On Operator-Valued Bi-Free Distributions

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Bi-Free with Amalgamation

- Let B be a unital algebra.
- Let $\mathcal{X}$ be a B-B-bimodule that may be decomposed as $\mathcal{X} = B \oplus \mathcal{X}^{\perp}$.
- The projection map $p : \mathcal{X} \to B$ is given by $p(b \oplus \eta) = b$.
- Thus $p(b \cdot \xi \cdot b') = bp(\xi)b'$.
- For $b \in B$, define $L_b, R_b \in \mathcal{L}(\mathcal{X})$ by $L_b(\xi) = b \cdot \xi$ and $R_b(\xi) = \xi \cdot b$.
- Define $E : \mathcal{L}(\mathcal{X}) \to B$ by $E(T) = p(T(1_B \oplus 0))$.
- $E(L_b R_{b'} T) = p(L_b R_{b'}(E(T) \oplus \eta)) = p(bE(T)b' \oplus \eta') = bE(T)b'$.
- $E(TL_b) = p(T(b \oplus 0)) = E(TR_b)$.

B-B-Non-Commutative Probability Space

$E(L_b R_{b'} T) = bE(T)b'$ and $E(TL_b) = E(TR_b)$.

Definition

A *B-B-non-commutative probability space* is a triple $(\mathcal{A}, E, \varepsilon)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, $\varepsilon : B \otimes B^{\mathrm{op}} \to \mathcal{A}$ is a unital homomorphism such that $\varepsilon|_{B\otimes I}$ and $\varepsilon|_{I\otimes B^{\mathrm{op}}}$ are injective, and $E : \mathcal{A} \to B$ is a linear map such that

$$E(\varepsilon(b_1 \otimes b_2)T) = b_1 E(T) b_2 \quad \text{and} \quad E(T\varepsilon(b \otimes 1_B)) = E(T\varepsilon(1_B \otimes b)).$$

Denote $L_b = \varepsilon(b \otimes 1_B)$ and $R_b = \varepsilon(1_B \otimes b)$.

Every B-B-non-commutative probability space can be embedded into $\mathcal{L}(\mathcal{X})$ for some B-B-bimodule $\mathcal{X}$.

B-NCPS via B-B-NCPS

Definition

Let $(\mathcal{A}, E, \varepsilon)$ be a B-B-ncps. The unital subalgebras of $\mathcal{A}$ defined by

$$\mathcal{A}_\ell := \{Z \in \mathcal{A} \mid ZR_b = R_b Z \text{ for all } b \in B\} \text{ and}$$
$$\mathcal{A}_r := \{Z \in \mathcal{A} \mid ZL_b = L_b Z \text{ for all } b \in B\}$$

are called the *left* and *right algebras of* $\mathcal{A}$ respectively. A pair of algebras (A_1, A_2) is said to be a *pair of B-faces* if

$$\{L_b\}_{b\in B} \subseteq A_1 \subseteq \mathcal{A}_\ell \quad \text{and} \quad \{R_b\}_{b\in B^{\mathrm{op}}} \subseteq A_2 \subseteq \mathcal{A}_r.$$

Note $(\mathcal{A}_\ell, E)$ is a B-ncps where $\{L_b\}_{b\in B}$ is the copy of B. Indeed for $T \in \mathcal{A}_\ell$ and $b_1, b_2 \in B$,

$$E(L_{b_1} T L_{b_2}) = E(L_{b_1} T R_{b_2}) = E(L_{b_1} R_{b_2} T) = b_1 E(T) b_2.$$

Similarly $(\mathcal{A}_r, E)$ is as B^{op}-ncps where $\{R_b\}_{b\in B^{\mathrm{op}}}$ is the copy of B^{op}.

Bi-Free Independence with Amalgamation

Definition

Let $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ be a B-B-ncps. Pairs of B-faces $(A_{\ell,1}, A_{r,1})$ and $(A_{\ell,2}, A_{r,2})$ of $\mathcal{A}$ are said to be *bi-freely independent with amalgamation over B* if there exist B-B-bimodules $\mathcal{X}_k$ and unital B-homomorphisms $\alpha_k : A_{\ell,k} \to \mathcal{L}(\mathcal{X}_k)_\ell$ and $\beta_k : A_{r,k} \to \mathcal{L}(\mathcal{X}_k)_r$ such that the following diagram commutes:

$$\begin{array}{ccccc}
A_{\ell,1} * A_{r,1} * A_{\ell,2} * A_{r,2} & \xrightarrow{\ i\ } & \mathcal{A} & \xrightarrow{\ E_{\mathcal{A}}\ } & B \\
\Big\downarrow {\scriptstyle \alpha_1 * \beta_1 * \alpha_2 * \beta_2} & & & & \Big\uparrow {\scriptstyle E_{\mathcal{L}(\mathcal{X}_1 * \mathcal{X}_2)}} \\
\mathcal{L}(\mathcal{X}_1)_\ell * \mathcal{L}(\mathcal{X}_1)_r * \mathcal{L}(\mathcal{X}_2)_\ell * \mathcal{L}(\mathcal{X}_2)_r & & \xrightarrow{\ \lambda_1 * \rho_1 * \lambda_2 * \rho_2\ } & & \mathcal{L}(\mathcal{X}_1 * \mathcal{X}_2)
\end{array}$$

Operator-Valued Bi-Freeness and Mixed Cumulants

Theorem (Charlesworth, Nelson, Skoufranis; 2015)

Let $(\mathcal{A}, E, \varepsilon)$ be a B-B-ncps and let $\{(A_{\ell,k}, A_{r,k})\}_{k\in K}$ be pairs of B-faces. Then the following are equivalent:

- $\{(A_{\ell,k}, A_{r,k})\}_{k\in K}$ *are bi-free over B.*
- *For all* $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to K$, *and* $Z_m \in A_{\chi(m),\epsilon(m)}$,

$$E(Z_1 \cdots Z_m) = \sum_{\pi \in BNC(\chi)} \left[\sum_{\substack{\sigma \in BNC(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma) \right] \mathcal{E}_\pi(Z_1, \ldots, Z_m)$$

- *For all* $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to K$ *non-constant, and* $Z_m \in A_{\chi(m),\epsilon(m)}$,

$$\kappa_\chi(Z_1, \ldots, Z_n) = 0.$$

Bi-Multiplicative Functions

κ and $\mathcal{E}$ are special functions where $\mathcal{E}_{1_\chi}(Z_1, \ldots, Z_n) = E(Z_1 \cdots Z_n)$.

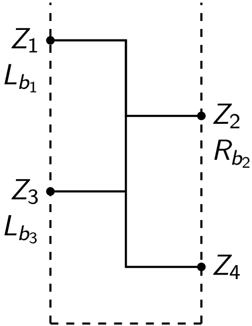
Given $(\mathcal{A}, E, \varepsilon)$, a *bi-multiplicative function* Φ is a map

$$\Phi : \bigcup_{n\geq 1} \bigcup_{\chi:\{1,\ldots,n\}\to\{\ell,r\}} BNC(\chi) \times \mathcal{A}_{\chi(1)} \times \cdots \times \mathcal{A}_{\chi(n)} \to B$$

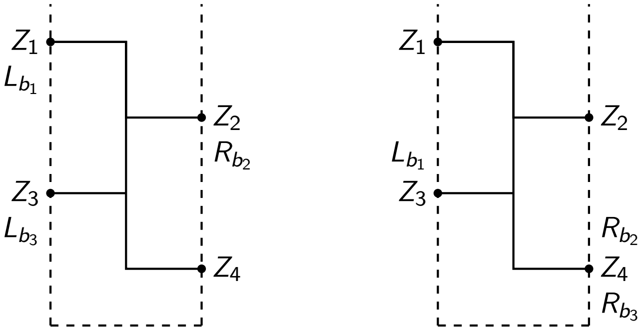
whose properties are described as follows:

Property 1 of Bi-Multiplicative Functions

$$\Phi_{1_\chi}(Z_1 L_{b_1}, Z_2 R_{b_2}, Z_3 L_{b_3}, Z_4)$$

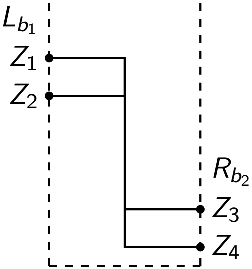


$$\Phi_{1_\chi}(Z_1 L_{b_1}, Z_2 R_{b_2}, Z_3 L_{b_3}, Z_4) = \Phi_{1_\chi}(Z_1, Z_2, L_{b_1} Z_3, R_{b_2} Z_4 R_{b_3}).$$



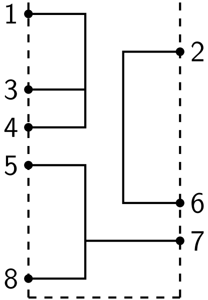
Property 2 of Bi-Multiplicative Functions

$$\Phi_{1_\chi}(L_{b_1}Z_1, Z_2, R_{b_2}Z_3, Z_4) = b_1\Phi_{1_\chi}(Z_1, Z_2, Z_3, Z_4)b_2.$$



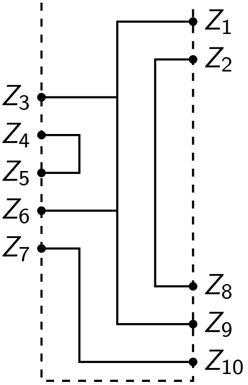
Property 3 of Bi-Multiplicative Functions

$$\Phi_\pi(Z_1, \ldots, Z_8) = \Phi_{1_{\chi_1}}(Z_1, Z_3, Z_4)\Phi_{1_{\chi_2}}(Z_5, Z_7, Z_8)\Phi_{1_{\chi_3}}(Z_2, Z_6).$$

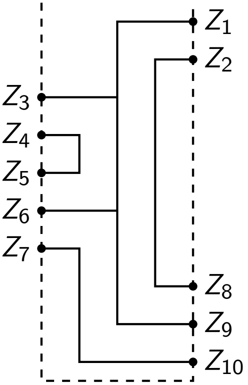


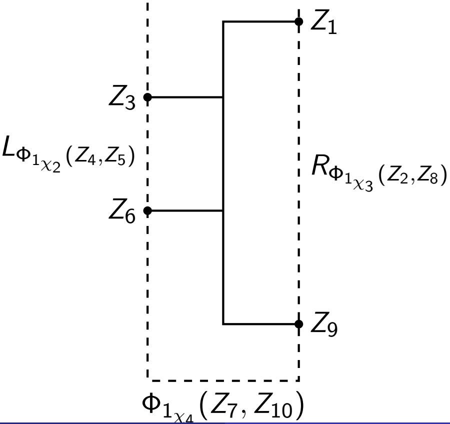
Property 4 of Bi-Multiplicative Functions

$\Phi_\pi(Z_1, \ldots, Z_{10})$



$$\Phi_\pi(Z_1, \ldots, Z_{10}) = \Phi_{1_{\chi_1}}\left(Z_1, Z_3, L_{\Phi_{1_{\chi_2}}(Z_4,Z_5)}Z_6, R_{\Phi_{1_{\chi_3}}(Z_2,Z_8)}Z_9 R_{\Phi_{1_{\chi_4}}(Z_7,Z_{10})}\right)$$





Amalgamating Over Matrices

- Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.
- $\mathcal{M}_N(\mathcal{A})$ is naturally a $\mathcal{M}_N(\mathbb{C})$-ncps where the expectation map $\varphi_N : \mathcal{M}_N(\mathcal{A}) \to \mathcal{M}_N(\mathbb{C})$ is defined via

$$\varphi_N([A_{i,j}]) = [\varphi(A_{i,j})].$$

- If A_1, A_2 are unital subalgebras of $\mathcal{A}$ that are free with respect to φ, then $\mathcal{M}_N(A_1)$ and $\mathcal{M}_N(A_2)$ are free with amalgamation over $\mathcal{M}_N(\mathbb{C})$ with respect to φ_N.
- Is there a bi-free analogue of this result?
- Is $\mathcal{M}_N(\mathcal{A})$ a $\mathcal{M}_N(\mathbb{C})$-$\mathcal{M}_N(\mathbb{C})$-ncps?

B-B-NCPS Associated to $\mathcal{A}$

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let B be a unital algebra. Then $\mathcal{A} \otimes B$ is a B-B-bi-module where

$$L_b(a \otimes b') = a \otimes bb', \qquad \text{and} \qquad R_b(a \otimes b') = a \otimes b'b.$$

If $p : \mathcal{A} \otimes B \to B$ is defined by

$$p(a \otimes b) = \varphi(a)b,$$

then $\mathcal{L}(\mathcal{A} \otimes B)$ is a B-B-ncps with

$$E(Z) = p(Z(1_{\mathcal{A}} \otimes 1_B)).$$

If $X, Y \in \mathcal{A}$, defined $L(X \otimes b) \in \mathcal{L}(\mathcal{A} \otimes B)_\ell$ and $R(Y \otimes b) \in \mathcal{L}(\mathcal{A} \otimes B)_r$ via

$$L(X \otimes b)(a \otimes b') = Xa \otimes bb' \qquad \text{and} \qquad R(Y \otimes b)(a \otimes b') = Ya \otimes b'b.$$

Bi-Freeness Preserved Under Tensoring

Theorem (Skoufranis; 2015)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $\{(A_{\ell,k}, A_{r,k})\}_{k\in K}$ be bi-free pairs of faces with respect to φ. If B is a unital algebra, then $\{(L(A_{\ell,k} \otimes B), R(A_{r,k} \otimes B))\}_{k\in K}$ are bi-free over B with respect to E as described above.

Proof Sketch.

If $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $Z_m = L(X_m \otimes b_m)$ if $\chi(m) = \ell$, and $Z_m = R(X_m \otimes b_m)$ if $\chi(m) = r$, then

$$E(Z_1 \cdots Z_n) = \varphi(X_1 \cdots X_n) \otimes b_{s_\chi(1)} \cdots b_{s_\chi(n)}$$

Also

$$\mathcal{E}_\pi(Z_1 \cdots Z_n) = \varphi_\pi(X_1, \ldots, X_n) \otimes b_{s_\chi(1)} \cdots b_{s_\chi(n)}. \qquad \square$$

Bi-Matrix Models - Creation/Annihilation on a Fock Space

Theorem (Skoufranis; 2015)

Given an index set K, an $N \in \mathbb{N}$, and an orthonormal set of vectors $\{h^k_{i,j} \mid i, j \in \{1, \ldots, N\}, k \in K\} \subseteq \mathcal{H}$, let

$$L_k(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} L(l(h^k_{i,j}) \otimes E_{i,j}), \quad L^*_k(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} L(l(h^k_{j,i})^* \otimes E_{i,j})$$

$$R_k(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} R(r(h^k_{i,j}) \otimes E_{i,j}), \quad R^*_k(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} R(r(h^k_{j,i})^* \otimes E_{i,j}).$$

If $E : \mathcal{L}(\mathcal{L}(\mathcal{F}(\mathcal{H})) \otimes \mathcal{M}_N(\mathbb{C})) \to \mathcal{M}_N(\mathbb{C})$ is the expectation, the joint distribution of $\{L_k(N), L^*_k(N), R_k(N), R^*_k(N)\}_{k \in K}$ with respect to $\frac{1}{N}\mathrm{Tr} \circ E$ is equal the joint distribution of $\{l(h^k), l^*(h^k), r(h^k), r^*(h^k)\}_{k \in K}$ with respect to φ where $\{h^k\}_{k \in K} \subseteq \mathcal{H}$ is an orthonormal set.

Bi-Matrix Models - q-Deformed Fock Space

- Moreover $(L(I_{\mathcal{F}(\mathcal{H})} \otimes \mathcal{M}_N(\mathbb{C})), R(I_{\mathcal{F}(\mathcal{H})} \otimes \mathcal{M}_N(\mathbb{C})))$ and $\{(L_k(N), L_k^*(N)), (R_k(N), R_k^*(N))\}_{k \in K}$ are bi-free.
- Considering the q-deformed Fock space, the joint distribution of the q-deformed versions

$$\{(L_k(N), L_k^*(N), L_k^t(N), L_k^{*,t}(N)), (R_k(N), R_k^*(N), R_k^t(N), R_k^{*,t}(N))\}_{k \in K}$$

with respect to $\frac{1}{N}\mathrm{Tr} \circ E$ asymptotically equals the joint distribution of

$$\{(l(h^k), l^*(h^k), l(h_0^k), l^*(h_0^k)), (r(h^k), r^*(h^k), r(h_0^k), r^*(h_0^k))\}_{k \in K}$$

with respect to φ where $\{h^k, h_0^k\}_{k \in K} \subseteq \mathcal{H}$ is an orthonormal set, and are asymptotically bi-free from

$$(L(I_{\mathcal{F}_q(\mathcal{H})} \otimes \mathcal{M}_N(\mathbb{C})), R(I_{\mathcal{F}_q(\mathcal{H})} \otimes \mathcal{M}_N(\mathbb{C}))).$$

Amalgamating over a Smaller Subalgebra

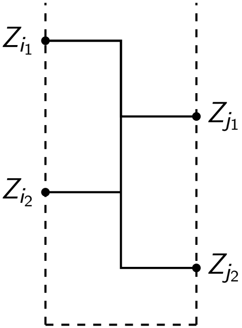
- Suppose $(\mathcal{A}, E, \varepsilon)$ is a B-B-ncps. Let D be a unital subalgebra of $\mathcal{B}$, and let $F : B \to D$ be such that $F(1_B) = 1_D$ and $F(d_1 b d_2) = d_1 F(b) d_2$ for all $d_1, d_2 \in D$ and $b \in B$.
- Note $(\mathcal{A}, F \circ E, \varepsilon|_{D \otimes D^{op}})$ is a D-D-ncps since

$$F(E(L_d R_{d'} Z)) = F(dE(Z)d') = dF(E(Z))d'$$
$$F(E(ZL_d)) = F(E(ZR_d))$$

  for all $d, d' \in D$ and $Z \in \mathcal{A}$. Note $\mathcal{A}_{\ell,B} \subseteq \mathcal{A}_{\ell,D}$ and $\mathcal{A}_{r,B} \subseteq \mathcal{A}_{r,D}$.
- How do the B-valued and D-valued distributions interact?
- How can one described said distributions?

Operator-Valued Bi-Free Distributions

Suppose $\{Z_i\}_{i \in I} \subseteq \mathcal{A}_\ell$ and $\{Z_j\}_{j \in J} \subseteq \mathcal{A}_r$. Suppose we wanted to describe all B-valued moments involving Z_{i_1}, Z_{j_1}, Z_{i_2}, and Z_{j_2} each occurring once in that order.



Operator-Valued Bi-Free Distributions

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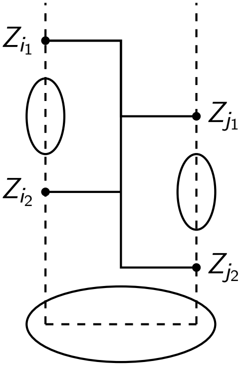

Operator-Valued Bi-Free Distributions

Suppose $\{Z_i\}_{i\in I} \subseteq \mathcal{A}_\ell$ and $\{Z_j\}_{j\in J} \subseteq \mathcal{A}_r$. Let $Z = \{Z_i\}_{i\in I} \sqcup \{Z_j\}_{j\in J}$. For $n \geq 1$, $\omega : \{1, \ldots, n\} \to I \sqcup J$, and $b_1, \ldots, b_{n-1} \in B$, let

$\mu^B_{Z,\omega}(b_1, \ldots, b_{n-1}) =$ Expectation of $Z_{\omega(1)}, \ldots, Z_{\omega(n)}$ in that order with $b_1, \ldots, b_{n-1}$ in-between gaps with respect to the χ-ordering.

$\kappa^B_{Z,\omega}(b_1, \ldots, b_{n-1}) =$ Cumulant of $Z_{\omega(1)}, \ldots, Z_{\omega(n)}$ in that order with $b_1, \ldots, b_{n-1}$ in-between gaps with respect to the χ-ordering.

Similarly, we can define $\mu^D_{Z,\omega}(d_1, \ldots, d_{n-1})$ and $\kappa^D_{Z,\omega}(d_1, \ldots, d_{n-1})$.

D-Valued Cumulants from B-Valued Cumulants

Theorem (Skoufranis; 2015)

If

$$\kappa^B_{Z,\omega}(d_1, \ldots, d_{n-1}) \in D$$

for all $n \geq 1$, $\omega : \{1, \ldots, n\} \to I \sqcup J$, *and* $d_1, \ldots, d_{n-1} \in D$, *then*

$$\kappa^D_{Z,\omega}(d_1, \ldots, d_{n-1}) = \kappa^B_{Z,\omega}(d_1, \ldots, d_{n-1})$$

for all $n \geq 1$, $\omega : \{1, \ldots, n\} \to I \sqcup J$, *and* $d_1, \ldots, d_{n-1} \in D$.

Bi-Free from B over D

Theorem (Skoufranis; 2015)

Assume that $F : B \to D$ satisfies the following faithfulness condition:

- *if $b_1 \in B$ and $F(b_2 b_1) = 0$ for all $b_2 \in B$, then $b_1 = 0$.*

Then $(\mathrm{alg}(\varepsilon(D \otimes 1_D), \{Z_i\}_{i \in I}), \mathrm{alg}(\varepsilon(1_D \otimes D^{\mathrm{op}}), \{Z_j\}_{j \in J}))$ is bi-free from $(\varepsilon(B \otimes 1_B), \varepsilon(1_B \otimes B^{\mathrm{op}}))$ with amalgamation over D if and only if

$$\kappa^B_{Z,\omega}(b_1, \ldots, b_{n-1}) = F\left(\kappa^B_{Z,\omega}(F(b_1), \ldots, F(b_{n-1}))\right) \tag{1}$$

for all $n \geq 1$, $\omega : \{1, \ldots, n\} \to I \sqcup J$, and $b_1, \ldots, b_{n-1} \in B$. Alternatively, equation (1) is equivalent to

$$\kappa^B_{Z,\omega}(b_1, \ldots, b_{n-1}) = \kappa^D_{Z,\omega}(F(b_1), \ldots, F(b_{n-1})). \tag{2}$$

This is a bi-free analogue of a result of Nica, Shlyakhtenko, and Speicher.

Bi-R-Cyclic Families

Definition

Let I and J be disjoint index sets and let

$$\{[Z_{k;i,j}]\}_{k\in I} \cup \{[Z_{k;i,j}]\}_{k\in J} \subseteq \mathcal{M}_N(\mathcal{A}).$$

The pair

$$(\{[Z_{k;i,j}]\}_{k\in I}, \{[Z_{k;i,j}]\}_{k\in J})$$

is said to be *R-cyclic* if for every $n \geq 1$, $\omega : \{1, \ldots, n\} \to I \sqcup J$, and $1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq d$,

$$\kappa^{\mathbb{C}}_{\chi_\omega}(Z_{\omega(1);i_1,j_1}, Z_{\omega(2);i_2,j_2}, \ldots, Z_{\omega(n);i_n,j_n}) = 0$$

whenever at least one of

$$j_{s_\chi(1)} = i_{s_\chi(2)}, j_{s_\chi(2)} = i_{s_\chi(3)}, \ldots, j_{s_\chi(n-1)} = i_{s_{\chi(n)}}, j_{s_\chi(n)} = i_{s_\chi(1)}$$

fail.

Bi-R-Cyclic Families and Bi-Free over the Diagonal

Theorem (Skoufranis; 2015)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let

$$\{[Z_{k;i,j}]\}_{k \in I} \cup \{[Z_{k;i,j}]\}_{k \in J} \subseteq \mathcal{M}_N(\mathcal{A}).$$

Then the following are equivalent:

- $(\{[Z_{k;i,j}]\}_{k \in I}, \{[Z_{k;i,j}]\}_{k \in J})$ *is R-cyclic.*
- $(\{L([Z_{k;i,j}])\}_{k \in I}, \{R([Z_{k;i,j}])\}_{k \in J})$ *is bi-free from* $(L(\mathcal{M}_N(\mathbb{C})), R(\mathcal{M}_N(\mathbb{C})^{\mathrm{op}}))$ *with amalgamation over* $\mathcal{D}_N$ *with respect to* $F \circ E_N$.

- This is a bi-free analogue of a result of Nica, Shlyakhtenko, and Speicher.
- One of the first non-trivial, concretely constructed examples of bi-freeness with amalgamation.

Bi-Free Partial R-Transform - Operator-Valued

If $(\mathcal{A}, E, \varepsilon)$ is a Banach B-B-ncps, $b, d \in B$, $X \in \mathcal{A}_\ell$, and $Y \in \mathcal{A}_r$, let

$$M^\ell_X(b) = 1 + \sum_{n\geq 1} E((L_b X)^n)$$

$$M^r_Y(d) = 1 + \sum_{n\geq 1} E((R_d Y)^n)$$

$$C^\ell_X(b) = 1 + \sum_{n\geq 1} \kappa^B_{\chi_{n,0}}(L_b X, \ldots, L_b X)$$

$$C^r_Y(d) = 1 + \sum_{n\geq 1} \kappa^B_{\chi_{0,n}}(R_d Y, \ldots, R_d Y)$$

Bi-Free Partial R-Transform - Operator-Valued

If $(\mathcal{A}, E, \varepsilon)$ is a Banach B-B-ncps, $b, d\!\!\!b, d \in B$, $X \in \mathcal{A}_\ell$, and $Y \in \mathcal{A}_r$, let

$$M_{X,Y}(b, d\!\!\!b, d) := \sum_{n,m\geq 0} E((L_b X)^n (R_d Y)^m R_{d\!\!\!b}) \quad \text{and}$$

$$C_{X,Y}(b, d\!\!\!b, d) := d\!\!\!b + \sum_{n\geq 1} \kappa^B_{\chi_{n,0}}(\underbrace{L_b X, \ldots, L_b X}_{n-1 \text{ entries}}, L_b X L_{d\!\!\!b})$$
$$+ \sum_{\substack{m\geq 1\\ n\geq 0}} \kappa_{\chi_{n,m}}(\underbrace{L_b X, \ldots, L_b X}_{n \text{ entries}}, \underbrace{R_d Y, \ldots, R_d Y}_{m-1 \text{ entries}}, R_d Y R_{d\!\!\!b}).$$

Theorem (Skoufranis; 2015)

With the above notation,

$$M^\ell_X(b) M_{X,Y}(b, d\!\!\!b, d) + M_{X,Y}(b, d\!\!\!b, d) M^r_Y(d)$$
$$= M^\ell_X(b) d\!\!\!b M^r_Y(d) + C_{X,Y}(M^\ell_X(b) b, M_{X,Y}(b, d\!\!\!b, d), d M^r_Y(d)).$$

Operator-Valued Free S-Transform

If $E(X)$ and $E(Y)$ are invertible, let

$$\Psi_{\ell,X}(b) = M_X^\ell(b) - 1 = \sum_{n\geq 1} E((L_b X)^n)$$

$$\Psi_{r,Y}(d) = M_Y^r(d) - 1 = \sum_{n\geq 1} E((R_d Y)^n)$$

$$\Phi_{\ell,X}(b) = C_X^\ell(b) - 1 = \sum_{n\geq 1} \kappa^B_{\chi_{n,0}}(L_b X, \ldots, L_b X)$$

$$\Phi_{r,Y}(d) = C_Y^r(d) - 1 = \sum_{n\geq 1} \kappa^B_{\chi_{0,n}}(R_d Y, \ldots, R_d Y)$$

$$S_X^\ell(b) = b^{-1}(b+1)\Psi_{\ell,X}^{\langle -1\rangle}(b) = b^{-1}\Phi_{\ell,X}^{\langle -1\rangle}(b)$$

$$S_Y^r(d) = \Phi_{r,Y}^{\langle -1\rangle}(d)(d+1)d^{-1} = \Phi_{r,Y}^{\langle -1\rangle}(d)d^{-1}.$$

Operator-Valued Free S-Transform

Theorem (Dykema; 2006)

Let $(\mathcal{A}, E, \varepsilon)$ be a Banach B-B-non-commutative probability space, let (X_1, Y_1) and (X_2, Y_2) be bi-free over B. Assume that $E(X_k)$ and $E(Y_k)$ are invertible. Then

$$S^{\ell}_{X_1 X_2}(b) = S^{\ell}_{X_2}(b) S^{\ell}_{X_1}(S^{\ell}_{X_2}(b)^{-1} b S_{X_2}(b)) \text{ and}$$
$$S^{r}_{Y_1 Y_2}(d) = S^{r}_{Y_1}(S^{r}_{Y_2}(d) d S_{Y_2}(d)^{-1}) S^{r}_{Y_2}(d)$$

each on a neighbourhood of zero.

Operator-Valued Bi-Free S-Transform

Let

$$K_{X,Y}(b, \text{ȸ}, d) = \sum_{n,m\geq 1} \kappa^{B}_{\chi_{n,m}}(\underbrace{L_bX, \ldots, L_bX}_{n \text{ entries}}, \underbrace{R_dY, \ldots, R_dY}_{m-1 \text{ entries}}, R_dYR_{\text{ȸ}})$$

$$\Upsilon_{X,Y}(b, \text{ȸ}, d) = K_{X,Y}\left(bS^{\ell}_{X}(b), \text{ȸ}, S^{r}_{Y}(d)d\right).$$

Definition (Skoufranis; 2015)

The *operator-valued bi-free partial S-transform of* (X, Y), denoted $S_{X,Y}(b, \text{ȸ}, d)$, is the analytic function

$$\text{ȸ} + b^{-1}\Upsilon_{X,Y}(b, \text{ȸ}, d) + \Upsilon_{X,Y}(b, \text{ȸ}, d)d^{-1} + b^{-1}\Upsilon_{X,Y}(b, \text{ȸ}, d)d^{-1}$$

for any bounded collection of ȸ provided b and d sufficiently small.

Operator-Valued Bi-Free S-Transform Formula

Theorem (Skoufranis; 2015)

If (X_1, Y_1) *and* (X_2, Y_2) *are bi-free over a unital algebra* B*, then*

$$S_{X_1X_2, Y_1Y_2}(b, db, d)$$

equals

$$Z_\ell S_{X_1, Y_1}\left(Z_\ell^{-1} b Z_\ell,\ Z_\ell^{-1} S_{X_2, Y_2}(b, db, d) Z_r^{-1},\ Z_r d Z_r^{-1}\right) Z_r$$

where $Z_\ell = S^\ell_{X_2}(b)$ *and* $Z_r = S^r_{Y_2}(d)$*.*

Thanks for Listening!