Chapter 5

The Trace

In the previous chapters we saw that the $M_n(\mathbb{C})$ group von Neumann algebras and the hyperfinite II_1 factor are all examples tracial von Neumann algebras. The main purpose of this chapter is to show admitting a trace that characterizes all finite von Neumann algebras.

The first section give some structural results for the projections on finite von Neumann algebras with an emphasis on the construction of the dyadic projections and the range of the trace on II_1 factors. The next section examines the outcome of the GNS contruction when applied to a trace on a factor, which is called the standard representation. There are many like it, but this one is ours. We also introduce ultrapower and ultraproduct constructions to help us define technical invariants for von Neumann algebras, namely the McDuff Property and Property Γ.

We leave the details of the construction of a center-valued trace to the very end of the chapter for those who want to punish themselves.

Lecture Preview: The content in this lecture will be covered over 2 days. The first of these lectures on the 10th of July will cover Pages 64–66 properties of the trace for finite von Neumann algebras. To prepare yourself for the lecture, it is highly encouraged that you know 5.4.1. Lemmas 5.1.1 through 5.1.3 will be briefly discussed, but proof will likely not be presented. Definition 5.1.4 onward will provide the bulk of the content. Please review the statements of Theorem 5.4.8, 5.4.9 , Theorem 5.4.10 as they will be referenced.

The lecture on Monday the 13th of July will describe the Standard Representation of a II_1 factor (Pages 67–70). If time allows, we will describe the ultraprocduct construction (see Definition 5.3.2).

5.1 Tracial von Neumann Algebras

Lemma 5.1.1. *Let $M \subset B(\mathcal{H})$ be a finite von Neumann algebra and $p \in \mathcal{P}(M)$ non-zero. If $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ is a family of pairwise orthogonal projections satisfying $p_i \sim p$ for all $i \in I$, then $|I| < \infty$.*

Proof. If I is infinite, then there exists a proper subset $J \subset I$ with $|J| = |I|$. But then

$$\sum_{i\in I} p_i \sim \sum_{j\in J} p_j < \sum_{i\in I} p_i,$$

contradicting M being finite. □

Lemma 5.1.2. *Let $M \subset B(\mathcal{H})$ be a type II_1 von Neumann algebra. Then there exists a projection $p_{1/2} \in \mathcal{P}(M)$ so that $p_{1/2} \sim 1-p_{1/2}$. Moreover, there exists a family of projections $\{p_r\}_r$ indexed by dyadic rationals $r \in [0,1]$ such that:*

(i) $p_r \le p_s$ *if* $r \le s$;

(ii) $p_s - p_r \sim p_{s'} - p_{r'}$ *whenever* $0 \le r \le s \le 1$ *and* $0 \le r' \le s' \le 1$ *satisfy* $s - r = s' - r'$;

(iii) $z(p_r) = 1$ *for every* r.

Proof. Let $\{p_i, q_i\}_{i\in I}$ be a maximal family of pairwise orthogonal projections such that $p_i \sim q_i$ for all $i \in I$. Define $p_{1/2} := \sum_i p_i$ and $q = \sum_i q_i$. Then $p_{1/2} \sim q$, and we further claim $q = 1 - p_{1/2}$. If not, then $1 - (p_{1/2} + q) \neq 0$. Since M is type II, $1 - (p_{1/2} + q)$ is not abelian and consequently there exists $p_0 \in \mathcal{P}([1 - (p_{1/2} + q)]M[1 - (p_{1/2} + q)])$ which is strictly less than its central support (in this corner), which we will denote by z. Therefore, if $q_0 = z - p_0$, then p_0 and q_0 are not centrally orthogonal, and consequently by Proposition 4.1.9 they have equivalent subprojections. However, this contradicts the maximality of $\{p_i, q_i\}_{i\in I}$. Thus $q = 1 - p_{1/2}$.

Now, we construct the family of projections indexed by dyadic radicals $r \in [0, 1]$ inductively. We let $p_{1/2}$ be as above, and set $p_1 := 1$ and $p_0 := 0$. Let $v \in M$ be such that $v^*v = p_{1/2}$ and $vv^* = 1 - p_{1/2}$. Note that $p_{1/2}Mp_{1/2}$ is type II by Remark 4.3.9. Moreover, it is type II_1 since $p_{1/2}$ is a finite projection: if $q \sim p_{1/2}$ with $q < p_{1/2}$ then $q+(1-p_{1/2}) \sim p_{1/2}+(1-p_{1/2}) = 1$ by Lemma 4.1.10, but $q+(1-p1/2) < p_{1/2}+(1-p_{1/2}) = 1$ contradicts 1 being finite. Thus $p_{1/2}Mp_{1/2}$ is type II_1 and so the above argument yields $p_{1/4} \leq p_{1/2}$ such that $p_{1/4} \sim p_{1/2} - p_{1/4}$. Set $p_{3/4} := p_{1/2} + vp_{1/4}v^*$. It is easily observed that $p_0 \leq p_{1/4} \leq p_{1/2} \leq p_{3/4} \leq p_1$ and $p_{1/4} \sim p_{(k+1)/4} - p_{k/4}$ for each $k = 0, 1, 2, 3$. Induction then yields a family satisfying (i) and (ii).

To see (iii), fix a dyadic rational r and set $z := 1 - z(p_r)$. Let $n \in \mathbb{N}$ be large enough so that $s := \frac{1}{2^n} \leq r$. Then by (i) we have $zp_s \leq zp_r = 0$. Using (ii), we have $zp_s \sim z(p_{ks} - p_{(k-1)s})$ for every $k = 1, \ldots, 2^n$, and so it must be that $z(p_{ks} - p_{(k-1)s}) = 0$. We then have

$$z = z\sum_{k=1}^{n}(p_{ks} - p_{(k-1)s}) = 0,$$

so that $z(p_r) = 1$ as claimed. □

Lemma 5.1.3. *Let $M \subset B(\mathcal{H})$ be a type II_1 von Neumann algebra, and let $\{p_r\}_r \subset \mathcal{P}(M)$ be the family of projections indexed by dyadic rationals $r \in [0, 1]$ as in the previous lemma. If $p \in \mathcal{P}(M)$ is non-zero, then there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ and a dyadic rational $r \in (0, 1]$ so that $p_rz \preceq pz$ and $p_rz, pz \neq 0$.*

Proof. By considering the compression $Mz(p)$, we may assume $z(p) = 1$. By the Comparison Theorem, for each dyadic rational $r \in (0, 1]$ there exists a central projection z_r such that $p_rz_r \preceq pz_r$ and $p(1 - z_r) \preceq p_r(1 - z_r)$. Suppose, towards a contradiction, $pz_r = 0$ for every r. Since $z(p) = 1$, it must be that $z_r = 0$ and so $p \preceq p_r$ for all r. In particular, we have for each $k \in \mathbb{N}$

$$p \preceq p_{2^{-(k+1)}} \sim p_{2^{-k}} - p_{2^{-(k+1)}}.$$

For each $k \in \mathbb{N}$, let $q_k \leq p_{2^{-k}} - p_{2^{-(k+1)}}$ be such that $p \sim q_k$. But then $\{q_k\}_{k\in\mathbb{N}}$ is an infinite family or pairwise orthogonal projections that contradicts Lemma 5.1.1. Thus there must be some r such that $pz_r \neq 0$. Consequently, $z_r \neq 0$ and so $p_rz_r \neq 0$ since $z(p_r) = 1$. □

THe existence of the dyadic projections is one of the first steps in constructing a trace on a type II_1 von Neumann algebra. The general idea would be to create a map from the dyadic projections mapping $\phi(p_r) \to r$ and then attempting to extend this map from M to $\mathcal{Z}(M)$. In Section 5.4, take an alternate route applying the Ryll-Nardjewski Theorem and other Banach space techniques. Unfortunatley, both paths we described as long and highly technical which is why we are instead choosing to accept that finite von Neuamm algebras have traces.

Definition 5.1.4. Let M be a von Neumann algebra. If $\tau : M \to \mathbb{C}$ is if there exists a normal, faithful state which also satisfies the trace condition, $\tau(xy) = \tau(yx)$, the τ is called a **trace** on M. We say M is **tracial** if M admits a trace.

Assuming that a trace exists, we know that M is automatically finite. The converse, however, is much more difficult and relies upon the construction of a center-valued trace, (see definition 5.4.1). This can be done, and the approach we take relies on heavy-handed Banach space techniques.

The upshot is that once you know the center valued state $\phi : M \to \mathcal{Z}(M)$ exists, we identify $\pi : \mathcal{Z}(M) \to L^\infty(X, \mu)$ (assuming that M has a cyclic vector). An even better situation comes up when M a factor because the center-valued trace is automatically a trace and we can stop here.

Theorem 5.1.5. *A von Neumann algebra M is finite if and only if M has a trace M is a finite factor if and only if M admits a unique trace $\tau : M \to \mathbb{C}$.*

Theorem 5.1.6. *Let M be a finite factor equipped with its unique trace τ.*

- *If M is of type I, then M is of type I_n with n finite and $\tau(\mathcal{P}(M)) = \{0, \frac{1}{n}, \ldots, 1\}$.*
- *If M is of type II_1, then $\tau(\mathcal{P}(M)) = [0, 1]$.*

One interpretation of the values of the trace on projections of a finite type I_n is that it tells us the size of the space onto which p projects relative to the ambient space. The trace on a II_1 factor is similar, except now, the relative size of a projection can be associated to a number in the continuum $[0, 1]$ and moreover, every value is realized.

I like to remind myself that every projection in $M_n(\mathbb{C})$ can be unitarily conjugated to a diagonal projection with the only non-zero entries being 1 somewhere along the diagonal. Here, we can view the trace as something akin to the normalized counting measure on a set of n points.

The picture that I have for II_1 factors is remarkably similar, except first I start with a "matrix" indexed by the interval $[0, 1]$ and mentally identity the diagonal with the interval $[0, 1]$. We might imagine an projection of trace t in a II_1 factor with "1's along the interval $[0, t]$. This allows to view the trace as a non-commutative analog of the Lebesgue measure on a $[0, 1]$.

The fact that τ is normal implies that for a countable collection of orthogonal projections, $\tau(\sum p_i) = \sum \tau(p_i)$. Since projections are the analogs of characteristic functions and the trace is similar to a measure, we interpret this as a kind of countable additivity .

If $M \subseteq B(\mathcal{H})$ is a II_1 factor with trace τ, then for any non-zero projection p we have that $pMp \subseteq B(p\mathcal{H})$ is also a type II_1 factor with trace given by $\tau(pxp)/\tau(p)$ (remember, p is the identity element of pMp). Now suppose that q is another projection such that $\tau(q) = \tau(p)$. Since M is a factor, we have that $p \sim q$ and $1 - p \sim 1 - q$ and thus, there is a unitary $u \in M$ so that $u^*pMpu = qMq$ and thus the isomorphism class of pMp depends only on $t = \tau(p)$ and not the choice of projection. Then for any $0 < t \leq 1$, we define define $M^t := pMp$ where p is any trace t projection.

It's also possible to extend the definition of M^t for any $t \geq 1$ by first choosing $n \in \mathbb{N}$ with $n \geq t$, and considering $M_n(N)$. $M_n(\mathbb{C})$ is again a II_1 factor with trace $\tau_n([x_{i,j}]) = \sum_{i=1}^n \tau(x_{i,i})$. Choosing a projection $p \in M_n(M)$ with trace $\tau_n(p) = t/n$, $M^t = pM_n(M)p$. We can check that up to isomorphism, M^t does not depend on our p or n and thus is well defined.

Definition 5.1.7. Let M be a type II_1 factor. The **fundamental group** of M is the subgroup of $\mathbb{R}_+$

$$\mathcal{F}(M) := \{t \in (0, \infty) : M^t \cong M\}.$$

The terminology here is unfortunate since this has concept no relation to the better-know fundamental group from topology. Mentioning the fundamental group of a II_1 factor in a talk or in casual conversation will almost surely result in someone asking if this has any connection to topology. My advice, just say "no" and then change the subject.

It is in fact a multiplicative subgroup of $\mathbb{R}$, which can be checked by verifying that for any $s, t > 0$ we have $(M^t)^s \cong M^{st}$.

When M is a tracial factor, there is another norm that one frequently encounters called the *2-norm.* Letting τ be the unique trace on M, via the formula

$$\|x\|_2 = \sqrt{\tau(x^*x)}.$$

Since τ is faithful, we see that M this formula indeed defines a norm on M. The trace also induces a Hilbert space structure on M via the formlula $\langle x, y\rangle = \tau(y^*x)$. Unfortunately, M is not complete with respect to this norm but it's completion is of interest. We delay that discussion for now. Instead, let's compare the 2-norm and the oeprator norm of a finte von Neumann algebra.

Theorem 5.1.8. *Let M be a tracial von Neumann algebra with trace τ. Then for any $x, y \in M$ we have that*

$$\|xy\|_2 \leq \|x\|\|y\|_2.$$

In particular, $\|x\|_2 \leq \|x\|$

Proof. We first prove that for any self-adjoint $w \in M$, $w \leq \|w\|1$, were $1 \in M$ is the identity element. Define $f(t) = \|w\| - t$ on $[-\|w\|, \|w\|]$. Then by the continuous functional calculus, we have that $\sigma(f(a)) \subset f(\sigma(a)) \subseteq [0, \infty)$ and thus $\|w\| - w \geq 0$. In particular $x^*x \leq \|x\|^2$.

Now let us compute:

$$\|xy\|_2^2 = \tau(y^*x^*xy) \leq \tau(\|x\|^2 y^*y) = \|x\|^2\tau(y^2y) = \|x\|^2\|y\|_2^2.$$

□

Exercises

5.1.1. Let $\mathcal{R}$ be the hyperfinite II_1 factor.

(a) Show for every dyadic rational $r \in [0,1]$, there exists a projection $p_r \in \mathcal{R}$ with $\tau(t_r) = r$. Hint: think about the construction of $\mathcal{R}$ as an inductive limit.

(b) Now if $t \in [0,1]$, show that there exists a projection $p_t \in \mathcal{R}$ with $\tau(p_t) = t$. Hint: if $t \in [0,1]$, there exists an increasing sequence (r_n) of dyadic rationals such $r_n \to t$.

5.1.2. Show that a von Neumann algebra M is finite if and only if for every $x, y \in M$ such that $xy = 1$ we have $yx = 1$, i.e. if X is right invertible, it is invertible.

5.1.3. Let M be a type II_∞ factor and p a finite projection in M. Show that there exists an infinite family of orthogonal projections $\{p_{i i\in I}\}$ with $p_i \sim p$ and $\sum_{i\in I} p_i = 1$. If $\tau : pMp \to \mathbb{C}$ is the trace on pMp and $v_i \in M$ with $v_i^*v_i = p, v_iv_i^* = p_i$, show that

$$\widetilde{\tau}(x) := \sum_{i\in I} \tau(v_i^*xv_i)$$

defines a normal tracial map. This is called a semi-finite trace on M.

5.1.4. Let M be a factor and $d : \mathcal{P}(M) \to [0, \infty]$ be a function such that

(i) $d(p+q) = d(p) + d(q)$ whenever $pq = 0$.

(ii) $d(p) = d(q)$ whenever $p \sim q$.

(iii) $d(p) = 0$ implies that $p = 0$.

Then any such d is called a dimension function.

(a) Show that M is finite if and only if there exists a dimension function d with $d(1) = 1$.

(b) When M is finite, show that $d = \tau|_{\mathcal{P}(M)}$ where τ is the trace on M.

(c) If M is type II_∞, show that any such function which is not identically 0 must take every value in $[0, \infty]$.

(d) If M is type III, show that $d(p) \in \{0, \infty\}$

5.1.5. Let M be a type II_1 factor.

(a) Show that $(M^t)^s \cong M^{ts}$.

(b) Conclude that the (poorly named IMO) fundamental group $\mathcal{F}(M)$ is in fact a subgroups of $\mathbb{R}_+$.

5.2 The Standard Representation

Let M be finite factor with unique faithful normal tracial state $\tau\colon M \to \mathbb{C}$. We denote by $L^2(M)$ the GNS Hilbert space associated to τ; that is,

$$\langle x, y\rangle_2 := \tau(y^*x) \qquad x, y \in M$$

defines an inner product on M and we take $L^2(M)$ to be its completion. For $x \in M$, we will sometimes add the decoration $\hat{x}$ when we want to emphasize that we are thinking of x as a vector in $L^2(M)$. We also obtain a faithful normal representation $\pi_\tau\colon M \to B(L^2(M))$ which is defined by $\pi_\tau(x)\hat{y} = \widehat{xy}$ for $x, y \in M$. Let us identify $M \cong \pi_\tau(M)$ so that we view M as a von Neumann algebra in $B(L^2(M))$, and for $x, y \in M$ we have $x\hat{y} = \widehat{xy}$.

Definition 5.2.1. For a finite factor M with unique trace τ, the representation $M \subset B(L^2(M))$ is called the **standard representation** of M.

Note that $x\hat{1} = \hat{x}$ implies $\hat{1}$ is a cyclic vector for M, and

$$\|x\hat{1}\|_2^2 = \langle \hat{x}, \hat{x}\rangle_2 = \tau(x^*x)$$

implies $\hat{1}$ is separating for M since τ is faithful.

Now, for $x \in M$ define $J\hat{x} := \widehat{x^*}$. We note that

$$\|J\hat{x}\|_2^2 = \|\widehat{x^*}\|_2^2 = \tau(xx^*) = \tau(x^*x) = \|\hat{x}\|_2^2.$$

Thus J extends to a conjugate linear isometry on $L^2(M)$.

Definition 5.2.2. For a finite factor M, the conjugate linear isometry J on $L^2(M)$ is called the **canonical conjugation operator**.

Note that since J is conjugate linear, we have $\langle J\xi, J\eta\rangle_2 = \langle \eta, \xi\rangle_2$ for $\xi, \eta \in L^2(M)$. You should also convince yourself that $(JxJ)^* = Jx^*J$ for $x \in B(L^2(M))$ (Exercise 5.2.1). Also observe that for $x, y, z \in M$ we have

$$\begin{aligned} x(JyJ)\hat{z} &= xJy\widehat{z^*} = xJ\widehat{yz^*} = x\widehat{zy^*} = \widehat{xzy^*} \\ &= J\widehat{yz^*x^*} = Jy\widehat{z^*x^*} = JyJ\widehat{xz} = (JyJ)x\hat{z}. \end{aligned}$$

Thus $x(JyJ) = (JyJ)x$ since $\widehat{M}$ is dense in $L^2(M)$. This implies $JMJ \subset M' \cap \mathcal{B}(L^2(M))$. We will show the reverse inclusion holds, but we first need to develop a few concepts. The following definition should remind you of left and right convolvers in $L(\Gamma)$ for a discrete group Γ (see Definition 1.3.4).

Definition 5.2.3. For $\xi \in L^2(M)$ define (potentially unbounded) linear operators $\lambda(\xi)\colon \widehat{M} \to L^2(M)$ and $\rho(\xi)\colon \widehat{M} \to L^2(M)$ by

$$\begin{aligned} \lambda(\xi)\hat{x} &:= (Jx^*J)\xi \qquad x \in M \\ \rho(\xi)\hat{x} &:= x\xi. \end{aligned}$$

We will call $\xi \in L^2(M)$ a **left bounded** (resp. **right bounded**) vector if $\lambda(\xi)$ (resp. $\rho(\xi)$) extends to a bounded operator on $L^2(M)$, and in this case we also denote this extension by $\lambda(\xi)$ (resp. $\rho(\xi)$). We denote by $LB(M)$ (resp. $RB(M)$) the collection of $\lambda(\xi)$ (resp. $\rho(\xi)$) for left-bounded (resp. right-bounded) vectors $\xi \in L^2(M)$.

We make a few observations about left and right bounded vectors. For $\lambda(\xi) \in LB(M)$ and $x \in M$

$$J\lambda(\xi)J\hat{x} = J\lambda(\xi)\widehat{x^*} = J(JxJ)\xi = xJ\xi = \rho(J\xi)\hat{x}.$$

This shows that $\rho(J\xi) \in RB(M)$ and $J\lambda(\xi)J = \rho(J\xi)$. Similarly, we have $J\rho(\xi)J = \lambda(J\xi)$ and hence $J(LB(M))J = RB(M)$. Additionally, for $\rho(\xi) \in RB(M)$ and $x, y \in M$

$$\langle \rho(J\xi)\hat{x}, \hat{y}\rangle_2 = \langle xJ\xi, \hat{y}\rangle_2 = \langle J\xi, x^*\hat{y}\rangle_2 = \left\langle J\widehat{x^*y}, \xi\right\rangle_2 = \left\langle \widehat{y^*x}, \xi\right\rangle_2 = \langle \hat{x}, y\xi\rangle_2 = \langle \hat{x}, \rho(\xi)\hat{y}\rangle_2 = \langle \rho(\xi)^*\hat{x}, \hat{y}\rangle_2 .$$

Thus $\rho(\xi)^* = \rho(J\xi)$, and using our previous identities we see that

$$\lambda(J\xi) = J\rho(\xi)J = J\rho(J\xi)^*J = (J\rho(J\xi)J)^* = \lambda(\xi)^*,$$

so $\lambda(xi)^* = \lambda(J\xi)$. Lastly, we observe that for $\lambda(\xi) \in LB(M)$, $\rho(\eta) \in RB(M)$, and $x, y \in M$

$$\begin{aligned}\langle \lambda(\xi)\rho(\eta)\hat{x}, \hat{y}\rangle_2 &= \langle x\eta, \lambda(J\xi)\hat{y}\rangle_2 = \langle x\eta, Jy^*J(J\xi)\rangle_2 = \langle JyJx\eta, J\xi\rangle_2 = \langle xJyJ\eta, J\xi\rangle_2 \\ &= \langle \xi, JxJyJ\eta\rangle_2 = \langle Jx^*J\xi, yJ\eta\rangle_2 = \langle \lambda(\xi)\hat{x}, \rho(J\eta)\hat{y}\rangle_2 = \langle \rho(\eta)\lambda(\xi)\hat{x}, \hat{y}\rangle_2 .\end{aligned}$$

Thus $\lambda(\xi)\rho(\eta) = \rho(\eta)\lambda(\xi)$, and so $LB(M) \subset RB(M)'$. We collect these observations in the following proposition.

Proposition 5.2.4. *Let M be a finite factor. For $\lambda(\xi) \in LB(M)$ and $\rho(\eta) \in RB(M)$ we have*

$$\begin{aligned}\lambda(\xi)^* &= \lambda(J\xi) = J\rho(\xi)J \\ \rho(\eta)^* &= \rho(J\eta) = J\lambda(\eta)J.\end{aligned}$$

Moreover, $J(LB(M)J = RB(M)$ and $LB(M) \subset RB(M)'$.

Just as left and right bounded vectors should remind of left and right convolvers, the proof of the following theorem should remind you of how we showed $R(\Gamma) = L(\Gamma)'$ (see Theorem 1.3.7).

Theorem 5.2.5. *Let M be a finite factor with trace τ. Under the standard representation $M \subset B(L^2(M))$, we have $M' = JMJ$ where J is the canonical conjugation operator on $L^2(M)$.*

Proof. For $x, y \in M$ we have

$$\lambda(\hat{x})\hat{y} = (Jy^*J)\hat{x} = Jy^*\widehat{x^*} = J\widehat{y^*x^*} = \widehat{xy} = x\hat{y},$$

so that $\lambda(\hat{x}) = x$. Hence $M \subset LB(M)$. Also, for $x \in M'$ and $y \in M$ we have

$$\rho(x\hat{1})\hat{y} = yx\hat{1} = xy\hat{1} = x\hat{y},$$

so that $\rho(x\hat{1}) = x$. Hence $M' \subset RB(M)$. Thus

$$M \subset LB(M) \subset RB(M)' \subset (M')' = M,$$

where the second inclusion follows from Proposition 5.2.4. Thus $M = LB(M) = RB(M)'$. Similarly,

$$M' \subset RB(M) \subset LB(M)' \subset M',$$

and so $M' = RB(M) = LB(M)'$. Thus $M' = RB(M) = J(LB(M))J = M$. □

One consequence of the above theorem is that $M' \cap B(L^2(M))$ is also a finite factor. Indeed, $\tau'(Jx^*J) := \tau(x)$ for $x \in M$ defines a trace on M'. However, this need not be true for an arbitrary representation $M \subset B(\mathcal{H})$ of a finite factor.

We change topics slightly here and derive another important concept from the standard representation.

Definition 5.2.6. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and $1_M \in N \subset M$ a von Neumann subalgebra. A **conditional expectation** from M to N is a linear map $E\colon M \to M$ satisfying

(i) $E(a) = a$ for all $a \in N$;

(ii) $E(axb) = aE(x)b$ for all $a, b \in N$ and $x \in M$;

(iii) $E(x) \geq 0$ whenever $x \geq 0$.

Observe for $x \in M$ that one has

$$\begin{aligned} 0 \leq E\left((x - E(x))^*(x - E(x))\right) &= E\left(x^*x - x^*E(x) - E(x)^*x + E(x)^*E(x)\right) \\ &= E(x^*x) - E(x^*)E(x) - E(x)^*E(x) + E(x)^*E(x) = E(x^*x) - E(x^*)E(x). \end{aligned}$$

So $E(x^*)E(x) \leq E(x^*x)$. Since E preserves positive elements, decomposing x as a linear combination of four positive elements yields $E(x^*) = E(x)^*$. Thus we have $E(x)^*E(x) \leq E(x^*x) \leq E(\|x\|^2 1_M) = \|x\|^2 1_M$, which implies $\|E(x)\| \leq \|x\|$. That is, E is automatically a contraction.

In general, a conditional expectation from M to a subalgebra N need not exist. However, when M is a finite factor the situation is quite nice:

Theorem 5.2.7. *Let M be a finite factor with trace τ. If $1_M \in N \subset M$ is a von Neumann subalgebra, then there exists a unique conditional expectation $E_N\colon M \to N$ satisfying $\tau \circ E_N = \tau$. Moreover, E_N is normal and faithful.*

The proof of this theorem is beyond the scope of these notes, but can be found in *An introduction to* II_1 *factors* by Claire Anantharaman-Delaroche and Sorin Popa. We mention that if $e_N := [N\hat{1}]$ then one can show $e_N\hat{x}$ is left bounded for all $x \in M$ and $E_N(x) = \lambda(e_N\hat{x})$ for $x \in M$. Observe that

$$\widehat{E_N(x)} = E_N(x)\hat{1} = \lambda(e_N\hat{x})\hat{1} = J1^*Je_N\hat{x} = e_N\hat{x}.$$

Since $\hat{1}$ is separating for M (and hence N), $E_N(x)$ is the unique $a \in N$ satisfying $\hat{a} = e_N\hat{x}$. Moreover, $E_N(x)$ is the unique $a \in N$ satisfying

$$\langle \hat{x}, \hat{b}\rangle_2 = \langle \hat{a}, \hat{b}\rangle_2 \qquad \forall b \in N.$$

Indeed,

$$\langle \hat{x}, \hat{b}\rangle_2 = \langle \hat{x}, e_N\hat{b}\rangle_2 = \langle e_N\hat{x}, \hat{b}\rangle_2 = \langle \widehat{E_N(x)}, \hat{b}\rangle_2,$$

implies that $\langle \hat{a} - \widehat{E_N(x)}, \hat{b}\rangle_2 = 0$ for all $b \in N$. Choosing $\hat{b} = \hat{a} - \widehat{E_N(x)}$ shows that $\hat{a} = E_N\hat{(}x)$ and so $a = E_N(x)$ since $\hat{1}$ is separating for M.

Exercises

5.2.1. For $x \in B(L^2(M))$, show that $(JxJ)^* = Jx^*J$.

5.2.2. For $n \in \mathbb{N}$, show that $L^2(M_n(\mathbb{C})) = M_n(\mathbb{C})$ with inner product

$$\langle A, B\rangle_2 = \frac{1}{n}\sum_{i,j=1}^{n} A_{i,j}\overline{B_{i,j}}.$$

5.2.3. For a discrete i.c.c. group Γ, let $M := L(\Gamma)$.

(a) Show that $L^2(M) = \ell^2(\Gamma)$.

(b) Show that $LB(M) = LC(\Gamma)$ and $RB(M) = RC(\Gamma)$.

5.2.4. Let M be a finite factor with trace τ. For $N := \mathbb{C} \subset M$, show that the conditional expectation $E_N\colon M \to N$ is given by $E_N(x) = \tau(x)1_M$.

5.2.5. For $n \in \mathbb{N}$, let $D \subset M_n(\mathbb{C})$ be the subalgebra of diagonal matrices. Show that the conditional expectation $E_D\colon M_n(\mathbb{C}) \to D$ is given by

$$E_D\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & & 0 \\ & \ddots & \\ 0 & & a_{n,n} \end{pmatrix}$$

5.2.6. Let Γ be a discrete i.c.c. group. Let $\Lambda < \Gamma$ be a subgroup, and view $L(\Gamma)$ as a von Neumann subalgebra of $L(\Gamma)$. Show that the conditional expectation $E_{L(\Lambda)}\colon L(\Gamma) \to L(\Lambda)$ satisfies $E_{L(\Lambda)}(\lambda(g)) = 1_\Lambda(g)\lambda(g)$.

5.3 The Tracial Ultraproduct and Ultrapowers

For this portion, we fix a family of von Neumann algebras $(M_n)_{n\in\mathbb{N}}$ such that every M_n is finite with trace τ_n. We fix an ultrafilter $\omega \in \beta(\mathbb{N})$, where $\beta(\mathbb{N})$. We let

$$\ell^\infty(\mathbb{N}, (E_n)) = \left\{ (x_n) \in \prod_{n=1}^{\infty} M_n : \sup_{n\in\mathbb{N}} \|x_n\| < \infty \right\},$$

denote the $*$-algebra of bounded sequences. We define the *trace ideal* to be

$$\mathcal{I} = \left\{ (x_n) \in \ell^\infty(\mathbb{N}, (M_n) : \lim_{n\to\omega} \tau_n(x_n^* x_n) = 0 \right\}$$

Lemma 5.3.1. *$\mathcal{I}$ as defined above is an operator closed 2-sided ideal of*

$$\ell^\infty(\mathbb{N}, (M_n))$$

.

Proof. Letting $(x_n) \in \mathcal{I}$, notice that for any $a_n, b_n \in M_n$, we have that

$$\|a_n x_n b_n\|_2 \leq \|a_n\| \|b_n\| \|x_n\|_2,$$

and hence

$$\lim_{n\to\omega} \tau_n(b_n^* x_n a_n^* a_n x_n b_n) \leq \lim_{n\to\omega} (\|a_n\| \|b_n\| \|x_n\|_2)^2 = 0$$

□

Definition 5.3.2. Consider a family of finite von Neumann algebras $(M_n)_{n\in\mathbb{N}}$ such that every M_n is finite with fixed trace τ_n, and fix an ultrafilter $\omega \in \beta(\mathbb{N})$ where $\beta(\mathbb{N})$ is the Stone-Cech compactification of $\mathbb{N}$. The algebra $\ell^\infty(\mathbb{N}, (M_n))/\mathcal{I}$, called the *ultraproduct of the family* (M_n). $\|(x_n)\| = \lim_{n\to\omega} \|x_n\|$ is a norm on the ultraproduct. When $M_n = M$ is a fixed finite von Neumann algebra, then this is called the ultrapower of M, and is denoted by M^ω.

There is a natural embedding of $M \subseteq M^\omega$ which is defined by mapping x to the equivalence class of the constant sequence $(x) \in M^\omega$.

Observe that since every element of $\ell^\infty(\mathbb{N}, (M_n))/\mathcal{I}$ with $\lim_{n\to\omega} \|x_n\| = 0$ is contained in the trace ideal $\mathcal{I}$, and thus the $\|(x_n)\| = \lim_{n\to\omega} \|x_n\|$.

Theorem 5.3.3. *The ultrapoduct of a family of finite von Neumann algebras is again a finite von Neumann algebra with trace $\tau_\omega := \lim_{n\to\omega} \tau_n$. Additionally, the ultraproduct is a factor whenever each of the M_n's are factors.*

Notice that the definition above is in some sense uninteresting when ω is a principle ultrafilter. Hence, we often make the standing assumption that an utlrafilter is non-principle.

Definition 5.3.4. Let M be a tracial von Neumann algebra. M has Property Gamma if and only if $M' \cap M^\omega \neq \mathbb{C}$ where ω is a non-principle ultrafilter on $\mathbb{N}$. M has the McDuff property if and only if $M' \cap M^\omega$ is non-abelian.

The advantage of working with an ultrapower von Neumann algebar is that it converts asymptotic behavior within a von Neumann algebra into something exact. To say the same thing more concretely, the key property of ulttrapowers is countable saturation, which essentially enables us to pass from approximately satisfying a certain property to exactly satisfying that property. On the flip side, if an ultrapower of a von Neumann M^ω algebra satisfies a certain property, then there should be some kind of sequential version of that same statement for M.

This is not the definition of Property Γ or the McDuff property that one usually encounters. However, the ultrapower version of these concepts simplifies things quite a bit. For example, here are the version of Property Γ and the McDuff property that is frequently found in the literature.

Definition 5.3.5. Let M be a tracial von Neumann algebra. M is said to have **property** Γ if there exist a sequence of unitaries $(u_n)_{n\in\mathbb{N}}$ with $\tau(u_n) = 0$ and

$$\lim_{n\to\infty} \|ux_n - xu_n\|_2 = 0$$

for every $x \in M$. This sequence $(u_n)_{n\in\mathbb{N}}$ is said to be an asymptotically central sequence of M. M is McDuff (or has the McDuff property) if $M \cong M \otimes \mathcal{R}$.

$\mathcal{R}$, the hyperfinite II_1 factor, has both Property Γ and the McDuff property, an hence any McDuff von Neumann algebra has property Γ, though my proof does depend on the model I create for $\mathcal{R}$ $\mathcal{R}$ is isomorphic to infinite tensor product of $M_2(\mathbb{C})$. Other examples McDuff von Neumann algebras include infinite tensor products of II_1 factors. Murray and von Neumann were able to show that $L(\mathbb{F}_2)$ does not have Property Γ and hence $L(\mathbb{F}_2) \ncong \mathcal{R}$. The issue here is that we have not talked about tensor products of von Neumann algebras,

We now have the terminology to state the infamous Connes Embedding Problem (sometimes called the Connes Embedding Conjecture). Does every II_1 factor M admit an embedding into $\mathcal{R}^\omega$ wehere R^ω is some ultrapower of $\mathcal{R}$? There are a myriad of equivalences that one can formulate here. In the language of free probability theory, the existence of an embedding of M into $\mathcal{R}^\omega$ is equivalent to M admitting *micorstates.* In C^* algebras, this question about embeddings of every possible II_1 factor is logically equivalent verifying that the tensor square of $C^*(\mathbb{F}_n)$ admits exactly one C^* norm. The language of operator spaces and quantum information theory allow for equivalent rephrasings of the Connes Embbedding Problem that, while notable, will not be discussed here. Talk to Roy...

5.4 Center-Valued traces

Fixing for a moment a finite dimensional factor, $M_n(\mathbb{C})$, there is a distinguished state $\tau_n : M_n(\mathbb{C}) \to \mathbb{C}$ which we call the trace and it is characterized by the so-called *tracial property* which means that $\tau(xy) = \tau(yx)$, c.f. Exercise 1.3.2. Now, suppose that $M = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C})$ is a direct sum of 2 finite dimensional factors. M admits multiple traces, for example $\tau = \frac{1}{2}(\tau_{n_1} \oplus \tau_{n_2})$ and $\tau' = \frac{1}{4}\tau_{n_1} \oplus \frac{3}{4}\tau_{n_2}$. We will soon see that the existence and uniqueness of a trace finite factors. Even when M is not a factorm, there is however a surrogate for the trace, a unique map $\phi : M \to \mathcal{Z}(M)$ that reduces to the trace when M is a factor. Moreover, the existence of such a map completely characterizes finite von Neumann algebras.

Definition 5.4.1. Let M be a von Neumann algebra and $\mathcal{Z}(M)$ its center. A map $\phi : M \to \mathcal{Z}(M)$ is a **center-valued state** if

(i) ϕ is linear and bounded,

(ii) and $\phi(zm) = z\phi(m)$ for any $z \in \mathcal{Z}(M)$,

If in addition we have that

(iii) $\phi(xy) = \phi(yx)$ for every $x, y \in M$,

then ϕ is called a **center-valued trace**.

Lemma 5.4.2. *Let M be a von Neumann algebra and $\phi : M \to \mathbb{C}$ be any linear functional. The following are equivalent:*

(i) $\phi(xy) = \phi(yx)$ *for all* $x, y \in M$.

(ii) $\phi(x) = \phi(u^*xu)$ *for all* $x \in M$ *and all unitaries* $u \in M$.

A linear functional as in Lemma 5.4.2 is called a *tracial* linear functional. A warning: some authors use the word *central* to describe linear functions which satisfy this property.

Proof. This is left as Exercise 5.4.2. ☐

We now discuss the structure theory of states, in particular the polar decomposition. If $\phi : M \to \mathbb{C}$ is a normal positive linear functional, then $\{x \in M : \phi(x^*x) = 0\}$ is a left ideal which is closed in the WOT, thus by Exercise 4.2.8 there exists a projection $p \in \mathcal{P}(M)$ such that $\phi(x^*x) = 0$ if and only if $x \in Mp$. We denote by $s(\phi) = 1 - p$ the support projection of ϕ. Note that if $q = s(\phi)$ then

$$\phi(xq) = \phi(qx) = \phi(x)$$

for all $x \in M$, and moreover, ϕ will be faithful when restricted to qMq.

Theorem 5.4.3 (Polar Decomposition for States)**.** *Suppose M is a von Neumann algebra and $\phi \in M_*$, then there exists a unique partial isometry $v \in M$ and positive linear functional $\psi \in M_*$ such that $\phi(x) = \psi(xv)$ for every $x \in M$ and $v^*v = s(\psi)$*

Proof. Assume for now that $\|\phi\| = 1$. There exists some $a \in (M)_1$ so that $\phi(a) = \|\phi\|$. Let $a^* = v|a^*|$ denote the polar decomposition of a^*. Letting $\psi(x) = \phi(xv)$, we have that $\psi(|a^*|) = \phi(a) = \|\phi\| = 1$. Since $0 \leq |a^*| \leq 1$, it follows that for every $t \in \mathbb{R}$

$$\||a^*| + e^{it}(1 - |a^*|)\| \leq 1.$$

Fix t so that $e^{it}\psi((1 - |a^*|) \geq 0$. Then we have

$$\psi(|a^*|) \leq \psi(|a^*|) + e^{it}\psi((1 - |a^*|) \leq \|\phi\| = \phi(|a^*|),$$

and thus $\psi(1) = \psi(|a^*|) = \|\phi\|$ implying that ψ is a positive linear functional.

Let $p = v^*v$. Since we may replace a with $avs(\phi)s$, we may assume that $p \leq s(\phi)$. For every $x \in M$ such that $\|x\| \leq 1$, we have that

$$\psi(|a^*| + (1 - p)x^*x(1 - p)) \leq \|\psi\|$$

which shows that $\psi((1 - p)x^*x(1 - p)) = 0$ and thus $p \geq s(\phi)$.

We leave out the proof of the uniqueness for now.

To see that$\phi(x) = \psi(xv)$ it suffices to show that $\phi(x(1 - p)) = 0$ for all $x \in M$. Suppose that $\|x\| = 1$ and $\phi(x(1 - p)) = \beta \geq 0$. Then for $n \in \mathbb{N}$ we have

$$\begin{aligned} n + \beta =& \phi(na + x(1 - p)) \\ \leq& \|na + x(1 - p)\| \\ =& \|(na + x(1 - p))(na + x(1 - p)^*\|^{1/2} \\ \leq& \|n^2|a^*|^2 + x(1 - p)x^*\|^{1/2} \\ \leq& \sqrt{n^2 + 1} \end{aligned}$$

implying that $\beta = 0$. ☐

Our goal with the next few lemmas is to characterize finite von Neumann algebras in terms of the existence of a center-valued state. The presentation contained here is an existence result that relies on the Ryll-Nardzewski fixed point theorem. We exclude the proof for now; instead, accept it as fact and acknowledge that it bestows upon us the existence of a fixed point in an appropriate setting.

Theorem 5.4.4 (Ryll-Nardzewski)**.** *Let X be a Hausdorff locally convex vector space, $K \subseteq X$ a non-empty, weakly compact, convex subset and E a non-contracting semigroup of weakly continuous affine mappings of K into K . Then there exists an $x_0 \in K$, such that $T(x_0) = x_0$ for every $T \in E$.*

Lemma 5.4.5. *Let M be a von Neumann algebra, $\mathcal{Z}(M)$ its center, and $\phi \in M_*$ a normal tracial linear functional. Then $\|\phi\| = \|\phi|_{\mathcal{Z}(M)}\|$. In particular, ϕ is positive if and only if $\phi|_{\mathcal{Z}(M)}$ is positive.*

Proof. Let $\phi = R_v|\phi|$ be the polar decomposition of ϕ. The for any unitary $u \in M$, we have that

$$\phi = R_{u^*vu}T_u|\phi|.$$

From the uniqueness of the polar decomposition for linear functionals and the centrality of ϕ, it follows that that $u^*vu = v$ and $T_u|\phi| = |\phi|$ for every unitary $u \in M$. Thus, $v \in \mathcal{Z}(M)$ and $|\phi|$ is also tracial. Thus, we have that

$$\|\phi\| = \| \, |\phi| \, \| = |\phi|(1) = \phi(v^*) \leq \|\phi|_{\mathcal{Z}(M)}\| \|v^*\| \leq \|\phi\|.$$

□

Lemma 5.4.6. *Let M be a finite von Neumann algebras with $\mathcal{Z}(M)$ its center. Then any normal linear functional $\omega : \mathcal{Z}(M) \to \mathbb{C}$ extends uniquely to a bounded normal tracial linear function ϕ_ω on M. Moreover, $\|\phi_\omega\| = \|\omega\|$, ϕ_ω is positive whenever ω is positive, and the map $\psi : \mathcal{Z}(M)_* \to M_*$ defined by $\omega \mapsto \phi_\omega$ is linear.*

Proof. The uniqueness will be left as an exercise (see 5.4.4). If we can indeed show that such an extension exists, then the norm preserving property, and positivity follow from the previous lemma. To show the existence, let $\phi \in M_*$ be any normal linear functional extending ω to M. For notational convenience, whenever u is a unitary in M we let $T_u : M_* \to M_*$ denote the transformation mapping $\psi \mapsto \psi \circ \mathrm{Ad}(u)$ where $\mathrm{Ad}(u)(x) = u^*xu$ for every $x \in M$. In the statement of the Ryll-Nardzewski theorem, let $X = M_*$, K be the norm closed convex hull of $= \{T_u\phi : u \in \mathcal{U}(M)\} \subseteq M_*$, and $E = \{T_u|_K\}$. We claim without proof that K is a weakly compact, convex, non-empty subset of $X = M_*$. Further, observe that $T_u|_K : K \to K$ and that T_u is an isometry, making E a collection of non-contracting semi-group of weakly continuous affine mappings of K to itself.

Then Ryll-Nardzewski Theorem provides the existence of a fixed point $\phi_\omega \in K$, i.e. $T_u\phi_\omega = \phi_\omega$ for every $u \in M$ implying that ϕ_ω is a normal tracial linear functional on M.

Finally, we show that $\phi_\omega|_{\mathcal{Z}(M)} = \omega$. Notice that by construction, $\phi|_{\mathcal{Z}(M)} = \omega$, and hence, $T_u\phi|_{\mathcal{Z}(M)} = \omega$ for every $u \in \mathcal{U}(M)$. Thus, any convex combination and therefore any element of K will also equal ω when restricted to the center of M.

Now to show linearity, assume that $\omega_1, \omega_2 \in \mathcal{Z}(M)_*$ and $c \in \mathbb{C}$. Then, $\psi_{\omega_1 + c\omega_2}$ and $\psi_{\omega_1} + c\psi_{\omega_2}$ are extensions of $\omega_1 + c\omega_2$, and by uniqueness they are equal. □

Theorem 5.4.7. *If M if a finite von Neumann algebra, then admits a center-valued trace, namely the adjoint of the map $\psi : \mathcal{Z}(M)_* \to M_*$ defines a center-valued state on a finite von Neumann algebra.*

Proof. Now consider a finite von Neumann algebra M. By Lemma 5.4.6, there is a linear and isometric map $\psi : \mathcal{Z}(M)_* \to M_*$ taking normal linear functionals on $\mathcal{Z}(M)$ to tracial linear functional on M. Since we may identify $\mathcal{Z}(M)_*)^*$ with $(\mathcal{Z}(M)$ and $(M_*)^*$ with M, we let $\phi : M \to \mathcal{Z}(M)$ be the map determined by the relation

$$\psi_\omega(x) = \omega(\phi(x))$$

for every $\omega \in \mathcal{Z}(M)_*$ and $x \in M$. In other words, $\phi : M \to \mathcal{Z}(M)$ is the (Banach space) adjoint of the map ψ. □

Theorem 5.4.8. *Let M be a von Neumann algebra, $\mathcal{Z}(M)$. If $\phi : M \to \mathcal{Z}(M)$ is a center-valued trace, then ϕ has the following additional properties:*

(i) *ϕ is unique.*

(ii) *$\|\phi\| = 1$,*

(iii) *ϕ is σ-WOT continuous (normal),*

(iv) *$\phi(zx) = z\phi(x)$ for every $x \in M$ and $z \in \mathcal{Z}(M)$ (bimodular),*

(v) *$\phi(x^*x) \geq 0$ (positive),*

(vi) *$\phi(x^*x) = 0 \implies x = 0$ (faithful),*

Proof. If we suppose that there was another center-value trace $\tilde{\phi}$ on M distinct from ϕ, there would exist $x \in M$ so that $\phi(x) \neq \tilde{\phi}(x)$. But this would imply that we can find a normal linear functional $\omega \in \mathcal{Z}(M)$ so that $\omega(\phi(x)) \neq \omega(\tilde{\phi}(x))$. However, since $\omega \circ \phi$ and $\omega \circ \tilde{\phi}$ are distinct bounded normal extensions of ω which are tracial, this contradicts Lemma 5.4.6. Hence, the center-valued trace from Theorem 5.4.7 is the unique such map on M.

The normality and the the fact that the ϕ has norm 1 arises from the fact that ϕ is the (Banach space) adjoint of the map from Lemma 5.4.6.

Now to prove the bimodularity, we start by fixing a unitary $u \in \mathcal{Z}(M)$ and defining $\psi : M \to \mathcal{Z}(M)$ by $\psi(x) = u^*\phi(ux)$. Notice that ψ is a center-valued trace and thus must equal ϕ, i.e. $\phi(x) = u^*\phi(ux)$ for every $x \in M$. Replacing x with u^*x shows that $\phi(u^*x) = u^*\phi(x)$ for every $x \in M$ and for every unitary $u \in \mathcal{Z}(M)$. Since every element $z \in \mathcal{Z}(M)$ is a linear combination of 4 unitaries and ϕ is a linear map, we now have that $\phi(zx) = z\phi(x)$ for every $z \in \mathbb{Z}(M)$ and every $x \in M$.

In order to verify postivity, we will show that $\omega(\phi(x^*x)) \geq 0$ for every positive linear functional $\omega \in \mathcal{Z}(M)_*$. Notice that ϕ must satisfy

$$\omega(\phi(x^*x)) = \phi_\omega(x^*x),$$

where ϕ_ω is the normal tracial linear functional extending ω. Since Lemma 5.4.6 shows that $\phi_\omega(x^*x) \geq 0$, which is what we wanted to show.

We will verify the definiteness of ϕ by proving the contrapositive. In particular, if $y \in M$ and $y > 0$ we will show that there exists $\omega \in \mathcal{Z}(M)_*$ so that $\omega(\phi(y)) \neq 0$. To this end, fix $y \in M$ be a positive element and $z = \mathbf{z}(y)$ its central support projection, choose ω a positive normal linear functional such that $p = s(\omega) \leq z$. If ψ_ω is the tracial extension of ω to M, it is invariant under conjugation by any unitary in M. Hence, its support projection is also invariant under conjugation by all unitaries of M implying that $s(\phi_\omega)$ is in the center of M and in particular $s(\phi_\omega) = s(\omega) = p$. If $\phi_\omega(y) = 0$, then $xp = 0$; however this is not possible since $0 \neq p \leq z$. Thus, $\phi_\omega(y) = \omega(\phi(x)) \neq 0$, finishing the final claim. ☐

In light of the first item in the previous lemma, we are justified calling $\phi : M \to \mathcal{Z}(M)$ *the canonical center valued trace* on a finite von Neumann algebra M, whenever such a map exists. We should observe that the canonical center valued trace ϕ is an example of a *conditional expectation.* That is, ϕ is a positive, bimodular, norm 1, linear functional from M to the subalgebra $\mathcal{Z}(M)$. We will explore general conditional expectations in a later section.

Corollary 5.4.9. *M is a finite von Neumann algebra if and only if M has a unique center-valued trace.*

Proof. Assume that ϕ is a center-valued trace on M. If p is a projection on in M such that $p \leq 1$ and $p \sim 1$. In this case, $0 \leq 1 - p$ and $1 = \phi(1) = \phi(p)$. It follows that $0 \leq \phi(1-p) = \phi(1) - \phi(p) = 0$. Thus, $1 = p$ and hence M is finite. ☐

One of the main uses of a center-valued trace is that it detects equivalence of projections.

Theorem 5.4.10. *Let M be a von Neumann algebra with center-valued trace ϕ. If p and q are projections in M , $p \preceq q$ if and only if $\phi(p) \leq \phi(q)$. Specifically, $p \sim q$ if and only if $\phi(p) = \phi(q)$.*

Proof. If $p \preceq q$ and v is a partial isometry such that $v^*v = p$ and $vv^* \leq q$, then $\phi(p) = \phi(v^*v) = \phi(vv^*) \leq \phi(q)$, where the last line follows from the fact that $q - vv^* \geq 0$ and the linearity of ϕ.

Conversely, assume that $\phi(p) \leq \phi(q)$. Linearity of ϕ shows that $\phi(q-p) \leq 0$. By the Comparison Theorem, there exists a central projection z so that $zp \preceq zq$ and $(1-z)q \preceq (1-z)p$. Using the tracial property of ϕ in conjunction with the , we have that $\phi((1-z)(q-p)) \geq 0$. The bimodularity of ϕ now implies $0 \leq (1-z)\phi(q-p)$ From here, we use the initial assumption to conclude that $(1-z)\phi(q-p) \leq 0$, which when combined with the positive definiteness implies that $(1-z)q \sim (1-z)p$. Thus, $p \preceq q$.

The fact that $p \sim q$ is logically equivalent to $\phi(p) = \phi(q)$ follows from an application of Proposition 4.1.5 (Cantor-Schoder-Bernstein for projections). ☐

For this portion, we fix a finite von Neumann algebra M with center valued trace $\phi : M \to \mathcal{Z}(M)$.

Definition 5.4.11. M is homogeneous of type I_n if there exists a family of n equivalent abelian mutually orthogonal projections,$e_1, \ldots, e_n$, such that $\sum_{i=1}^n p_i = 1$.

An elementary example of a homogenous von Neumann algebra of type I_n is the $n \times n$ matrices, $M_n(\mathbb{C})$.

Theorem 5.4.12. *Let M be a finite homogeneous type I_n von Neumann algebra and $\phi : M \to \mathcal{Z}(M)$ its center valued trace. Then the range of ϕ restricted to the projections of M coincides with*

$$\sum_{k=1}^{n} \frac{k}{n} z_k$$

where $z_1, \ldots, z_n$ are mutually orthogonal central projections

Proof. First, we show that there exists a projection p_0 so that $z(p_0) = 1$. Letting $\{p_1, \ldots, p_k\}$ be a maximal family of mutually centrally orthogonal abelian projections ($\mathbf{z}(p_i)\mathbf{z}(p_j) = 0$ whenever $i \neq j$). Then $p_0 = \bigvee p_i$ is also abelian, and by maximality we must have that $\mathbf{z}(p_0)) = 1$.

Since M is homogeneous, of type I_n, there exists a family of n abelian, mutually orthogonal projections, equivalent to p_0 whose sum equals 1. Hence

$$1 = \phi(1) = \phi\left(\sum_{i=1}^{n} q_i\right) = \sum_{i=1}^{n} \phi(q_i) = n\phi(p_0).$$

Now, for any central projection z we see that $\phi(p_0 z) = \frac{1}{n} z$ and thus the range of $\phi|_{P(M)}$ contains elements of the form indicated above.

□

A center valued trace on a II_1 von Neumann algebra satisfies an analog of the intermediate value property.

Theorem 5.4.13. *Let M be a type II_1 von Neumann algebra with center valued trace ϕ. If p, q are projections in M and $z \in \mathcal{Z}(M)$ is a central projection with $\phi(q) \leq z \leq \phi(q)$, then there exists some projection r with $p \leq r \leq f$ and $\phi(r) = q$.*

Proof. First an observation about II_1 von Neumann algebras: if s is any projection and $\varepsilon > 0$ then there exists a non-zero projection $s_\varepsilon \leq s$ such that $\phi(s_\varepsilon) \leq \varepsilon \mathbf{z}(s_\varepsilon)$. To this end, choose n so that $\frac{1}{2^n} \leq \varepsilon$. Since M is type II_1, Lemma 5.1.2 shows that there is a family of 2^n equivalent, mutually orthogonal, non-zero subprojections of s whose sum is s. Letting s_ε be any one of these, this now gives that $\phi(s_\varepsilon) = \frac{1}{2^n}\phi(s) \leq \varepsilon\phi(s) \leq \varepsilon \mathbf{z}(e_\varepsilon)$.

Now, let P a maximal family of totally ordered projections in M such that if $s \in P$ then $p \leq s \leq q$ and $\phi(s) \leq z$. Such a collection exists and is non-empty since $p \in P$. Letting $r = \bigvee_{s \in P} s$, we have that $p \leq r \leq q$ and $\phi(p) \leq z$.

Let's suppose that $z - \phi(r) > 0$. Then in this case, there is some $\varepsilon > 0$ and a no-zero central projection w so that

$$z - \phi(r)w \geq \varepsilon w.$$

Notice that this would imply that $(q - r)w \neq 0$; otherwise we would have that $\phi(r)w = \phi(p) \geq zp$, a contradiction . So, we can find some non-zero projection s_ε with $s_\varepsilon \leq (p - r)w$ and $\phi(s_\varepsilon) \leq \varepsilon w$. But this would imply that $r + S_\varepsilon \in P$, contradicting the maximality of P. So we must have that $z = \phi(r)$.

□

Exercises

5.4.1. Show that if M is finite and separable, then M is tracial. When M is not a factor show that a trace is not unique.

5.4.2. Prove Lemma 5.4.2. [**Hint:** use Exercise 3.1.7.]

5.4.3. Let $n_1, \ldots, n_k$ be a collection of natural numbers and consider $M = \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$.

(a) Compute the center of M. Show that M has a continuum of faithful states $\phi : M \to \mathbb{C}$ with the tracial property.

(b) $M = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ has two non-zero orthogonal central projections which sum to the identity of M, which we call z_1, z_2. For each $a \in \{0, \frac{1}{2,} 1\}$ and $b \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, find a projection $p \in \mathcal{P}(M)$ such that $\phi(p) = az_1 + bz_2$ where ϕ is the center-valued trace.

5.4.4. Show that under the conditions in 5.4.6, the extension of ω to a tracial state defined all of M is unique.