9. Completely Positive Maps

This section gives a very quick introduction to completely positive maps for C*-algebraists. If this is your first time seeing such maps defined, we recommend ignoring the non-unital generalities for this go around. Once you have a grasp of the unital setting, you'll understand what's going on, and you will know where to look if you ever need the non-unital generalizations in the future. With the exception of a few examples, we will stick with the unital assumption in lecture.

The lecture will focus mostly on understanding key examples of completely positive maps (Examples 9.5, 9.8, 9.9 and Exercise 9.10), the characterization of completely positive maps afforded by Stinespring's Dilation theorem (Theorem 9.22), and an understanding Arveson's Extension theorem (9.28) for completely positive maps into $B(\mathcal{H})$.

With time, we will give an overview of the proof of Stinespring's Dilation Theorem, which is a direct generalization of the GNS construction. In which case, it will be beneficial to have the GNS construction proof handy. This proof goes through some algebraic tensor products for vector spaces. If it feels too confusing, try revisiting it after we've had a treatment of tensor products next week.

Section 9.1 establishes some preliminary results and delves into dilation techniques. We encourage you to read through the various dilation tricks and try the corresponding exercises in Section 9.1. These are valuable tools, which we will not address in lecture.

This section concerns maps that preserve positivity even after matrix amplification. We will have to forego several important facts and results on (completely) positive maps. For a full treatment, we highly recommend Vern Paulsen's book: [8, Chapters 2,3,6,7].

We begin with what we mean by matrix amplification. Ignoring the norm for a moment, given a $*$-algebra A and some $1 \leq n < \infty$, we define $\mathrm{M}_n(A)$ to be the $n \times n$ matrices with entries in A (just as we would in more general ring theory).

$$\mathrm{M}_n(A) := \{[a_{ij}]_{1\leq i,j\leq n} \ : \ a_{ij} \in A, 1 \leq i,j \leq n\} \tag{9.1}$$

We will usually suppress the usual subscripts on the matrices, i.e. we write $[a_{ij}]$ for $[a_{ij}]_{1\leq i,j\leq n}$ (sometimes also $[a_{ij}]_{ij}$).

This also comes with a natural involution where $[a_{i,j}]^* = [a_{j,i}^*]$ for all $[a_{i,j}] \in \mathrm{M}_n(A)$.

Definition 9.1. For a linear map $\phi : A \to B$ between $*$-algebras we define, for each $n \geq 1$, the linear map

$$\phi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B), \quad \phi^{(n)}([a_{ij}]) = [\phi(a_{ij})].$$

The map $\phi^{(n)}$ is often called a *matrix amplification* of ϕ.

When A is a C*-algebra, there is a natural C*-norm on $\mathrm{M}_n(A)$, which is inherited from the norm on A in the following sense:

Recall from Exercise 7.50 from Day 1 Lectures that $\mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$ for any Hilbert space $\mathcal{H}$. Now (using Theorem 8.1), we faithfully represent A on some Hilbert space $\mathcal{H}$ with an injective $*$-homomorphism $\pi : A \to B(\mathcal{H})$. This induces a $*$-homomorphism $\pi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$, which is also injective (check). Then we can define a norm on $\mathrm{M}_n(A)$ by $\|[a_{ij}]\| := \|\pi^{(n)}([a_{ij}])\|$ (injectivity implies this is a norm and not just a semi-norm), which will satisfy the C*-identity (because $(\pi^{(n)})^{-1} : \pi^{(n)}(\mathrm{M}_n(A)) \to \mathrm{M}_n(A)$ is a $*$-homomorphism).

The following inequality is a useful exercise, but we already have plenty of exercises. The argument is outlined in [12, Exercise 1.13].

Proposition 9.2. *For any* C*-*algebra* A*,* $n \geq 1$*, and* $[a_{ij}] \in \mathrm{M}_n(A)$*, we have*

$$\max_{i,j}\{\|a_{ij}\|\} \leq \|[a_{ij}]\| \leq \sum_{ij} \|a_{ij}\|.$$

9.1. Preliminary results on cp maps.

Unlike with the Gelfand-Naimark Theorem for commutative C*-algebras, we will not start from scratch here. However, results in this section are developed nicely in [8, Chapter 2]. The proofs therein are well-written and easy to follow, but we are after bigger fish and therefore will just take these as means to an end.

Definition 9.3. We say a linear map $\phi : A \to B$ between C*-algebras is *positive* if it maps positive elements to positive elements. We say it is *n-positive* if $\phi^{(n)}$ is positive, and we say that it is *completely positive* (c.p. or cp) if it is n-positive for all $n \geq 1$. A completely positive map that is unital is abbreviated *ucp*.

Remark 9.4. For notation and terminology: often the word "linear" is dropped when discussing cp maps, and $\phi^{(n)}$ is sometimes denoted by ϕ_n.

One important class of examples that we have already seen is positive linear functionals (such as the states used in the GNS representation theorem).

Example 9.5. For a unital C*-algebra A, a positive linear functional $\phi \in A^*$ is completely positive. Indeed, (for the unital case) note that $\phi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(\mathbb{C})$, so we check positivity by checking for positive-definiteness. To that end, let $\xi \in \mathbb{C}^n$ and $[a_{ij}] \in \mathrm{M}_n(A)$ positive. Then by Exercise 3.11,

$$\begin{bmatrix} \bar{\xi}_1 & \dots & \bar{\xi}_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \xi_n & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} \sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix} \tag{9.2}$$

is positive in $\mathrm{M}_n(A)$. Then $\sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij}$ is positive in A,[7] which means its image under ϕ is positive by assumption. Then we compute

$$\begin{aligned} \langle \phi^{(n)}([a_{ij}])\xi, \xi\rangle = \langle [\phi(a_{ij})]\xi, \xi\rangle &= \left\langle \begin{bmatrix} \sum_{j=1}^n \phi(a_{1j})\xi_j \\ \vdots \\ \sum_{j=1}^n \phi(a_{n,j})\xi_j \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle \\ &= \sum_{i,j=1}^n \overline{\xi_i}\xi_j \phi(a_{ij}) = \phi\Big(\sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij}\Big) \geq 0 \end{aligned}$$

Exercise 9.6. Show that the composition of completely positive maps is completely positive.

Exercise 9.7. Let $\phi : A \to B$ be a positive map between C*-algebras. Show that ϕ is $*$-preserving, i.e. $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

Exercise 9.8. Show that the matrix amplification of any $*$-homomorphism between C*-algebras is again a $*$-homomorphism. Conclude that any $*$-homomorphism is completely positive.

Example 9.9. To get more examples of completely positive maps we build them out of known examples.

The idea is to conjugate another cp map: Let $\psi : A \to B$ be a cp map between C*-algebras and $b \in B$. Then the map $\phi := b^*\psi(\cdot)b : A \to B$ is linear and positive by Exercise 3.11. It is moreover completely positive. Indeed, for each $n \geq 1$ and positive element $[a_{ij}] \in \mathrm{M}_n(A)$,

$$\phi^{(n)}([a_{ij}]) = \begin{bmatrix} b^*\phi(a_{11})b & \dots & b^*\phi(a_{1n})b \\ \vdots & \ddots & \vdots \\ b^*\phi(a_{n1})b & \dots & b^*\phi(a_{nn})b \end{bmatrix} = \begin{bmatrix} b^* & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b^* \end{bmatrix} \begin{bmatrix} \phi(a_{11}) & \dots & \phi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(a_{n1}) & \dots & \phi(a_{nn}) \end{bmatrix} \begin{bmatrix} b & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b \end{bmatrix}.$$

Observe (**exercise**) that when $\|b\| \leq 1$, ϕ is then cpc.

Exercise 9.10. Now consider a more concrete setting of $B(\ell^2)$, and consider the rank n projection P defined on the basis vectors by $Pe_i = 1$ if $i \leq n$ and $Pe_i = 0$ if $i > n$. If we write an operator $A \in B(\ell^2)$ as a matrix array, what does its image under the completely positive map $A \mapsto PAP$ look like? (This is where the word "compression" comes from.)

Now, we identify $PB(\ell^2)P \simeq B(P\ell^2) \simeq \mathrm{M}_n(\mathbb{C})$ (like in Example 6.5). These are $*$-isomorphisms, which means their composition with the above compression by P gives a completely positive map $B(\ell^2) \to \mathrm{M}_n(\mathbb{C})$.

[7] Perhaps there is a quicker argument, but here is one through tensor products. We'll go ahead and record it so you can come back after we've covered them. Exercise 11.7 tells us that $\mathrm{M}_n \odot A = \mathrm{M}_n(A)$. So, the positive matrix in (9.2) is of the form $x = p \otimes b \in \mathrm{M}_n \otimes A$, where p is the projection onto the first coordinate. Then $x = |x| = \sqrt{p \otimes b^*b}$, which must also equal $|b| \otimes p$ by uniqueness of positive square roots. Then $p \otimes b - p \otimes |b| = 0$ implies $b = |b| \geq 0$.

Example 9.11. One important class of completely positive maps are conditional expectations, which feature more prominently in von Neumann algebras. Recall from the von Neumann lecture notes that a conditional expectation is a contractive linear projection $E : A \to B$ from a C*-algebra onto a C*-subalgebra $B \subset A$ such that $Eb = b$ for all $b \in B$. By a theorem of Tomiyama, any conditional expectation is automatically completely positive and contractive. In this exercise, we consider a class of these that we will use a few times in these notes.

Recall that a finite dimensional C*-algebra has the form $F = \oplus_{j=1}^m \mathrm{M}_{l_j}(\mathbb{C}) \subset \mathrm{M}_L(\mathbb{C})$ where $L = \sum l_j$. We define a conditional expectation $\mathrm{M}_L(\mathbb{C}) \to F$ as follows: for each j, let P_j denote the projection onto the jth component of F, and define $\rho_j : \mathrm{M}_L \to \mathrm{M}_{l_j}$ as the compression $E_j(\cdot) = P_j \cdot P_j$ (where we identify $\mathrm{M}_{l_j}(\mathbb{C})$ with its copy in $\mathrm{M}_L(\mathbb{C})$). Then $E : \mathrm{M}_L(\mathbb{C}) \to F$, given by $\sum_j E_j$, is a ucp map (**exercise** check). (Why do we automatically know F is unital?)

Theorem 9.12 (Russo-Dye)**.** *Let A and B be unital* C*-*algebras and $\phi : A \to B$ a positive map. Then $\|\phi\| = \|\phi(1)\|$.*

This is [8, Corollary 2.9], where it appears as a Corollary to von Neumann's inequality [8, Corollary 2.7], which we will not treat here.

In the subsection on nonunital C*-algebras in [8, Chapter 2], Paulsen gives this non-unital extension of the Russo-Dye theorem.

Proposition 9.13. *Any positive map between* C*-*algebras is bounded.*

Finally, we record the following examples for future use. The proof is short, but we leave it for [8, Theorem 3.9].

Proposition 9.14. *For any unital* C*-*algebra A and any compact Hausdorff space X, any unital positive map $\phi : A \to C(X)$ is ucp.*

Remark 9.15*.* The converse holds too. This is a theorem of Stinespring (not to be confused with his dilation theorem in the next section). ([8, Theorem 3.11])

Dilation Tricks:
Though our goals are Theorems 9.22 and 9.28, we would be doing a disservice to come this close to dilation tricks and not give you a feel for the techniques. Also, we'll want some of these facts later.

Lemma 9.16. *Let A be a unital* C*-*algebra and $a, b \in A$. Then $\|a\| \leq 1$ iff $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$ is positive in $M_2(A)$.*

Proof. We assume A is faithfully (and unitally) represented on a Hilbert space $B(\mathcal{H})$, whence we check for positive-definiteness. For $a \in A$, if $\|a\| \leq 1$, then for any $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}\left\langle \begin{pmatrix} 1_{\mathcal{H}} & a \\ a^* & 1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle &= \langle \xi, \xi \rangle + \langle a\eta, \xi \rangle + \langle \xi, a\eta \rangle + \langle \eta, \eta \rangle \\ &\geq \|\xi\|^2 - 2\|a\|\|\eta\|\|\xi\| + \|\eta\|^2 \geq 0.\end{aligned}$$

On the other hand, if $\|a\| > 1$, then there exist unit vectors $\xi, \eta \in A$ such that $\langle a\eta, \xi \rangle < -1$, which would make the inner product above negative.

□

Definition 9.17. We say a linear map $\phi : A \to B$ between C*-algebras is *completely bounded* if

$$\sup_n \|\phi^{(n)}\| < \infty.$$

Corollary 9.18. *Any completely positive map is completely bounded. Moreover, if A and B are unital* C*-*algebras and $\phi : A \to B$ is a completely positive map, then*

$$\|\phi(1)\| = \|\phi\| = \sup_n \|\phi^{(n)}\|.$$

We prove the case where ϕ is unital, i.e. $\phi(1) = 1$, which also means $\phi^{(n)}(1) = 1_{\mathrm{M}_n(A)}$ for all $n \geq 1$. The more general case needs one additional fact and is addressed in [8, Proposition 3.6], but the main idea is already in the unital case.

Proof. We know already that $\|\phi(1)\| \leq \|\phi\| \leq \sup_n \|\phi^{(n)}\|$. Moreover, we know $\|\phi(1)\| = \|1_{\mathrm{M}_n(B)}\| = 1$ for all $n \geq 1$. So, we want to prove that $\sup_n \|\phi^{(n)}\| \leq 1$. So, let $a = [a_{ij}] \in \mathrm{M}_n(A)$ with $\|a\| \leq 1$. Then by Lemma 9.16,

$$\begin{pmatrix} 1_{\mathrm{M}_n(A)} & a \\ a^* & 1_{\mathrm{M}_n(A)} \end{pmatrix} \in \mathrm{M}_{2n}(A)$$

is positive. Since ϕ is completely positive, $\phi^{(n)}$ is 2-positive, and so

$$\phi^{(2n)}\left(\begin{pmatrix} 1_{\mathrm{M}_n(A)} & a \\ a^* & 1_{\mathrm{M}_n(A)} \end{pmatrix}\right) = \begin{pmatrix} 1_{\mathrm{M}_n(B)} & \phi^{(n)}(a) \\ \phi^{(n)}(a)^* & 1_{\mathrm{M}_n(B)} \end{pmatrix}$$

is positive. By Lemma 9.16, this implies $\|\phi^{(n)}(a)\| \leq 1$, as desired. □

More abbreviations:
Corollary 9.18 says that any ucp map is completely positive and completely contractive, abbreviated by cpc (or some permutation of those letters).

Exercise 9.19. Let A be a unital C*-algebra and $a \in A$ such that $\|a\| \leq 1$. Show that the following is a unitary in $\mathrm{M}_2(A)$:

$$\begin{pmatrix} a & \sqrt{1 - aa^*} \\ \sqrt{1 - a^*a} & a^* \end{pmatrix}.$$

This is sometimes referred to Halmos' Dilation

Now that we've tried a few dilation tricks, we (you) are ready to show a powerful result in C*-algebras that relies heavily on the functional calculus and dilation tricks. Don't worry, it's just assembling pieces at this point.

Exercise 9.20. Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras and $b \in \pi(A)$. Show that there exists $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$. (Aloud we usually say something like, "contractions lift to contractions"– with the assumption that we can scale to get the full result) Here are some steps:

(1) Consider the element $x = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \in \mathrm{M}_2(B)$. Show that $\|x^*x\| = \|b^*b\|$.
(2) Apply Exercise 2.21 to x and $\pi^{(2)}$ to get some lift $y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \in \mathrm{M}_2(A)$ (i.e. $\pi^{(2)}(y) = x$) with y self-adjoint and $\|y\| = \|x\| = \|b\|$.
(3) Show that y_{12} is a lift of b.
(4) Now use Proposition 9.2 to finish the argument. (Don't forget to mention why $\|y_{12}\| \leq b$.)

We close with one important fact that holds for cpc maps that does not hold in general is that any cpc maps between C*-algebras extends to a ucp map between their unitizations. The proof is short but digs into some surprisingly technical aspects of double duals of C*-algebras, so we leave it to [3, Proposition 2.2.1].

Proposition 9.21. *Let A and B be C*-algebras with A non-unital and B unital, and let $\phi : A \to B$ be a cpc map. Then ϕ extends to a ucp map $\tilde{\phi} : \tilde{A} \to B$, which is given by*

$$\tilde{\phi}(a + \lambda 1_{\tilde{A}}) = \phi(a) + \lambda 1_B.$$

9.2. **Stinespring's Dilation Theorem.** We saw in the previous section that compressing a $*$-homomorphism gives a completely positive map. What Stinespring's Dilation Theorem tells us is that that's basically how *every* completely positive map arises! That's right, when we are working with completely positive maps, we are really just looking at "compressed" $*$-homomorphisms.[8] That's what makes Stinespring's theorem so powerful: cp (ucp) maps are more abundant than $*$-homomorphisms, but when you have a cp map, you can draw a lot of conclusions pertaining to its structure by appealing to its "Stinespring Dilation" $*$-homomorphism.

Enough prelude. Here's the theorem.

[8] "Compressed" is in quotations because in the non-unital setting it will be conjugation but not necessarily by a projection as in Definition 4.2.1 in the von Neumann notes.

Theorem 9.22 (Stinespring's Dilation Theorem)**.** *Let A be a unital* C**-*algebra and $\phi : A \to B(\mathcal{H})$ a cp map. Then there exists a Hilbert space $\mathcal{H}'$, a unital $*$-representation $\pi : A \to B(\mathcal{H}')$ and a linear map $V : \mathcal{H} \to \mathcal{H}'$ such that*

$$\phi(a) = V^*\pi(a)V$$

for every $a \in A$. In particular, $\|\phi\| = \|V\|^2 = \|V^*V\| = \|\phi(1)\| = \sup_n \|\phi^{(n)}\|$.

Moreover, if ϕ is unital, then V is an isometry and $V^* = P_{V\mathcal{H}}$ is the projection onto $V\mathcal{H} \subset \mathcal{H}'$. In this case we identify $\mathcal{H}$ with a subspace $V\mathcal{H} \subset \mathcal{H}'$ and have

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}.$$

Remark 9.23. We have a few remarks on this.

(1) When ϕ is unital, we think of $\pi(a)$ as

$$\pi(a) = \begin{bmatrix} \phi(a) & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where $T_{12} : \mathcal{H}^\perp \to \mathcal{H}$, $T_{21} : \mathcal{H} \to \mathcal{H}^\perp$ and $T_{22} : \mathcal{H}^\perp \to \mathcal{H}^\perp$ are some bounded linear maps.

Notice how the unital case generalizes Example 9.10 (with $\pi = id$).

(2) There is a non-unital version. Follow [3, Remark 1.5.4].

(3) One usually hears the term "minimal Stinespring dilation." Consider a Stinespring representation $(\pi, \mathcal{H}', V)$ for $\phi : A \to B(\mathcal{H})$. Let $\mathcal{H}_0 \subset \mathcal{H}$ be the closed linear span of $\pi(A)V\mathcal{H}$, which reducing for $\pi(A)$ (as in vN notes) and hence the co-restriction $\pi : A \to B(\mathcal{H}_0)$ is a representation. Whenever $\pi(A)V\mathcal{H}$ is dense in $\mathcal{H}'$, (i.e. its closure is $\mathcal{H}_0$), then the Stinespring dilation is unique up to unitary equivalence. (See [8, Proposition 4.2].)

The proof is exactly a generalization of the GNS construction of a representation corresponding to a state. The technique in general is sometimes called "separation and completion": first you define a semi-norm (or semi-inner-product in this case), then you mod out by the null set– hence making it a genuine norm (or inner product), then complete the quotient space with respect to your new norm. Since we have already seen the technical side of the GNS proof, let's see the overarching idea this time around in order to better understand how to potentially use this technique in other settings. (For a proof that checks all the details, see [8, Theorem 4.1].)

Proof of Stinespring's Dilation Theorem. Let $\phi : A \to B(\mathcal{H})$ be a cp map, and consider the algebraic tensor product

$$A \odot \mathcal{H} := \operatorname{span}\{a \odot \xi \ : \ a \in A, \xi \in \mathcal{H}\}.$$

We define a symmteric bilinear function $\langle \cdot, \cdot \rangle$ by

$$\langle a \odot \xi, b \odot \eta \rangle = \langle \phi(b^*a)\xi, \eta \rangle_{\mathcal{H}},$$

for $a, b \in A$ and $\xi, \eta \in \mathcal{H}$ (extending linearly to $A \odot \mathcal{H}$). One then checks that this is positive semidefinite (i.e. $\langle x, x \rangle \geq 0$), which means it's an inner product modulo the fact that we could potentially have $\langle x, x \rangle = 0$ for non-zero $x \in A \odot \mathcal{H}$. No worries. It turns out the space consisting of such elements $\mathcal{N} = \{x \in A \odot \mathcal{H} \ : \ \langle x, x \rangle = 0\}$ is a subspace of $A \odot \mathcal{H}$, which means we can take the quotient $(A \odot \mathcal{H})/\mathcal{N}$. The symmetric bilinear function $\langle \cdot, \cdot \rangle$ from before now induces a genuine inner product on $(A \odot \mathcal{H})/\mathcal{N}$ given by

$$\langle x + \mathcal{N}, y + \mathcal{N} \rangle := \langle x, y \rangle.$$

So, when we complete $(A \odot \mathcal{H})/\mathcal{N}$ with respect to this inner product, we get a Hilbert space. Let's suggestively call it $\mathcal{H}'$.

For $a \in A$, we define the linear map $\pi(a) : A \odot \mathcal{H} \to A \odot \mathcal{H}$ by left multiplication, i.e. on for $a \in A$ and $b \odot \xi \in A \odot \mathcal{H}$, we have

$$\pi(a)(b \odot \xi) = ab \odot \xi,$$

and we extend linearly. A computation shows that $\pi(a) : \mathcal{N} \to \mathcal{N}$, and so it induces a linear map on the quotient $(A \odot \mathcal{H})/\mathcal{N}$, which we still denote by $\pi(a)$. Moreover, it turns out that $\|\pi(a)(x + \mathcal{N})\| \leq \|a\|\|x + \mathcal{N}\|$ for all $x + \mathcal{N} \in (A \odot \mathcal{H})/\mathcal{N}$ (where $\|x + \mathcal{N}\|^2 = \langle x + \mathcal{N}, x + \mathcal{N} \rangle$), which means we can extend $\pi(a)$ to a bounded linear operator on all of $\mathcal{H}'$. One then checks that this is indeed a unital $*$-homomorphism.

Define $V : \mathcal{H} \to \mathcal{H}'$ by $V(\xi) = (1_A \odot \xi) + \mathcal{N}$. Then we compute for each unit vector $\xi \in \mathcal{H}$, using Exercise 7.48 from the Day 1 Lecture Notes,

$$\|V\xi\|^2 = \langle 1_A \odot \xi, 1_A \odot \xi\rangle = \langle \phi(1_A^* 1_A)\xi, \xi\rangle \leq \|\phi(1)\|\|\xi\|^2 = \|\phi(1)\|.$$

It follows that $\|V\| = \|\phi(1)\|$ and moreover that V is an isometry when ϕ is unital.

Finally, since V is an isometry, we conclude that for all $\xi, \eta \in \mathcal{H}$,

$$\langle V^*\pi(a)V\xi, \eta\rangle_{\mathcal{H}} = \langle \pi(a)V\xi, V\eta\rangle = \langle \pi(a)((1 \odot \xi) + \mathcal{N}), (1 \odot \eta) + \mathcal{N}\rangle = \langle \phi(a)\xi, \eta\rangle_{\mathcal{H}},$$

and hence $V^*\pi(a)V = \phi(a)$. □

Exercise 9.24. Describe in words how (the proof of) Stinespring's Dilation Theorem generalizes the Gelfand Naimark Segal Theorem. In particular, when ϕ is a state, what is $\mathcal{H}$? $A \odot \mathcal{H}$?

Yes, there is a generalization of Stinespring's Dilation Theorem called the Kasparov-Stinespring Dilation Theorem. This is phrased in either the language of Hilbert C*-modules (see [6] for a nice introduction) or multiplier algebras (in Kasparov's original paper).

Frankly, Stinespring's theorem admits several generaliztions. For instance, there is one for maps that are just considered completely bounded, i.e. linear maps with $\sup_n \|\phi^{(n)}\| < \infty$.

For the sake of seeing Stinespring's Dilation theorem in action, we introduce another useful concept for ucp maps: multiplicative domains. Here's how we define a multiplicative domain.

Definition 9.25. Let A and B be unital C*-algebras and $\phi : A \to B$ ucp. Then the set

$$\{a \in A : \phi(a)\phi(b) = \phi(ab) \text{ and } \phi(b)\phi(a) = \phi(ba)\ \forall\ b \in A\}$$

is a C*-subalgebra of A called the *multiplicative domain of* ϕ.

Notice that ϕ is a $*$-homomorphism when restricted to this set. In fact, this is the largest C*-subalgebra on which the ucp map acts as a $*$-homomorphism, though the fact that it is a C*-algebra requires proof. To prove this, we use Stinespring's Dilation theorem to prove the following alternative description.

Proposition 9.26. *Let A and B be unital* C*-*algebras and $\phi : A \to B$ ucp. Then the multiplicative domain of ϕ is equal to the set*

$$\{a \in A : \phi(a)^*\phi(a) = \phi(a^*a) \text{ and } \phi(a)\phi(a)^* = \phi(aa^*)\}.$$

Proof of Proposition 9.26. Let A be a unital C*-algebra and $\phi : A \to B$ a ucp map. One inclusion is immediate. We will work through the other inclusion.

Since B can be faithfully represented on some $B(\mathcal{H})$ (and the composition of that representation with ϕ is still cp), we assume $B \subset B(\mathcal{H})$ and view ϕ as a map into $B(\mathcal{H})$. Let $(\pi, V, \mathcal{H}')$ be a Stinespring Dilation for $\phi : A \to B(\mathcal{H})$, i.e. $\pi : A \to B(\mathcal{H}')$ is a representation of A and $V : \mathcal{H} \hookrightarrow \mathcal{H}'$ an isometric embedding so that $\phi(a) = V^*\pi(a)V$ for all $a \in A$. Then for any $a \in A$, we have

$$\begin{aligned}\phi(a^*a) - \phi(a)^*\phi(a) &= V^*\pi(a^*a)V - V^*\pi(a)^*VV^*\pi(a)V \\ &= V^*\pi(a)^*1_{\mathcal{H}'}\pi(a)V - V^*\pi(a)^*VV^*\pi(a)V \\ &= V^*\pi(a)^*(1_{\mathcal{H}'} - VV^*)\pi(a)V\end{aligned}$$

Now, suppose $a \in A$ so that $\phi(a^*a) = \phi(a)^*\phi(a)$ and $\phi(aa^*) = \phi(a)\phi(a)^*$. Since V is an isometry, VV^* is a positive contraction, and so by Exercise 3.11, $1_{\mathcal{H}'} - VV^*$ is a positive contraction, which has a unique positive square root. With that observation, we compute

$$\begin{aligned}0 = \phi(a^*a) - \phi(a)^*\phi(a) &= V^*\pi(a)^*(1_{\mathcal{H}'} - VV^*)\pi(a)V \\ &= V^*\pi(a)^*(\sqrt{1_{\mathcal{H}'} - VV^*})^2\pi(a)V \\ &= [\sqrt{1_{\mathcal{H}'} - VV^*}\pi(a)V]^*[\sqrt{1_{\mathcal{H}'} - VV^*}\pi(a)V].\end{aligned}$$

It follows (from say the C*-identity) that $\sqrt{1_{\mathcal{H}'} - VV^*}\pi(a)V = 0$.

With that, we let $b \in A$ and compute

$$\phi(ba) - \phi(b)\phi(a) = V^*\pi(b)(1_{\mathcal{H}'} - VV^*)\pi(a)V = 0.$$

A symmetric argument shows that $\phi(ab) = \phi(b)\phi(a)$ for all $b \in A$, which completes the argument. □

Exercise 9.27. Conclude that the multiplicative domain of a cpc map $\phi : A \to B$ from a unital C*-algebra is a C*-subalgebra.

9.3. **Arveson's Extension Theorem.** The other major theorem for completely positive maps (as far as C*-algebraists are usually concerned) is Arveson's Dilation Theorem. Just as Stinespring's Dilation Theorem was a generalization of GNS, which was a generalization of GN, Arveson's Extension Theorem is a generalization of Krein's Theorem, which is a strengthening of the Hahn-Banach Theorem for C*-algebras. On the other hand, where Stinespring's proof was a generalization of the proofs that came before, Arveson's proof builds on the proofs that came before.

Theorem 9.28 (Arveson's Extension Theorem)**.** *Let B be a unital* C*-*algebra, $A \subset B$ a unital* C*-*subalgebra, and $\phi : A \to B(\mathcal{H})$ a cp map. Then there exists a cp map $\tilde{\phi} : B \to B(\mathcal{H})$ extending ϕ, i.e. $\tilde{\phi}|_A = \phi$.*

Remark 9.29. In an abuse of categorical language, $B(\mathcal{H})$ is often called *injective* in the category of C*-algebras with morphisms given by cpc maps. (It's an abuse of language because we always assume an embedding $A \subset B$ is a $*$-homomorphism embedding.)

This theorem plays a big role in the next section when we see a characterization of nuclear C*-algebras in terms of completely positive maps. For now, we just give an idea of the proof via the results it generalizes.

Theorem 9.30 (Krein)**.** *Let B be a unital* C*-*algebra, $A \subset B$ a unital* C*-*subalgebra, and $\phi : A \to \mathbb{C}$ a positive linear map. Then ϕ extends to a positive map on B.*

This intermediate result is [8, Theorem 6.2].

Proposition 9.31. *Let B be a unital* C*-*algebra, $n \geq 1$, $A \subset B$ a unital* C*-*subalgebra, and $\phi : A \to \mathrm{M}_n(\mathbb{C})$ completely positive. Then ϕ extends to a completely positive map $B \to \mathrm{M}_n(\mathbb{C})$.*

From this to Arveson's theorem, we take a completely positive map $\phi : A \to B(\mathcal{H})$ and an increasing net of finite rank projections $P_i \in B(\mathcal{H})$. Then each compression $\phi_i : A \to P_i B(\mathcal{H}) P_i \simeq \mathrm{M}_{\mathrm{rank} P_i}(\mathbb{C})$, given by $P_i \phi(\cdot) P_i$, is a completely positive map with completely positive extension. From here you take a point-ultraweak cluster point of the ϕ_i's (ask Brent and Rolando), and that's your cp extension of ϕ!

Exercise 9.32. Suppose $C \subset B(\mathcal{H})$ is a unital C*-subalgebra of $B(\mathcal{H})$ (meaning its unit is $1_{\mathcal{H}}$) and $E : B(\mathcal{H}) \to C$ is a conditional expectation (which we recall from Exercise 9.11 is completely positive by Tomiyama's theorem). Show that Arveson's Extension theorem holds for C as well, i.e. for any unital C*-algebras $A \subset B$ and cp map $\phi : A \to C$, there exists a cp map $\tilde{\phi} : B \to C$ extending ϕ, i.e. $\tilde{\phi}|_A = \phi$.

Using Example 9.11, conclude that Arveson's Extension theorem holds for all finite dimensional C*-algebras.

Remark 9.33. If you've peeked at some of the reference texts, you'll notice that many of the theorems from this section are given for operator systems. What are those? You'll learn more about them in Sam Kim's expository lecture next week, but for now, here's an idea.

Notice that completely positive maps completely preserve the structure of positive elements in a C*-algebra. So, there is a lot to be gained from considering self-adjoint unital subspaces of C*-algebras.

An *operator system* is a unital self-adjoint subspace of a C*-algebra. (Not necessarily closed.) Arveson's extension theorem is actually stated where we assume that $A \subset B$ is not a C*-algebra but an operator system inside B.

10. Completely Positive Approximations

This section introduces what is historically known as the "completely positive approximation property," which, in the hindsight provided by a major theorem of Choi-Effros and also Kirchberg (which we give next week), is now called nuclearity. In essence, a C*-algebra has the completely positive approximation property when it can be well approximated by cpc maps that factor through finite dimensional C*-algebras. This is, at its heart, a property of maps, which is where we start in section 10.1.

However, in the lecture, we will focus on nuclearity of C*-algebras (10.2) and hence will take the material in Section 10.1 for granted. We will go through the discussion on $K(\ell^2)$ at the beginning of this section in the context of the definition of nuclearity. We will prove Proposition 10.15 and Proposition 10.9 in the separable setting. Arveson's Extension theorem will feature prominently.

Though we will not be able to treat it in lecture, we highly recommend reading the argument that commutative C*-algebras are nuclear (Proposition 10.10) and working out the hands-on example in Exercise 10.12.

Many of the C*-algebras we can get our hands on have some reasonable connection to finite-dimensional C*-algebras. AF algebras in particular were built out of finite-dimensional subalgebras. More generally, they can be approximated by their finite dimensional subalgebras in a way that can be generalized to a much larger class of C*-algebras. To get a better feeling for what we mean, let us start with a motivating example.

We know (Example 6.5) that $K(\ell^2)$ is built as a union of finite-dimensional algebras as follows:

$$K(\ell^2) = \overline{\bigcup_n P_n K(\ell^2) P_n}$$

where P_n is the projection onto $\operatorname{span}\{e_1, \dots, e_n\}$. Since the projections $(P_n)_n$ form an approximate unit for $K(\ell^2)$, we have for each $T \in K(\ell^2)$,

$$\|T - P_n T P_n\| \to 0.$$

We saw in the previous section that the map $T \mapsto P_n T P_n$ is a completely positive comtractive map. Compose that with the $*$-isomorphism $P_n K(\ell^2) P_n \simeq \mathrm{M}_n(\mathbb{C})$, and we have a cpc map $\psi_n : K(\ell^2) \to \mathrm{M}_n(\mathbb{C})$. Moreover, when we compose that with the $*$-homomorphism embedding $\phi_n : \mathrm{M}_n \to P_n K(\ell^2) P_n \subset K(\ell^2)$, we can write

$$\|T - \phi_n \psi_n(T)\| \to 0.$$

This is called a *completely positive approximation* of $K(\ell^2)$, and the existence of such an approximation is what it means (in modern terms) to be nuclear.

For the sake of simplicity, many results here are not stated in their full generality. If you find this section interesting, we suggest [3, Chapter 2], which covers this material quite well, save a dearth of hands-on examples.

10.1. **Nuclear Maps.** We start with the definition of a nuclear map between C*-algebras.

Definition 10.1. A cpc map $\theta : A \to B$ between C*-algebras is called *nuclear* if there exist cpc maps $\psi_i : A \to \mathrm{M}_{k(i)}(\mathbb{C})$ and $\phi_i : \mathrm{M}_{k(i)}(\mathbb{C}) \to B$, for $i \in I$, so that $\phi_i \circ \psi_i \to \theta$ in the point norm topology, i.e. for each $a \in A$,

$$\lim_{i \in I} \|\phi_i(\psi_i(a)) - \theta(a)\| = 0.$$

Remark 10.2. There's lots to say here. This idea is thoroughly researched and nuanced, and there are so many variations on the above definition. We'll keep these remarks brief.

- If A is separable, then it can be written as a countable union of finite subsets. Then we can choose the net I in Definition 10.1 to be a sequence.
- The requirements placed on the maps ψ_i and ϕ_i can vary. It turns out we could equivalently relax the contractive requirement. On the other hand, we could equivalently keep the requirement that they are cpc and demand moreover that they have certain (approximate) orthogonality preserving properties (known as order zero). There's a fair bit of research in this direction by Winter, Zacharias, Kirchberg, Hirchberg, Brown, and Carrion to name a few. (FYI: Nate Brown will be one of our expository speakers, and José Carrion will speak at our conference.)

- The convergence in Definition 10.1 could have be given with respect the point-ultraweak (aka σ-weak) topology (in which case the map would be called *weakly nuclear*). This is the first step on the bridge between nuclearity for C*-algebras and semidiscreetness/ hyperfiniteness for von Neumann algebras (ask Brent and Rolando what those terms mean), but we are getting ahead of ourselves.

This is really a local property, as the following proposition shows.

Proposition 10.3. *A cpc map $\theta : A \to B$ is nuclear iff for any $\epsilon > 0$ and finite set $F \subset A$, there exists $n \geq 0$ and cpc maps $\psi : A \to \mathrm{M}_n(\mathbb{C})$ and $\phi : \mathrm{M}_n(\mathbb{C}) \to B$ such that*

$$\|\phi(\psi(a)) - \theta(a)\| < \epsilon$$

for each $a \in F$.

Proof. Suppose there exist cpc maps $\psi_i : A \to \mathrm{M}_{k(i)}(\mathbb{C})$ and $\phi_i : \mathrm{M}_{k(i)}(\mathbb{C}) \to B$ for $i \in I$ so that $\phi_i \circ \psi_i \to \theta$ in the point norm topology. Then for any $\epsilon > 0$ and $F \subset A$ finite, we choose $i \in I$ so that $\|\phi_i(\psi_i(a)) - \theta(a)\| < \epsilon$ for each $a \in F$.

Now, we assume the localized version. As in Examples 5.3 and 5.8 in the Prerequisite materials, we form a directed set

$$\{(\epsilon, F) \ : \ \epsilon > 0, F \subset A \text{ finite}\}.$$

For each (ϵ, F), let $\phi_{(\epsilon,F)}$ be a cpc map so that $\|\phi_{(\epsilon,F)}(\psi_{(\epsilon,F)}(a)) - \theta(a)\| < \epsilon$ for each $a \in F$. Then for each $a \in A$, we have the desired convergence. □

Exercise 10.4. Show that a map $\theta : A \to B$ is nuclear if there exist finite dimensional C*-algebras F_i and cpc maps $\psi_i : A \to F_i$ and $\phi_i : F_i \to B$ so that $\phi_i \circ \psi_i$ converges pointwise in norm to θ.
Hint: Recall that a finite dimensional C*-algebra has the form $F = \oplus_{j=1}^m \mathrm{M}_{l_j}(\mathbb{C}) \subset \mathrm{M}_L(\mathbb{C})$ where $L = \sum l_j$, and use Example 9.11.

Exercise 10.5. Let A and B be C*-algebras and $C \subset B$ a C*-subalgebra. Show that if $\theta : A \to C$ is a nuclear map, then so is θ when viewed as a map from A to B. Suppose we have a map $\rho : A \to C$ that is nuclear as a map from A to B. What could prevent ρ from being a nuclear map as a map from A to C?

10.2. Completely Positive Approximation Property.

Definition 10.6. A C*-algebra is *nuclear* if the identity map $\mathrm{id}_A : A \to A$ is nuclear, i.e. there exists cpc maps $A \xrightarrow{\psi_i} \mathrm{M}_{k(i)}(\mathbb{C}) \xrightarrow{\phi_i} A$ for $i \in I$ such that for each $a \in A$,

$$\|a - \phi_i(\psi_i(a))\| \to 0.$$

In the separable setting, the usual image one presents is something like the following approximately commutative diagram.

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_0"] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_1"] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_2"] & & \cdots \\
& \mathrm{M}_{k(0)}(\mathbb{C}) \arrow[ur, "\phi_0"] & & \mathrm{M}_{k(1)}(\mathbb{C}) \arrow[ur, "\phi_1"] & & \mathrm{M}_{k(2)}(\mathbb{C}) & \cdots
\end{tikzcd}$$

Remark 10.7. Sometimes these C*-algebras are called amenable. Sometimes for mathematical reasons—sometimes because the word "nuclear" in a grant application means one must fill out many many more forms.

A C*-algebra satisfying Definition 10.6 is also said to satisfy the *completely positive approximation property* (CPAP).

Example 10.8. It follows from Exercise 10.4 that finite dimensional C*-algebras are nuclear.

Proposition 10.9. *Ideals of nuclear C*-algebras are nuclear.*

Proof. Suppose A is nuclear with completely positive approximation $A \xrightarrow{\psi_i} \mathrm{M}_{k(i)}(\mathbb{C}) \xrightarrow{\phi_i} A$ for $i \in I$. Let $J \triangleleft A$ be an ideal and $\{e_\lambda\}_\Lambda$ an approximate unit of J (with $0 \leq e_\lambda \leq e_\gamma \leq 1$ when $\lambda \leq \gamma$). Let $\iota : J \to A$ denote the inclusion of J into A (i.e. $\iota(a) = a$ for all $a \in J$). For each λ, define $\rho_\lambda : A \to J$ by $\rho_\lambda(a) = e_\lambda a e_\lambda$. Since each e_λ is self-adjoint and contractive, the maps ρ_λ are cpc by Exercise 9.9. Since the compositions of cpc maps are cpc (Exercise 9.6), for each i, λ, the maps $\psi'_{i,\lambda} := \psi_i \circ \iota : J \to \mathrm{M}_{k(i)}$ and $\phi'_{i,\lambda} := \rho_\lambda \circ \phi_i : \mathrm{M}_{k(i)} \to J$

are cpc. (Yes, the λ is a superflous index on ψ'_i.) Moreover, $\{(i,\lambda)\}_{I\times\Lambda}$ is a directed set with $(i,\lambda)\le(j,\gamma)$ when $i\le j$ and $\lambda\le\gamma$.

Let $a\in J$ and $\epsilon>0$, and choose $(i_0,\lambda_0)\in I\times\lambda$ so that $\|a-\phi_i\circ\psi_i\|<\epsilon/2$ and $\|a-\rho_\lambda(a)\|<\epsilon/2$ for each $i\ge i_0$ and $\lambda\ge\lambda_0$. Then for each $(i,\lambda)\ge(i_0,\lambda_0)$,

$$\begin{aligned}\|a-\phi'_{i,\lambda}\circ\psi'_{i,\lambda}\| &= \|a-e_\lambda(\phi_i\circ\psi_i(a))e_\lambda\|\\ &\le \|a-e_\lambda a e_\lambda\|+\|e_\lambda a e_\lambda-e_\lambda(\phi_i\circ\psi_i(a))e_\lambda\|\\ &\le \|a-e_\lambda a e_\lambda\|+\|e_\lambda\|\|a-\phi_i\circ\psi_i(a)\|\|e_\lambda\|\\ &\le \|a-e_\lambda a e_\lambda\|+\|a-\phi_i\circ\psi_i(a)\|\\ &<\epsilon.\end{aligned}$$

□

In approximately commutative diagrams, the picture from the above proof looks like this.

$$\begin{tikzcd}
J \arrow[rrrr, "id_J"] \arrow[dr, "\iota"] & & & & J \\
& A \arrow[rr, "id_A"] \arrow[dr, "\psi_i"] & & A \arrow[ur, "\rho_\lambda"] & \\
& & \mathrm{M}_{k(i)}(\mathbb{C}) \arrow[ur, "\phi_i"] & &
\end{tikzcd}$$

It's not a proof, but it's a good intuition to guide the proof.

Proposition 10.10. *Abelian* C^**-algebras are nuclear.*

The proof uses what is known as a "partition of unity argument." Generalizing the idea of a partition of unity has proved very fruitful in certain areas of research in recent years, so we give this proof as an example.

We take for granted the fact from topology that, given any compact Hausdorff space X with open cover $U_1,...,U_n$, there exist continuous functions $h_1,...,h_n : X\to[0,1]$ so that $\mathrm{supp}h_j\subset U_j$ and $\sum_j h_j=1$. (See [Theorem 2.13, Rudin, Real and Complex Analysis].) This is a *partition of unity* (in fact a rather nice one).

Proof. Let A be an abelian C^*-algebra. If A is not unital, then $A\triangleleft\tilde{A}$, and by Propositon 10.9, it suffices to show that $\tilde{A}$ is nuclear. So, we assume A is unital and moreover that $A=C(X)$ for some compact Hausdorff space X. Combining Proposition 10.3 and Exercise 10.4, we conclude that it suffices to show that for any $F\subset C(X)$ finite and $\epsilon>0$, there exists a finite dimensional C^*-algebra C (in our case, it will be $\mathbb{C}^n=\oplus_1^n\mathrm{M}_1(\mathbb{C})$) and cpc maps $C(X)\xrightarrow{\psi}C\xrightarrow{\phi}C(X)$ so that $\|f-\phi\circ\psi(f)\|<\epsilon$ for every $f\in F$.

Let $F\subset C(X)$ be a finite subset and $\epsilon>0$. For each $x\in X$, let

$$U_x:=\bigcap_{f\in F}f^{-1}(B_{\epsilon/2}(f(x))).$$

Then $U_x\subset X$ is an open neighborhood of x such that for each $y\in U_x$ and $f\in F$, we have $|f(y)-f(x)|<\epsilon/2$. Since X is compact, we choose $x_1,...,x_n$ so that a finite subcover $U_{x_1},...,U_{x_n}$ covers X, and moreover for each $f\in F$ and $y\in U_i$,

$$|f(y)-f(x_i)|<\epsilon.$$

Then we choose a partition of unity $h_1,...,h_n : X\to[0,1]$ so that $\mathrm{supp}h_j\subset U_{x_j}$ and $\sum_j h_j=1$.

Define $\psi : C(X)\to\mathbb{C}^n$ by $\psi(g)=(g(x_1),...,g(x_n))=\oplus_{j=1}^n ev_{x_j}$, where ev_{x_j} denotes the point evaluation $g\mapsto g(x_j)$. Then ψ is a unital $*$-homomorphism. Define $\phi:\mathbb{C}^n\to C(X)$ by

$$(\lambda_1,...,\lambda_n)\mapsto\sum\lambda_i h_i.$$

Then ϕ is a positive map, which is moreover unital since $\phi(1)=\sum h_i=1$. Hence by Proposition 9.14, it is ucp, and, in particular, cpc by Corollary 9.18.

So, we estimate for $f \in F$,

$$\begin{aligned}\|f - \phi \circ \psi(f)\| &= \| \left(\sum h_i\right) f - \sum f(x_i) h_i \| = \| \sum f h_i - f(x_i) h_i \| \\ &= \sup_{y \in X} | \sum (f(y) - f(x_i)) h_i(y) | \le \sup_{y \in X} \sum |f(y) - f(x_i)| h_i(y) \\ &\le \sum \epsilon h_i(y) = \epsilon.\end{aligned}$$

□

Remark 10.11. There has been a significant push in the classification program for nuclear C*-algebras (that satisfy a nice list of adjectives) to come up with a non-commutative version of this partition of unity argument. With it comes certain non-commutative dimension theories (see for example Winter and Zacharias's paper on Nuclear Dimension). Some of this may be featured in José Carrion's conference talk.

Exercise 10.12. Partitions of unity are nicer when you have a concrete example. For each $n \ge 2$, cover $[0,1]$ by $2^n - 1$ open intervals of equal length. (What are they? Also, we could start with $n = 1$, but it's too simple to pick up on a pattern.) Call this cover $\mathcal{U}_n$. Define (sketch) a partition of unity for $\mathcal{U}_n$. (Hint: think zig-zags.)

Now, construct a sequence of completely positive maps $C([0,1]) \xrightarrow{\psi_n} \mathbb{C}^{k_n} \xrightarrow{\phi_n} C([0,1])$, (what is k_n?) that give a completely positive approximation of $C([0,1])$.

Proposition 10.13. *Suppose for each finite subset $F \subset A$ and $\epsilon > 0$, there exists a nuclear* C*-*subalgebra $B \subset A$ such that for each $a \in F$, there exists a $b \in B$ such that $\|a - b\| < \epsilon$. Then A is nuclear.*

Proof. By Proposition 10.3, it suffices to show that for any $\epsilon > 0$ and finite set $F \subset A$, there exists $n \ge 0$ and cpc maps $\psi : A \to \mathrm{M}_n(\mathbb{C})$ and $\phi : \mathrm{M}_n(\mathbb{C}) \to B$ such that

$$\|\phi(\psi(a)) - \theta(a)\| < \epsilon$$

for each $a \in F$. Let $\{a_1, ..., a_m\} \subset A$ be a finite subset $\epsilon > 0$ and let $B \subset A$ nuclear so that for each a_j, there exists a $b_j \in B$ such that $\|a_j - b_j\| < \epsilon/3$. Let $n \ge 0$ and $\psi_B : B \to \mathrm{M}_n(\mathbb{C})$ and $\phi_B : B \to \mathrm{M}_n(\mathbb{C})$ be cpc maps so that $\|b_j - \phi_B \psi_B(b_j)\| < \epsilon/3$ for each $1 \le j \le m$.

But how do we get a map ψ defined on all of A? Easy, since $\mathrm{M}_n(\mathbb{C}) = B(\mathbb{C}^n)$, the cpc map $\psi_B : B \to \mathrm{M}_n(\mathbb{C})$ extends to a cpc map $\psi : A \to \mathrm{M}_n(\mathbb{C})$ by Arveson's Extension Theorem.[9] Since $\phi_B : \mathrm{M}_n(\mathbb{C}) \to B \subset A$, we don't need to change it, so we choose $\phi = \phi_B$.

Now, all that's left is to compute for each $1 \le j \le m$:

$$\begin{aligned}\|a_j - \phi\psi(a_j)\| &\le \|a_j - b_j\| + \|b_j - \phi\psi(b_j)\| + \|\phi\psi(b_j) - \phi\psi(a_j)\| \\ &\le \|a_j - b_j\| + \|b_j - \phi\psi(b_j)\| + \|b_j - a_j\| \\ &< \epsilon\end{aligned}$$

□

Exercise 10.14. Using the above proposition, show that nuclearity is closed under taking direct limits. Conclude that AF algebras are nuclear.

The above proof is perhaps a little abstract. Here's a version that's a little more tangible. First, we recall once more the construction of the CAR algebra:

Let $\mathrm{M}_{2^n}(\mathbb{C})$ be the algebra of $2^n \times 2^n$ matrices with maps $\phi_{n,n+1} : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ defined by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

The inductive limit is denoted $\mathrm{M}_{2^\infty} = \overline{\bigcup_n \mathrm{M}_{2^n}(\mathbb{C})}$. Note that by construction, for each $n \ge 0$, the copys of M_{2^n} in M_{2^∞} is unital.

Proposition 10.15. *The CAR algebra is nuclear.*

[9]Acutally, it's overkill here– one of the preliminary results leading up to Arveson's would work in finite dimensions.

Proof. For each $n \geq 0$, define $\phi_n : \mathrm{M}_{2^n}(\mathbb{C}) \to \mathrm{M}_{2^\infty}$ be the inclusion where we identify M_{2^n} with its copy inside M_{2^∞}. The restriction of this map to its image is a $*$-isomorphism, so we call its inverse $\phi_n^{-1} : \phi_n(\mathrm{M}_{2^n}(\mathbb{C})) \to \mathrm{M}_{2^n}(\mathbb{C})$. This is a unital $*$-homomorphism from a C*-subalgebra of M_{2^∞} to $\mathrm{M}_{2^n}(\mathbb{C}) = B(\mathbb{C}^{2^n})$. So Arveson's Extension theorem says ϕ_n^{-1} has a ucp extension $\psi_n : \mathrm{M}_{2^\infty} \to \mathrm{M}_{2^n}(\mathbb{C})$. So, we have ucp maps $\psi_n : \mathrm{M}_{2^\infty} \to \mathrm{M}_{2^n}(\mathbb{C})$ and $\phi_n : \mathrm{M}_{2^n}(\mathbb{C}) \to \mathrm{M}_{2^\infty}$. Moreover, for each $a \in \bigcup_n \mathrm{M}_{2^n}(\mathbb{C})$, there exists an $N \geq 0$ so that $a \in \mathrm{M}_{2^n}(\mathbb{C})$ for all $n \geq N$, which means $\phi_n \circ \psi_n(a) = \phi_n \circ \phi_n^{-1}(a) = a$ for all $n \geq N$.

Now, suppose $a \in \mathrm{M}_{2^\infty}$, and $a_0 \in \bigcup_n \mathrm{M}_{2^n}(\mathbb{C})$ so that $\|a - a_0\| < \epsilon/2$. Choose $N \geq 0$ so that $\phi_n \circ \psi_n(a_0) = a_0$ for all $n \geq N$. Then for all $n \geq N$,

$$\|a - \phi_n \circ \psi_n(a)\| \leq \|a - a_0\| + \|a_0 - \phi_n \circ \psi_n(a_0)\| + \|\phi_n \circ \psi_n(a_0 - a)\| < \epsilon$$

□

Exercise 10.16. Generalize the proof or Proposition 10.15 to get another proof that all separable AF algebras are nuclear.
Hint: Consider a inductive (aka directed) system of finite dimensional C*-algebras (A_n, ι_{mn}) where $\iota_{mn} : A_n \to A_m$ is the inclusion map, and let A be the direct (inductive) limit of this system. Then use Exercise 9.32.

Chapter 2 in [3] does an excellent job of introducing the operations that do and do not preserve nuclearity. Since we do not wish to re-write their book. We will just collect them here. These range from easy exercises to deep theorems.

(1) Nuclearity passes to direct limits and direct sums ($\bigoplus_i A_i$) (but not direct products $\prod_i A_i$).
(2) Nuclearity passes to quotients.
There are essentially two proofs for this. The first is a consequence of Connes' Fields Medal work involving showing hyperfinite ⇔ injective– ask Brent and Rolando. Otherwise, it follows from the fact that exactness (defined soon) passes to quotients. The proof of this (due to Kirchberg) is one of the most difficult proofs in C*-algebras, resting some of the deepest and most difficult theorems in von Neumann algebra theory. – See [3, Chapter 9] for an (not self-contained) outline.
(3) Nuclearity does not necessarily pass to subalgebras.
The easiest examples come from crossed products, which we'll see next week. (See [3, Remark 4.4.4].) For a more sophisticated appeal, we have Kirchberg's $\mathcal{O}_2$ embeddability theorem, which implies that the non-nuclear C*-algebra $\mathrm{C}^*_r(\mathbb{F}_2)$ embeds into the nuclear C*-algebra $\mathcal{O}_2$. (We will see next week why $\mathrm{C}^*_r(\mathbb{F}_2)$ is not nuclear. We take for granted that the Cuntz-Pimsner algebras are nuclear.)
(4) Nuclearity passes to ideals (Proposition 10.9) (even hereditary subalgebras) and C*-subalgebras to which there exists a conditional expectation.
(5) Nuclearity passes to extensions, i.e. if $0 \to J \to A \to B \to 0$ is short exact and both J and B are nuclear, then so is A. (This one is easier with next week's characterization.)

We wrap up this section with a slight weakening of nuclearity that is still a very powerful property.

As we saw in Exercise 10.5, the range of a cpc map has a lot of bearing on whether or not it is nuclear. It may be that a C*-algebra fails to be nuclear but still has a faithful nuclear representation. These are still a nice class of C*-algebras.

Definition 10.17. A C*-algebra A is *exact* if there exists a faithful nuclear representation $\pi : A \to B(\mathcal{H})$.

Every nuclear C*-algebra is exact– moreover for nuclear C*-algebras, the map $\pi : A \to \pi(A)$ is nuclear. A non-nuclear example is $\mathrm{C}^*(\mathbb{F}_2)$ (due to Wasserman).

Exercise 10.18. Show that exactness *does* pass to C*-subalgebras. What does that tell you about every C*-subalgebra of a nuclear C*-algebra?

The name "exact" is hardly justified here. We will see it again later in the tensor product section, where it will make more sense.

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