Problem Session July 10th, 2020

C*.1 Show that the matrix amplification of any $*$-homomorphism between C*-algebras is again a $*$-homomorphism. Conclude that any $*$-homomorphism is completely positive.

C*.2 Let A and B be C*-algebras and $C \subset B$ a C*-subalgebra. Show that if $\theta : A \to C$ is a nuclear map, then so is θ when viewed as a map from A to B. Suppose we have a map $\rho : A \to C$ that is nuclear as a map from A to B. What could prevent ρ from being a nuclear map as a map from A to C?

C*.3 Partitions of unity are nicer when you have a concrete example. For each $n \geq 2$, cover $[0, 1]$ by $2^n - 1$ open intervals of equal length. (What are they? Also, we could start with $n = 1$, but it's too simple to pick up on a pattern.) Call this cover $\mathcal{U}_n$. Define (sketch) a partition of unity for $\mathcal{U}_n$. (Hint: think zig-zags.)

Now, construct a sequence of completely positive maps $C([0, 1]) \xrightarrow{\psi_n} \mathbb{C}^{k_n} \xrightarrow{\phi_n} C([0, 1])$, (what is k_n?) that give a completely positive approximation of $C([0, 1])$.

W*.1 Let Γ be a countable discrete group. Show that all projections in $L(\Gamma)$ are finite.

[**Hint:** use the trace.]

W*.2 Let $\pi\colon M \to N$ be a $*$-isomorphism between von Neumann algebras and let $p \in \mathcal{P}(M)$.

(a) Show p is finite in M if and only if $\pi(p)$ is finite in N.

(b) Assuming π is normal, show p is semi-finite in M if and only if $\pi(p)$ is finite in N.

(c) Show p is purely infinite in M if and only if $\pi(p)$ is purely infinite in N.

(d) Show p is properly infinite in M if and only if $\pi(p)$ is properly infinite in N.

W*.3 In this exercise, you will show that $M_n(\mathbb{C})$ can be realized via a crossed-product construction. Consider $\Gamma := \mathbb{Z}_n$, the countable cyclic group of order n, and also set $X := \mathbb{Z}_n$ which we view as simply a space and equip with the counting (probability) measure.

(a) Show that $\alpha_g(f) := f(\cdot - g)$ for $g \in \Gamma$ defines an action $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$.

(b) Show that $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ is free, ergodic, and probability measure preserving.

(c) Show that $1_{\{1\}}, \ldots, 1_{\{n\}} \in L^\infty(X, \mu)$ are pairwise orthogonal and equivalent minimal projections.

(d) Show that $L^\infty(X, \mu) \rtimes_\alpha \Gamma \cong M_n(\mathbb{C})$. What is the preimage of $E_{i,j}$ under this isomorphism?

(e) Explain why there does not exist a discrete group Γ such that $L(\Gamma) \cong M_n(\mathbb{C})$.