Problem Session

June 29th, 2020

1. Show that $v \in B(\mathcal{H})$ is a partial isometry if and only if v^*v is a projection.
[**Hint:** expand $\|(v - vv^*v)\xi\|^2$ for $\xi \in \mathcal{H}$.]

2. For $x \in B(\mathcal{H})$ with $x = x^*$, show that

$$\sup_{\|\xi\|=1} |\langle x\xi, \xi\rangle| = \|x\|.$$

[**Hint:** show Re $\langle x\xi, \eta\rangle = \frac{1}{2}\langle x(\xi+\eta), \xi+\eta\rangle + \frac{1}{2}\langle x(\xi-\eta), \xi-\eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.]

3. For a Hilbert space $\mathcal{H}$, prove the inclusions

$$FR(\mathcal{H}) \subset L^1(B(\mathcal{H})) \subset HS(\mathcal{H}) \subset K(\mathcal{H}).$$

[**Hint:** approximate by finite-rank operators in the appropriate norm.]

4. Show that $v \in B(\mathcal{H})$ is a partial isometry if and only if there exists a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $v|_{\mathcal{K}}$ is an isometry and $v|_{\mathcal{K}^\perp} \equiv 0$.

5. Let $x \in B(\mathcal{H})$. We say x is *bounded below* if there exists $\epsilon > 0$ such that $\|x\xi\| \geq \epsilon\|\xi\|$ for all $\xi \in \mathcal{H}$. Determine the implications between the following properties for $x \in B(\mathcal{H})$:

(i) x is injective (i.e. $\ker(x) = \{0\}$)
(ii) x is left-invertible (i.e. $\exists\, y \in B(\mathcal{H})$ with $yx = 1$)
(iii) x is bounded below.

6. Let $\mathcal{H}$ be a Hilbert space and $1 \leq n < \infty$. We denote $\mathcal{H}^n = \bigoplus_{j=1}^n \mathcal{H}$. For $x_{i,j} \in B(\mathcal{H})$ for $1 \leq i,j \leq n$, define $[x_{i,j}] : \mathcal{H}^n \to \mathcal{H}^n$ by

$$[x_{i,j}] \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n x_{1,j}\xi_j \\ \vdots \\ \sum_{j=1}^n x_{n,j}\xi_j \end{pmatrix}.$$

Check that this gives an operator in $B(\mathcal{H}^n)$ (in fact $\|[x_{i,j}]\| \leq (\sum \|x_{i,j}\|)^{1/2}$). We denote by $M_n(B(\mathcal{H}))$ the operators in $B(\mathcal{H}^n)$ that can be written as $[x_{i,j}]$ for some $x_{i,j} \in B(\mathcal{H})$. Show $M_n(B(\mathcal{H})) = B(\mathcal{H}^n)$.
[**Hint:** How would you do this for $\mathcal{H} = \mathbb{C}^m$?]

7. Here's an intuition building exercise to think about for Wednesday:

(a) Show that all maximal ideals in $C([0,1])$ are of the form $\{f \in C([0,1]) \ : \ f(t) = 0\}$ for some $t \in [0,1]$.
(b) For each $t \in [0,1]$, define the map $ev_t : C([0,1]) \to \mathbb{C}$ by $ev_t(f) = f(t)$. Show that $\widehat{C([0,1])} = \{ev_t \ : \ t \in [0,1]\}$.
(c) Recall that for $A = C_0((0,1])$, its unitization is $\tilde{A} := C([0,1])$. That means we can identify $C_0((0,1])$ with a maximal ideal inside $C([0,1])$. To which character $\phi \in \hat{\tilde{A}}$ does this ideal correspond?
Show that this character agrees with the functional $\phi_0 : \tilde{A} \to \mathbb{C}$ given by $\phi(f + \lambda 1) = \lambda$ for all $f \in A$.