Mixed q-Gaussian algebras and free transport

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Free probability and large N limit, V
Berkeley, March 2016

Outline

1. Mixed q-Gaussian algebras $\Gamma_Q(\mathbb{R}^N)$
2. Isomorphism theorem for $\Gamma_Q(\mathbb{R}^N)$
3. Free transport for infinite variables

The object to study

- In 1993, Speicher introduced the mixed q-commutation relation.

$$l_i^* l_j - q_{ij} l_j l_i^* = \delta_{i=j}$$

  where $q_{ij} = q_{ji} \in [-1, 1]$, $i, j = 1, 2, ..., N$.
- Speicher (1993), Bożejko–Speicher (1994) showed that it has Fock space representation: l_i, l_i^* can be represented as left creation and annihilation operators on a certain Fock space.
- $\Gamma_Q(\mathbb{R}^N) := vN(X_i^Q, i = 1, ..., N)$, $X_i^Q := l_i + l_i^*$, $Q := (q_{ij})_{1 \leq i,j \leq N}$. We call $\Gamma_Q(\mathbb{R}^N)$ the mixed q-Gaussian algebras.

Why consider mixed q-Gaussian?

- Some examples:
  - $q_{ij} \equiv q \in [-1,1]$, the q-Gaussian algebra $\Gamma_q(\mathbb{R}^N)$.
  - Some special pattern of Q: $*_i\Gamma_{q_i}(\mathbb{R}^{n_i})$, $*_i(\Gamma_{q_i}(\mathbb{R}^{n_i})\overline{\otimes}\Gamma_{p_i}(\mathbb{R}^{m_i}))$, ...
- Mixed q-commutation relation still verifies the braid relation (or Yang–Baxter equation): Let $T \in B(H \otimes H)$, $T(e_i \otimes e_j) = q_{ij} e_j \otimes e_i$. Then $(T \otimes 1)(1 \otimes T)(T \otimes 1) = (1 \otimes T)(T \otimes 1)(1 \otimes T)$.
- It provides an example to study free transport for infinite variables (extending Guionnet–Shlyakhtenko's theorem).
- Many results for q-Gaussian algebras and Bożejko–Speicher's algebras: Bożejko, Kümmerer, Speicher, Dykema, Nica, Biane, Krolak, Lust-Piquard, Shlyakhtenko, Nou, Śniady, Ricard, Anshelevich, Belinschi, Lehner, Kennedy, Avsec, Dabrowski, Guionnet, et al.

Speicher's central limit theorem: Notation

- $J_{N,m} := [N] \times [m]$, $J_N := [N] \times \mathbb{N}$.
- $\varepsilon : J_N \times J_N \to \{-1, 1\}$, $\varepsilon(x, y) = \varepsilon(y, x)$, $\varepsilon(x, x) = -1$.
- $\mathcal{A}_m = \mathrm{alg}(x_i(k), (i, k) \in J_{N,m})$, where $x_i(k)$'s satisfy $x_i(k)^* = x_i(k)$ and

$$x_i(k)x_j(l) - \varepsilon((i, k), (j, l))x_j(l)x_i(k) = 2\delta_{(i,k),(j,l)}$$

  for $(i, k), (j, l) \in J_{N,m}$. $\mathcal{A}_m$ can be represented as a matrix subalgebra of $M_{2^{Nm}}$.
- A word of $\mathcal{A}_m$

$$x_B = x_{i_1}(k_1) \cdots x_{i_d}(k_d),$$

$$B = \{(i_1, k_1), \cdots , (i_d, k_d)\} \subset J_{N,m}.$$

- A normalized trace τ_m on $\mathcal{A}_m$: $\tau_m(x_B) = \delta_{B,\emptyset}$.

Speicher's central limit theorem

- Consider independent random variables $\varepsilon((i,k),(j,l)) : \Omega \to \{-1,1\}$ for $(i,k) < (j,l)$ on $(\Omega, \mathbb{P})$ with distribution

$$\mathbb{P}(\varepsilon((i,k),(j,l)) = -1) = \frac{1-q_{ij}}{2}, \quad \mathbb{P}(\varepsilon((i,k),(j,l)) = 1) = \frac{1+q_{ij}}{2},$$

$(i,k),(j,l) \in [N] \times \mathbb{N}$. $\tilde{x}_i(m) := \frac{1}{\sqrt{m}} \sum_{k=1}^{m} x_i(k)$.

Theorem (Speicher '93)

Let $\underline{i} \in [N]^s$. *Then*

$$\lim_{m\to\infty} \tau_m(\tilde{x}_{i_1}(m) \cdots \tilde{x}_{i_s}(m)) = \delta_{s\in 2\mathbb{Z}} \sum_{\substack{\sigma\in P_2(s) \\ \sigma \le \sigma(\underline{i})}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)) \quad a.s.$$

Here and in what follows, we understand $\prod_{\{i,j\}\in\emptyset} q(i,j) = 1$.

Speicher's central limit theorem: ctd

- $\sigma \le \pi$ or $\pi \ge \sigma$ iff σ is a refinement of π.
- Given $\underline{i} = (i_1, \cdots, i_d) \in [N]^d$, we associate a partition $\sigma(\underline{i})$ to $\underline{i}$ by requiring $k, l \in [d]$ belonging to the same block of $\sigma(\underline{i})$ iff $i_k = i_l$.
- $P_2(d)$ consists of $\pi = \{V_1, \cdots, V_{d/2}\}$ such that $|V_k| = 2$.
- Write $V_k = \{e_k, z_k\}$ with $e_k < z_k$ and $e_1 < e_2 < \cdots < e_{d/2}$. Given $\pi \in P_2(d)$, the set of crossings of π is

$$I(\pi) = \{\{k, l\} | 1 \le k, l \le d/2, e_k < e_l < z_k < z_l\}.$$

Fock space representation

- Inner product:

$$\langle e_{i_1} \otimes \cdots \otimes e_{i_m}, e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle_Q$$
$$= \delta_{m,n} \sum_{\sigma \in S_n} a(\sigma, \underline{j}) \langle e_{i_1}, e_{j_{\sigma^{-1}(1)}} \rangle \cdots \langle e_{i_m}, e_{j_{\sigma^{-1}(n)}} \rangle$$

on $\mathbb{C}\Omega \oplus \bigoplus_{d \geq 1} (\mathbb{R}^N)^{\otimes d}$, and denote the completion by $\mathcal{F}_Q(\mathbb{R}^N)$.

- $l(e_n)$ denotes the left creation operator

$$l(e_n)\Omega = e_n$$
$$l(e_n) e_{i_1} \otimes \cdots \otimes e_{i_d} = e_n \otimes e_{i_1} \otimes \cdots \otimes e_{i_d},$$

Fock space representation: ctd

- Its adjoint is given by the left annihilation operator $l(e_n)^*\Omega = 0$ and

$$l(e_n)^* e_{i_1} \otimes \cdots \otimes e_{i_d} = \sum_{k=1}^{d} \delta_{n=i_k} q_{ni_1} \cdots q_{ni_{k-1}} e_{i_1} \otimes \cdots \otimes \hat{e}_{i_k} \otimes \cdots \otimes e_{i_d},$$

  where $\hat{e}_{i_k}$ means that e_{i_k} is omitted in the tensor product.

- We will also need the right creation operator $r(e_n)$ defined by

$$r(e_n)(e_{i_1} \otimes \cdots \otimes e_{i_d}) = e_{i_1} \otimes \cdots \otimes e_{i_d} \otimes e_n.$$

- Bożejko–Speicher ('94)

$$\|l(e_n)\| = \|r(e_n)\| = \begin{cases} \frac{1}{\sqrt{1-q_{nn}}} & \text{if } q_{nn} \in [0,1), \\ 1 & \text{if } q_{nn} \in (-1,0]. \end{cases}$$

Ultraproduct construction

- Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We get a finite vNa $\mathcal{A}_{\mathcal{U}} := \prod_{m,\mathcal{U}} \mathcal{A}_m$ with normal faithful tracial state $\tau_{\mathcal{U}} = \lim_{m,\mathcal{U}} \tau_m$.
- $\mathcal{A}_{\mathcal{U}}^{\infty} := \cap_{p<\infty} L_p(\mathcal{A}_{\mathcal{U}})$. For each $\omega \in \Omega$,

$$(\tilde{x}_i(m)(\omega))^{\bullet} \in \mathcal{A}_{\mathcal{U}}^{\infty}.$$

  Here $(\tilde{x}_i(m)(\omega))^{\bullet}$ is the element represented by $(\tilde{x}_i(m)(\omega))_{m\in\mathbb{N}}$ in the ultraproduct.
- Speicher's CLT implies

$$\tau_{\mathcal{U}}((\tilde{x}_{i_1}(m)(\omega))^{\bullet} \cdots (\tilde{x}_{i_s}(m)(\omega))^{\bullet}) = \delta_{s\in 2\mathbb{Z}} \sum_{\substack{\sigma\in P_2(s) \\ \sigma\leq\sigma(\underline{i})}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)$$

  and $\tau_{\mathcal{U}}(|(\tilde{x}_i(m)(\omega))^{\bullet}|^p) \leq Cp^p$.

Ultraproduct construction: ctd

- Junge ('06): the von Neumann algebras generated by the spectral projections of $(\tilde{x}_i(m)(\omega))^\bullet, i = 1, \cdots, N$ for different $\omega \in \Omega$ are isomorphic.
- $\Gamma_Q(\mathbb{R}^N)$ is any von Neumann algebra in the isomorphic class with generators $X_i^Q := (\tilde{x}_i(m)(\omega))^\bullet, i = 1, \cdots, N$.
- X_i^Q may be unbounded, therefore may not be in Γ_Q. But it belongs to $\Gamma_Q^\infty := \cap_{p<\infty} L_p(\Gamma_Q, \tau_{\mathcal{U}})$.
- $\tau_Q = \tau_{\mathcal{U}}|_{\Gamma_Q}$.

Wick word decomposition

Theorem (Junge-Z, '15)

Let $(\tilde{x}_j(m))^\bullet \in \cap_{p<\infty} L_p(\prod_{m,\mathcal{U}} L_\infty(\Omega;\mathcal{A}_m))$ *for* $j = 1, \cdots, d$. *Then*

$$(\tilde{x}_{i_1}(m))^\bullet \cdots (\tilde{x}_{i_d}(m))^\bullet = \sum_{\substack{\sigma \in P_{1,2}(d) \\ \sigma \le \sigma(\underline{i})}} w_\sigma(\underline{i})$$

holds for all $\omega \in \Omega$.

Here $w_\sigma(\underline{i}) = \left(\frac{1}{m^{d/2}} \sum_{\underline{k}\in[m]^d : \sigma(\underline{k})=\sigma} E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\right)^\bullet$, $\mathcal{N}_s(\underline{k})$ denotes the von Neumann algebra generated by all $x_{i_\alpha}(k_\alpha)$'s, where k_α corresponds to singleton blocks in $\sigma(\underline{k})$.

Proof is a bit technical: Based on NC Khintchine and martingale inequalities, decoupling, Pisier's method for multi-index summations (Möbius inversion), etc.

The Ornstein–Uhlenbeck semigroup

- The Wick word (or Wick product) of $e_{i_1} \otimes \cdots \otimes e_{i_n}$, $W(e_{i_1} \otimes \cdots \otimes e_{i_n})$, is the unique element in Γ_Q satisfying

$$W(e_{i_1} \otimes \cdots \otimes e_{i_n})\Omega = e_{i_1} \otimes \cdots \otimes e_{i_n}.$$

- $W(e_{i_1} \otimes \cdots \otimes e_{i_n})$ can be identified with

$$w(\underline{i}) = \Big(\frac{1}{m^{s/2}} \sum_{\underline{k}\in[m]^s:\sigma(\underline{k})\in P_1(s)} x_{i_1}(k_1)\cdots x_{i_s}(k_s)\Big)^{\bullet}.$$

- The O-U semigroup is defined by

$$T_t(W(e_{i_1} \otimes \cdots \otimes e_{i_n})) = e^{-t|\underline{i}|} W(e_{i_1} \otimes \cdots \otimes e_{i_n}).$$

Equivalently, for $\underline{i} \in [N]^s$,

$$T_t w(\underline{i}) = \Big(\frac{1}{m^{s/2}} \sum_{\underline{k}:\sigma(\underline{k})\in P_1(s)} e^{-ts} x_{i_1}(k_1)\cdots x_{i_s}(k_s)\Big)^{\bullet} = e^{-ts} w(\underline{i}).$$

Analytic properties: Hypercontractivity

Theorem (Junge–Z '15)

Let $q_{ij} \in [-1, 1]$. *Then for* $1 \le p, r < \infty$,

$$\|T_t\|_{L_p \to L_r} = 1 \quad \text{if and only if} \quad e^{-2t} \le \frac{p-1}{r-1}.$$

- Let A be the generator of T_t. Meyers "carr du champs" is defined by

$$\Gamma(f, g) = \frac{1}{2}[A(f^*)g + f^*A(g) - A(f^*g)].$$

Analytic properties: Riesz transform

Theorem (Lust-Piquard '99, Junge–Z '15)

(a) Let $2 \leq p < \infty$. Then for every $f \in \mathrm{Dom}(A)$,

$$c_p^{-1}\|A^{1/2}f\|_p \leq \max\{\|\Gamma(f,f)^{1/2}\|_p, \|\Gamma(f^*,f^*)^{1/2}\|_p\} \leq K_p\|A^{1/2}f\|_p$$

where $c_p = O(p^2)$ and $K_p = O(p^{3/2})$.

(b) Let $1 < p \leq 2$. Then for every $f \in \mathrm{Dom}(A)$,

$$K_{p'}^{-1}\|A^{1/2}f\|_p \leq \inf_{\substack{\delta(f)=g+h \\ g\in G_p^c, h\in G_p^r}} \{\|E(g^*g)^{1/2}\|_p + \|E(hh^*)^{1/2}\|_p\} \leq C_p\|A^{1/2}f\|,$$

where $K_{p'} = O(1/(p-1)^{3/2})$ and $C_p = O(1/(p-1)^2)$.

Analytic properties: L_p Poincaré inequalities

Theorem (Junge–Z '15)

Let $2 \leq p < \infty$. *Then for every* $f \in \mathrm{Dom}(A)$,

$$\|f - \tau_Q(f)\|_p \leq C\sqrt{p}\max\{\|\Gamma(f,f)^{1/2}\|_p, \Gamma(f^*,f^*)^{1/2}\|_p\}.$$

Proofs follow from the Wick word decomposition theorem and the corresponding results in the matrix level.

This idea was originally used by Biane ('97) to deduce free hypercontractivity, and was later used by many authors. e.g. Kemp, Lee, Ricard, Junge, Palazuelos, Parcet, Perrin, et al.

CMAP and strong solidity

- A (finite) vNa $\mathcal{M}$ has the weak* completely bounded approximation property (w*CBAP) if there exists a net of normal, completely bounded, finite rank maps $\phi_\alpha : \mathcal{M} \to \mathcal{M}$ such that $\|\phi_\alpha\|_{cb} \leq C$ for all α and $\phi_\alpha \to \mathrm{id}$ in the point weak* topology.
- The infimum of such constants C is called the Cowling–Haagerup constant and is denoted by $\Lambda_{cb}(\mathcal{M})$.
- A vNa with w*CBAP is also said to be weakly amenable.
- If $\Lambda_{cb}(\mathcal{M}) = 1$, $\mathcal{M}$ is said to have the weak* completely contractive approximation property (w*CCAP) or CMAP.
- Following Ozawa–Popa, $\mathcal{M}$ is called strongly solid if the normalizer $\mathcal{N}_{\mathcal{M}}(P) := \{u \in \mathcal{U}(\mathcal{M}) : uPu^* = P\}$ of any diffuse amenable subalgebra $P \subset \mathcal{M}$ generates an amenable vNa. Here $\mathcal{U}(\mathcal{M})$ is the set of unitary operators in $\mathcal{M}$.

Operator algebraic properties

Theorem (Junge–Z '15)

Γ_Q *has w*CCAP and is strongly solid provided* $\max_{1\le i,j\le N}|q_{ij}|<1$.

- Some ideas in proof of CCAP: Find q such that $\max_{i,j}|q_{ij}|<q<1$. Let $Q=q\tilde{Q}$, where $\tilde{Q}=(\tilde{q}_{ij})$ satisfies $\max_{i,j}|\tilde{q}_{ij}|<1$. Let $X_i^q=l_i+l_i^*$ and $x_{i,j}=X^{\tilde{Q}\otimes\mathbb{1}_n}(f_i\otimes e_j)$.
- Let $\pi_{\mathcal{U}}:\Gamma_Q(\mathbb{R}^N)\to\prod_{m,\mathcal{U}}\Gamma_q(\mathbb{R}^m)\bar{\otimes}\Gamma_{\tilde{Q}\otimes\mathbb{1}_m}$ be a $*$-homomorphism given by

$$\pi_{\mathcal{U}}(X_i^Q)=\Big(\frac{1}{\sqrt{m}}\sum_{k=1}^m X_k^q\otimes x_{i,k}\Big)^{\bullet}.$$

Then $\pi_{\mathcal{U}}$ is trace preserving. Therefore, Γ_Q is isomorphic to the von Neumann algebra generated by $\pi_{\mathcal{U}}(X_i^Q)$.

Some ideas in the proof

- Avsec ('11): $\exists$ a net of finite rank maps $\varphi_\alpha(A) : \Gamma_q(\mathbb{R}^m) \to \Gamma_q(\mathbb{R}^m)$, $\varphi_\alpha(A) \to \mathrm{id}$ in the point weak* topology, $\|\varphi_\alpha(A)\|_{cb} \leq 1 + \varepsilon$.
- $$\begin{array}{ccc} \Gamma_Q & \overset{\pi_{\mathcal{U}}}{\hookrightarrow} & \prod_{m,\mathcal{U}} \Gamma_q(\mathbb{R}^m)\bar{\otimes}\Gamma_{\tilde{Q}\otimes \mathbb{1}_m} \\ \Big\downarrow \psi_\alpha & & \Big\downarrow \varphi_\alpha(A)\otimes \mathrm{id} \\ \Gamma_Q & \overset{\pi_{\mathcal{U}}}{\hookrightarrow} & \prod_{m,\mathcal{U}} \Gamma_q(\mathbb{R}^m)\bar{\otimes}\Gamma_{\tilde{Q}\otimes \mathbb{1}_m} \end{array}$$
- Strong solidity is more complicated and follows literally the same strategy of Houdayer–Shlyakhtenko ('11) which is an extension of Ozawa–Popa ('10).

Free transport: Background

- A general question: What's the relation between $\Gamma_Q(\mathbb{R}^N)$ and $L(\mathbb{F}_N) \cong \Gamma_0(\mathbb{R}^N)$.
- A breakthrough of Guionnet–Shlyakhtenko (2013(4)) develops a free transport theory and, together with Dabrowski's result on the existence of conjugate variables, proves that $\Gamma_q(\mathbb{R}^N) \cong L(\mathbb{F}_N)$ provided q is small enough.
- Suppose $X = (X_n)_{n\in I}$ is a sequence of algebraically free self-adjoint operators generating a tracial von Neumann algebra (M, τ). Let $\mathscr{P}$ denote the noncommutative polynomials in X_n, $n \in I$. Voiculescu defined for each n the n-th free difference quotient $\partial_n \colon \mathscr{P} \to \mathscr{P} \otimes \mathscr{P}^{op}$ by

$$\partial_n(X_k) = \delta_{n=k} 1 \otimes 1$$

$$\partial_n(AB) = \partial_n(A) \cdot B + A \cdot \partial_n(B), \qquad A, B \in \mathscr{P}.$$

Free transport: Set-up

- Voiculescu ('02) defined for each $n \in \mathbb{N}$ the n-th cyclic derivative $\mathscr{D}_n \colon \mathscr{P} \to \mathscr{P}$ by

$$\mathscr{D}_n(p) = \sum_{p = AX_nB} BA,$$

for $p \in \mathscr{P}$ a monomial and extend linearly to $\mathscr{P}$. For $P \in \mathscr{P}$, the sequence $\mathscr{D}P := (\mathscr{D}_n P)_{n \in I}$ is called the cyclic gradient of P.

- The n-th conjugate variable is $\xi_n \in L^2 M$ such that

$$\langle P, \xi_n \rangle_\tau = \langle \partial_n P, 1 \otimes 1 \rangle_{\tau \otimes \tau^{op}}, \qquad \forall P \in \mathscr{P}.$$

Clearly, $\xi_n = \partial_n^*(1 \otimes 1)$, provided it exists.

- For a polynomial $P = \sum_{p \text{ monomial}} c_p p \in \mathscr{P}$ and for each $R > 0$, define

$$\|P\|_R = \sum_p |c_p| R^{\deg(p)}.$$

Free transport

- Let $X = (X_n)_{n\in I}$ be operators in a vNa $\mathcal{M}$ with a faithful normal state φ. Recall that the joint law of X with respect to φ is a linear functional φ_X defined on noncommutative polynomials by

$$\varphi_X(T_{i_1}\cdots T_{i_d}) = \varphi(X_{i_1}\cdots X_{i_d}), \qquad \forall \underline{i} \in I^d.$$

- Let $Z = (Z_n)_{n\in I}$ be another sequence in a vNa $\mathcal{N}$ with a faithful normal state ψ, and let ψ_Z be the joint law of Z with respect to ψ. Observe that if $\varphi_X = \psi_Z$ then

$$W^*(X_n : n \in I) \cong W^*(Z_n : n \in I),$$

since φ and ψ are faithful normal states.

Definition (Guionnet–Shlyakhtenko)

Transport from φ_X to ψ_Z is a sequence $Y = (Y_n)_{n\in I} \subset W^*(X_n : n \in I)$ whose joint law with respect to φ is equal to ψ_Z. That is, $\varphi_Y = \psi_Z$.

Free transport theorem of Guionnet–Shlyakhtenko

Theorem (Guionnet–Shlyakhtenko '14)

Let $R > R' > 4$. Let $X_1, ..., X_N \in (M, \tau)$ be semicircular variables. Then there exists a universal constant $C = C(R, R') > 0$ such that whenever $W \in \mathscr{P}^{(R+1)}$ satisfies $\|W\|_{R+1} < C$, there is $G \in \mathscr{P}^{(R')}$ so that

1. *If we set $Y_j = \mathscr{D}_j G$, then $Y_1, ..., Y_N \in \mathscr{P}^{(R')}$ has law τ_V, with $V = \frac{1}{2}\sum X_j^2 + W$;*
2. *$S_j = H_j(Y_1, ..., Y_n)$ for some $H \in \mathscr{P}^{(R')}$;*

In particular, there are trace preserving isomorphisms

$$C^*(\tau_V) = C^*(X_1, ..., X_N), \quad W^*(\tau_V) \cong L(\mathbb{F}_N).$$

Observation: Assume moreover that $\xi_i^* = \xi_i = \partial_i^*(X_i)$ belongs to $\mathscr{P}^{(R+1)}$ for some $R > 4$. Voiculescu showed that $V = \frac{1}{2}\Sigma(\sum_{j=1}^n X_j\xi_j + \xi_j X_j)$ satisfies $\xi_j = \mathscr{D}_j V$ and thus the theorem applies. Here $\Sigma = \mathscr{N}^{-1}$.

Isomorphism theorem

- Dabrowski showed that the conjugate variables $\xi_i, i = 1, ..., N$ exist for q-Gaussian variables (X_i^q) provided $q < q_0(N)$ small.

Theorem (Guionnet–Shlyakhtenko '14)

$\Gamma_q(\mathbb{R}^N) \cong \Gamma_0(\mathbb{R}^N) \cong L(\mathbb{F}_N)$ *and* $C^*(X_1^q, ..., X_N^q) \cong C^*(S_1, ..., S_N)$ *as long as* $q < q_0(N)$.

- How about $\Gamma_Q(\mathbb{R}^N)$?

Theorem (Nelson–Z '15a)

Let $Q = (q_{ij})$ *be a symmetric* $N \times N$ *matrix with* $N \in \{2, 3, \ldots\}$ *and* $q_{ij} \in (-1, 1)$. *Then* $\Gamma_Q(\mathbb{R}^N) \cong \Gamma_0(\mathbb{R}^N) \cong L(\mathbb{F}_N)$ *and* $C^*(X_1^Q, \ldots, X_N^Q) \cong C^*(S_1, ..., S_N)$ *as long as* $\max_{i,j} |q_{ij}| < q_0(N)$.

Proof of isomorphism theorem: Some ideas

- With Guionnet–Shlyakhtenko's transport theorem, we only need to construct suitable conjugate variables. Or, extend the argument of Dabrowski to the mixed q case.
- The derivation $\partial_j^{(Q)}$

$$\partial_j^{(Q)} : \mathbb{C}\langle X_1^Q, \dots, X_N^Q\rangle \to \mathcal{B}(L^2(\Gamma_Q(\mathbb{R}^N))),$$
$$\partial_j^{(Q)}(X) = [X, r_j] := Xr_j - r_jX.$$

- The derivation Ξ_i

$$\Xi_i : \mathcal{F}_Q(\mathbb{R}^N) \to \mathcal{F}_Q(\mathbb{R}^N),$$
$$\Xi_i(e_{j_1} \otimes \cdots \otimes e_{j_n}) = q_{ij_1} \cdots q_{ij_n} e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

(NB: If $q_{ij} \equiv q$, $\Xi_1 = \Xi_2 = \cdots$.)

Proof of isomorphism theorem: Some ideas

- If q is small enough, Ξ_i is invertible and Ξ_i^{-1} can be written as a noncommutative power series. This is based on a crucial estimate of Bożejko ('98) (see Dykema–Nica '93 for fixed q case):

$$\|(P^{(n)})^{-1}\| \leq \left[(1-q)\prod_{k=1}^{\infty}\frac{1+q^k}{1-q^k}\right]^n.$$

Here $\langle \xi, \eta\rangle_Q = \delta_{n,m}\langle \xi, P^{(n)}\eta\rangle_0$ for $\xi \in (\mathbb{R}^N)^{\otimes n}, \eta \in (\mathbb{R}^N)^{\otimes m}$; $q = \max_{1\leq i,j\leq N}|q_{ij}|$.

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$$\xi_j(Y_1,\dots,Y_N) := (\Xi_j^{-1})^*(Y_1,\dots,Y_N)\#Y_j$$
$$- m\circ(1\otimes\tau_Q\otimes 1)\circ(1\otimes\partial_j^{(Q)} + \partial_j^{(Q)}\otimes 1)[(\Xi_j^{-1})^*(Y_1,\dots,Y_N)],$$

where $(a\otimes b)\#x = axb$ and $m(a\otimes b) = ab$.

Proof of isomorphism theorem: Some ideas

- Voiculescu ('98) implies

$$\xi_j := \xi_j(X_1, \dots, X_N) = \partial_j^{(Q)*}((\Xi_j^{-1})^*).$$

- $\partial_n^{(Q)}(\cdot) = \partial_n(\cdot)\#\Xi_n$ and

$$\langle \xi_j, P\rangle_{\tau_Q} = \langle (\Xi_j^{-1})^*, \partial_j^{(Q)}(P)\rangle_{HS} = \langle 1 \otimes 1^\circ, \partial_j(P)\rangle_{\tau_Q \otimes \tau_Q^{op}};$$

- Define

$$V(Y_1, \dots, Y_N) = \Sigma\left(\frac{1}{2}\sum_{i=1}^{N} \xi_i(Y_1, \dots, Y_N)Y_i + Y_i\xi_i(Y_1, \dots, Y_N)\right).$$

Guionnet–Shlyakhtenko's theorem yields the isomorphism result.

The case of infinite variables

- Question (Voiculescu, others(?)): Are the methods of Guionnet–Shlyakhtenko valid for an infinite number of variables?
- The difficulty: In the fixed q case, Dabrowski's theorem on the existence of conjugate variables requires $|q| < q_0(N)$ and $q_0(N) \to 0$ as $N \to \infty$. So free transport for infinite variables (assume it is valid) does not apply to $\Gamma_q(\ell^2)$.
- Note also that the structure array $(q_{ij} \equiv q)_{i,j\in\mathbb{N}}$ of $\Gamma_q(\ell^2)$ is not even bounded as an operator on ℓ^2 unless $q = 0$. So $(q_{ij} \equiv q)_{i,j\in\mathbb{N}}$ is not small which is required in free transport.
- However, if q_{ij} decays very fast, one may have hope.

Some infinite variable formalism

- Let $\mathscr{P}^{(R)}_\infty$ denote $\ell^\infty(\mathbb{N}, \mathscr{P}^{(R)})$, the set of uniformly bounded sequences of elements of $\mathscr{P}^{(R)}$, with norm

$$\|(P_n)_{n\in\mathbb{N}}\|_{R,\infty} := \sup_n \|P_n\|_R.$$

Definition

Given $W \in \mathscr{P}^{(R)}$ for $R > \sup_n \|X_n\|$, we say that τ_X is a *free Gibbs state with quadratic potential perturbed by* W (or a *free Gibbs state with perturbation* W) if

$$\tau([X_n + \mathscr{D}_n W]P) = \tau \otimes \tau^{op}(\partial_n P) \qquad \forall P \in \mathscr{P},\ \forall n \in \mathbb{N}.$$

- Note that $V_0 = \frac{1}{2}\sum_{i=1}^N X_i^2 \in \mathbb{C}\langle X_1, \ldots, X_N\rangle$ does not converge in R-norm as $N \to \infty$, we have modified the definition to refer only to the perturbation $W \in \mathscr{P}^{(R)}$.

Free transport for infinite variables

Theorem (Nelson-Z '15b)

Let $X = (X_n)_{n\in\mathbb{N}}$ *be free semicircular variables generating the von Neumann algebra* $\mathcal{M} \cong L(\mathbb{F}_\infty)$, *with trace* τ, *and let* $R > S > 4$. *Suppose* $\mathcal{N}$ *is a von Neumann algebra with a faithful normal state* ψ, *and the joint law of* $Z = (Z_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ *with respect to* ψ *is a free Gibbs state with perturbation* $W = W^* = \mathscr{P}^{(R+1)}$. *If* $\|W\|_{R+1} \leq \frac{e\log\left(\frac{R+1}{S+1}\right)}{2}$, *then transport from* τ_X *to* ψ_Z *is given by* $Y = X + \mathscr{D}g \in \mathscr{P}_\infty^{(S)}$ *for some* $g = g^* \in \mathscr{P}^{(S)}$. *This transport satisfies* $\|Y - X\|_{S,\infty} \to 0$ *as* $\|W\|_{R+1} \to 0$, *and is invertible in the sense that* $H(Y) = X$ *for some* $H \in \mathscr{P}^{(2.4)}$. *In particular, there are trace-preserving isomorphisms:*

$$C^*(Z_n : n\in\mathbb{N}) \cong C^*(X_n : n\in\mathbb{N}) \qquad \text{and} \qquad W^*(Z_n : n\in\mathbb{N}) \cong L(\mathbb{F}_\infty).$$

Isomorphism result

- $Q_i(p) := \sum_{j\geq 1} |q_{ij}|^p$.
- $\pi(Q,n,R) := \frac{[(R(1-q)+1)Q_n(\frac{1}{2})]^2}{(1-2q)^2-[(R(1-q)+1)Q_n(\frac{1}{2})]^2}$.

Theorem (Nelson-Z '15b)

Let $R > 5$. If the structure array Q for the mixed q-Gaussian algebra Γ_Q satisfies $0 < \pi(Q,n,R) < 1$ for all $n \in \mathbb{N}$, and

$$\sum_{n\in\mathbb{N}} \frac{\pi(Q,n,R)}{1-\pi(Q,n,R)} < \frac{e\log\left(\frac{R}{5}\right)}{R\left(R+\frac{4}{R-\sup_n \|X_n^Q\|}\right)},$$

then $\Gamma_Q \cong L(\mathbb{F}_\infty)$ and $C^*(X_n^Q : n\in\mathbb{N}) \cong C^*(X_n : n\in\mathbb{N})$, where $\{X_n\}_{n\in\mathbb{N}}$ is a free semicircular family.

- Example: If $q_{ij} = q^{i+j-1}$, then one can take $|q| < 0.0002488$ and $R = 6.7$.

Some words about the proof

- The construction of free transport follows the same steps as Guionnet–Shlyakhtenko along with some technical extensions and modification suitable to infinite variable setting. e.g. if $R > S > \max\{1, \sup_n \|X_n\|\}$,

$$\sum_{n\in I} \|\partial_n \Sigma P\|_{R\otimes_\pi R} \leq \frac{1}{R}\|P\|_R \quad \text{and} \quad \sum_{n\in I} \|\partial_n P\|_{S\otimes_\pi S} \leq \mathscr{C}(R,S)\|P\|_R,$$

where $\|\cdot\|_{R\otimes_\pi R}$ is the projective tensor norm on $\mathscr{P}^{(R)} \otimes (\mathscr{P}^{(R)})^{op}$:

$$\|\eta\|_{R\otimes_\pi R} := \inf\left\{\sum_i \|A_i\|_R \|B_i\|_R \colon \eta = \sum_i A_i \otimes B_i\right\}.$$

Some words about the proof

- The proof of the existence of conjugate variables (and potential) follows similar strategy used by Dabrowski and our previous work on finite variable case along with more careful analysis of certain estimation.
- For example, in the finite variable case, if $q := \max_{1\le i,j\le N} |q_{ij}|$ satisfies $q^2 N < 1$ then $\Xi_j \in HS(\mathcal{F}_Q)$.
- In the infinite variable case, suppose $Q_n(2) < 1$. Then Ξ_n is a Hilbert–Schmidt operator with $\|\Xi_n\|_{\mathrm{HS}} = (1 - Q_n(2))^{-1/2}$.

Open problems

So far certain small perturbations of free semicircular systems in Bożejko–Speicher algebras have been understood. How about regimes far away from $q = 0$?

Thank you for your attention!