does not matter what tensor product we choose.)

Since $\mathcal{K}$ is stable and since $(A \otimes \mathcal{K}) \otimes \mathcal{K} \simeq A \otimes (\mathcal{K} \otimes \mathcal{K}) \simeq A \otimes \mathcal{K}$ for any C*-algebra A[17], we call $A \otimes \mathcal{K}$ the *stabilization* of A. This is a basic object in many results and theories in C*-algebras, such as multiplier algebras, K-theory and classification, and is closely tied to Morita equivalence for C*-algebras. It turns out that the stabilization of A is very similar to A from the perspective of many C*-algebraic invariants, and so replacing A by its stabilization gives one more "wiggle room" for computations without affecting the underlying structure very much.

There is another fundamental class of nuclear C*-algebras: commutative C*-algebras. This was not so hard to prove with the completely positive approximation property definition of nuclearity (Proposition

[17] In fact, the associativity for the minimal and maximal tensor product norms holds for all C*-algebras, i.e. for C*-algebras A, B, C, we have $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ and $(A \otimes_{\max} B) \otimes_{\max} C \simeq A \otimes_{\max} (B \otimes_{\max} C)$. This is normally an exercise, but we have plenty already.

10.10). Before the Choi-Effros/Kirchberg theorem, Takesaki showed that tensor products with commutative C*-algebras always have a unique C*-norm, but the proof was much more involved.

Theorem 11.52 (Takesaki)**.** *Let A and C be* C*-*algebras with C commutative. Then there is a unique* C*-*tensor norm on $C \odot A$.*

11.6. $C_0(X, A)$ **as tensor products.** Let us spend a little more time on this last class of nuclear C*-algebras. Recall from the Gelfand Naimark Theorem that any commutative C*-algebra is $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. With this in mind look into another description of the tensor product of a C*-algebra with a commutative C*-algebra.

Definition 11.53. Let A be a C*-algebra and X a locally compact Hausdorff space (when X is not compact, we denote by $X \cup \{\infty\}$ its one point compactification). Just as we did for $A = \mathbb{C}$, we define

$$C_0(X, A) := \{f : X \cup \{\infty\} \to A \ : \ f \text{ continuous and } f(\infty) = 0\}.$$

When X is moreover compact, this is the same as $C(X, A)$.

Lemma 11.54. *Let A be a* C*-*algebra and X a locally compact Hausdorff space. Define the $*$-homomorphism $\phi : C_0(X) \odot A \to C_0(X, A)$ on simple tensors by $f \odot a \mapsto f(\cdot)a$. This gives a $*$-homomorphism, which then extends to a surjective $*$-homomorphism $C_0(X) \otimes_{\max} A \to C_0(X, A)$. Moreover, ϕ is injective on $C_0(X) \odot A$.*

The proof that the image of ϕ is dense in $C_0(X, A)$ is another example of a "partition of unity argument." We will give the argument from [7, Lemma 6.4.16] in the case where X is compact. The non-compact case amounts to identifying $C_0(X, A) = \{f \in C(X \cup \{\infty\}, A) \ : f(\infty) = 0\}$ (see [7, Lemma 6.4.16] for full details).

Recall that we take for granted the fact from topology that, given any compact Hausdorff space X with open cover $U_1, ..., U_n$, there exist continuous functions $h_1, ..., h_n : X \to [0, 1]$ so that $\mathrm{supp}(h_j) \subset U_j$ and $\sum_j h_j = 1$. (See [Theorem 2.13, Rudin, Real and Complex Analysis].) This is a *partition of unity* subordinate to $U_1, ..., U_n$ (in fact a rather nice one).

Proof of Lemma 11.54. Since there is nothing surprising in checking that it is a $*$-homomorphism, which by universality extends to a $*$-homomorphism on $A \otimes_{\max} B$, we move straight to the surjective $*$-isomorphism claim.

For the surjectivity argument, we assume X is compact (or work in its one point compactification as aforementioned). Since the image of a $*$-homomorphism from a C*-algebra is closed, it suffices to show that $C(X, A)$ is the closed linear span of functions of the form $f(\cdot)a$ for $f \in C(X)$ and $a \in A$. Let $g \in C(X, A)$ and $\varepsilon > 0$. Since X is compact and g continuous, $g(X)$ is compact, which means we can find a finite collection $a_1, ..., a_n \in g(X) \subset A$ so that $\{B_\varepsilon(a_j)\}_j$ covers $g(X)$, and hence $U_j = g^{-1}(B_\varepsilon(a_j))$ forms a finite open cover of X. Since X is compact, the aforementioned fact from topology tells us there exist continuous functions $h_j : X \to [0, 1]$, $1 \le j \le n$ so that for each j, $\mathrm{supp}(h_j) \subset U_j$ and $\sum_j h_j(x) = 1$ for all $x \in X$. Notice that, by our choice of U_j, that means that for each $x \in X$, either $h_j(x) = 0$ or $\|g(x) - a_j\| < \varepsilon$. Then we compute for each $x \in X$,

$$\begin{aligned} \|g(x) - \sum_j h_j(x)a_j\| &= \| \left(\sum_j h_j(x) \right) g(x) - \sum_j h_j(x)a_j\| \\ &= \| \sum_j h_j(x)(g(x) - a_j)\| \le \sum_j h_j(x)\|g(x) - a_j\| \\ &\le \sum_j h_j(x)\varepsilon = \varepsilon. \end{aligned}$$

This establishes our claim.

For injectivity, on $C_0(X) \odot A$, suppose $c = \sum_{j=1}^n f_j \odot a_j \in \ker(\phi)$ where $f_1, ..., f_n \in C_0(X)$ and $a_1, ..., a_n$ are linearly independent elements of A. Then $\phi(c) = 0$ implies that $\sum f_j(x)a_j = 0$ for all $x \in X$. But now these $f_j(x)$ are just complex numbers, and so the linear independence of the $a_1, ..., a_n$ implies that $f_j(x) = 0$ for each $1 \le j \le n$ and every $x \in X$. That means $f_1 = ... = f_n = 0$ and so $c = 0$. Hence ϕ is injective on $C_0(X) \odot A$. ∎

Theorem 11.55. *If A is a* C*-*algebra and X is a locally compact Hausdorff space, then for any* C*-*tensor norm, we have $\overline{C_0(X) \odot A}^{\|\cdot\|} \simeq C_0(X, A)$.*

Proof. Since the map ϕ from Lemma 11.54 is injective, the pull-back of the norm from $C_0(X, A)$ (i.e. $\|c\| = \|\pi(c)\|$) gives a C*-norm on $C_0(X) \odot A$ (as opposed to just a semi-norm). By Theorem 11.52, there is a unique C*-tensor norm on $C_0(X) \odot A$, which means this norm agrees with $\|\cdot\|_{\max}$. Hence the surjective $*$-homomorphism $C_0(X) \otimes_{\max} A \to C_0(X, A)$ is isometric, and hence a $*$-isomorphism. By identifying $C_0(X) \otimes_{\max} A$ with the closure of $C_0(X) \odot A$ under any other C*-norm, the claim follows. □

Example 11.56. Three particularly interesting cases are when $X = [0, 1]$, $X = (0, 1]$, and $X = (0, 1)$. [18] For a C*-algebra A, the *cone* over A is the C*-algebra

$$CA := C_0((0, 1], A) = \{f : (0, 1] \to A \ : \ f \text{ is continuous and } \lim_{t\to 0} f(t) = 0\},$$

and the *suspension*[19] over A is the C*-algebra,

$$SA := C_0((0, 1), A) := \{f : (0, 1)] \to A \ : \ f \text{ is continuous and } \lim_{t\to 0} f(t) = 0 = \lim_{t\to 1} f(t)\}.$$

The suspension will become very important when we get to K-theory. It is also sometimes denoted by ΣA.

11.7. **Continuous linear maps on tensor products.** In Takesaki's proof that $\|\cdot\|_{\min}$ is the smallest C*-norm, a delicate and crucial part of the argument is showing that states extend to tensor products, i.e. for $\phi_i \in S(A_i)$, $\phi_1 \odot \phi_2$ extends to a state on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ for any C*-norm $\|\cdot\|$ (mapping into $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$).

Given a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, we have a $*$-homomorphism

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

defined on the dense $*$-subalgebra $A_1 \odot A_2$ of $\overline{A_1 \odot A_2}^{\|\cdot\|}$ where $\|\cdot\|$ is any C*-norm. By Proposition 11.33, this extends to a $*$-homomorphism on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ iff $\phi_1 \odot \phi_2$ is contractive on sums of simple tensors. Naturally, this depends on the norm we put on $B_1 \odot B_2$ (e.g. if $A_i = B_i$ and we give $A_1 \odot A_2$ the minimal norm and $B_1 \odot B_2$ the maximal norm).

We already saw in Corollary 11.28 that this holds when we consider both $A_1 \odot A_2$ and $B_1 \odot B_2$ with their respective minimal tensor product norms.

Exercise 11.57. Show that for a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism on

$$\phi_1 \otimes_{\max,\beta} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_\beta B_2$$

for any C*-tensor product $B_1 \otimes_\beta B_2$.

However, many maps that we want to work with (e.g. states) are not necessarily $*$-homomorphisms. Hence it is important to understand which class of bounded linear maps extend to tensor products, in particular, for which bounded linear maps $\phi_i : A_i \to B_i$ does $\phi_1 \odot \phi_2$ extend to continuous linear maps

$$\phi_1 \otimes_{\max} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2$$

and

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2?$$

Let us consider an example where this fails.

Example 11.58. Consider $\mathcal{K} = \mathcal{K}(\ell^2)$. As we saw in Example 11.50, $\mathcal{K}$ is nuclear, meaning in particular that the completion of $\mathcal{K} \odot \mathcal{K}$ under any tensor norm can be identified with the completion of $\mathcal{K} \odot \mathcal{K}$ with respect to the norm on $B(\ell^2 \otimes \ell^2)$ (via the tensor product of faithful representations $id_{\mathcal{K}} \odot id_{\mathcal{K}}$). For each i, j, we define the rank one operator $P_{i,j} = \langle \cdot, e_i \rangle e_j$. (Think of these as an infinite-dimensional version of the matrix units for $\mathrm{M}_n(\mathbb{C})$.) For each $n \geq 1$, define $V_n \in \mathcal{K} \otimes \mathcal{K}$ by

$$V_n := \sum_{i,j=1}^{n} P_{i,j} \otimes P_{j,i}.$$

Then V_n is a partial isometry. (Indeed, since $P_{i,j} P_{l,k} = \delta_{j,l} P_{i,k}$, we can compute that $V_n^* V_n = P_n \odot P_n$ where P_n is the rank n projection sending $e_j \mapsto e_j$ for $1 \leq j \leq n$ and $e_j \mapsto 0$ for $j > n$.) So $\|V_n\| = 1$ for all n.

[18] Depending on how we like to define our functions these intervals are sometimes replaced with homeomorphic copies, e.g., sometimes $\mathbb{R}$ is used in place of $(0, 1)$. This certainly makes the "∞" notation more natural!

[19] "Cone" and "suspension" are not to be confused with the notions from topology, in case you are wondering.

Now considering each $T = [t_{ij}] \in \mathcal{K}$ as an array, we let $Tr : \mathcal{K} \to \mathcal{K}$ denote the transpose map, which is given by $Tr([t_{ij}]) = [t_{ji}]$. This is a linear $*$-preserving isometric map (since $T^* = [\bar{t}_{ji}]$), and

$$Tr \odot 1_{\mathcal{K}}(V_n) = \sum_{i,j=1}^{n} e_{ji} \otimes e_{ji}.$$

Now, consider the vector $\xi = \sum_{k=1}^{n} e_k \otimes e_k$. One computes

$$\begin{aligned}\|Tr \odot 1_{\mathcal{K}}(V_n)\xi\| &= \|\sum_{i,j=1}^{n}\sum_{k=1}^{n} \langle e_k, e_j\rangle e_i \otimes \langle e_k, e_j\rangle e_i\| \\ &= \|\sum_{i=1}^{n}\sum_{k=1}^{n} \langle e_k, e_k\rangle e_i \otimes \langle e_k, e_k\rangle e_i\| \\ &= \|\sum_{i=1}^{n} n(e_i \otimes e_i)\| = \|n\xi\| = n\|\xi\|.\end{aligned}$$

In particular, this means that $\|Tr \odot 1_{\mathcal{K}}(V_n)\| \geq n$ and hence $\|Tr \odot 1_{\mathcal{K}}\| \geq n$ for all $n \in \mathbb{N}$. This is an unbounded operator and hence not continuous.

So what kinds of bounded linear maps on C*-algebras yield continuous tensor product maps? Notice that the above example is $*$-preserving, so that's not enough. We have remarked several times that much of the structure of the C*-algebra is preserved by positive elements. Perhaps we need to consider linear maps $\phi : A \to B$ that send positive elements in A to positive elements in B? But even that isn't enough. It turns out that the transpose map above does send positive elements to positive elements. So, what gives? This is where we finally motivate the idea of *completely* positive maps. Recall that a linear map $\phi : A \to B$ between C*-algebras is completely positive if (equivalently) the linear map

$$\phi^{(n)} : \mathrm{M}_n(\mathbb{C}) \otimes A \to \mathrm{M}_n(\mathbb{C}) \otimes B$$

is positive for all $n \geq 1$.

Theorem 11.59. *Let $\phi_i : A_i \to B_i$ be linear cp maps. Then the algebraic tensor product map*

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

extends to a linear cp map (which is then also bounded and hence continuous) map on both the maximal and minimal tensor products:

$$\begin{aligned}\phi_1 \otimes \phi_2 :& A_1 \otimes A_2 \to B_1 \otimes B_2 \\ \phi_1 \otimes_{\max} \phi_2 :& A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2.\end{aligned}$$

Moreover, we have $\|\phi_1 \otimes_{\max} \phi_2\| = \|\phi_1 \otimes \phi_2\| = \|\phi_1\|\|\phi_2\|$.

Remember that we have already proved this for $*$-homomorphisms. Stinespring's Dilation theorem will allow us to transfer this fact to cpc maps.

In full disclosure, we need a stronger version of this to prove the $\otimes_{\max}$ part of Theorem 11.59, so we direct you to [3, Proposition 1.5.6] and its use in the proof of [3, Theorem 3.5.3]. But for the sake of seeing Stinespring's Theorem in action, let's prove that the algebraic tensor product of cp maps extends to a cp map between spatial tensor products.

Proof of Theorem 11.59 (for spatial tensor). Let A_1, A_2, B_1, B_2 be C*-algebras and $\phi_i : A_i \to B_i$ cp maps. First, by taking faithful representations, it suffices to assume that $B_i \subset B(\mathcal{H}_i)$ for $i = 1, 2$ (why?). Then $\phi_i : A_i \to B(\mathcal{H}_i)$ are cp maps, which have Stinespring dilations $(\pi_i, \mathcal{H}'_i, V_i)$ for $i = 1, 2$. Since these are $*$-homomorphisms, $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}'_1) \odot B(\mathcal{H}'_2) \subset B(\mathcal{H}'_1 \otimes \mathcal{H}'_2)$ extends to $A_1 \otimes A_2$. Define the map $\phi_1 \otimes \phi_2 : A_1 \otimes A_2 \to B_1 \otimes B_2 \subset B(\mathcal{H}'_1 \otimes \mathcal{H}'_2)$ by

$$\phi_1 \otimes \phi_2(x) = (V_1 \otimes V_2)^*(\pi_1 \otimes \pi_2)(x)(V_1 \otimes V_2).$$

By Example 9.9, this is a cp map. Moreover, for elementary tensors $a_1 \odot a_2 \in A_2 \odot A_2$, we have

$$\phi_1 \otimes \phi_2(a_1 \odot a_2) = (V_1^*\pi_1(a_1)V_1) \otimes (V_2^*\pi_2(a_2)V_2) = \phi_1(a_1) \odot \phi_2(a_2),$$

which means (by linearity) that $\phi_1 \otimes \phi_2|_{A_1 \odot A_2} = \phi_1 \odot \phi_2$. ∎

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