Asymptotic moments of random Vandermonde matrix

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Vandermonde matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{bmatrix}$$

If $m = n$ then the determinant is

$$\prod_{1 \le i < j \le n} (x_j - x_i).$$

Vandermonde matrix can be used to find the interpolating polynomial with given data.

Random Vandermonde matrix

$$X_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \zeta_1 & \zeta_1^2 & \dots & \zeta_1^{N-1} \\ 1 & \zeta_2 & \zeta_2^2 & \dots & \zeta_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_N & \zeta_N^2 & \dots & \zeta_N^{N-1} \end{bmatrix},$$

where $\zeta_1, \dots, \zeta_N$ are i.i.d. random variables uniformly distributed on the unit circle.

The rows of X_N are i.i.d. copies of

$$v = \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{N-1} \end{bmatrix},$$

where ζ is uniformly distributed on the unit circle.

The random vector is isotropic:

$$\mathbb{E}|\langle v, x\rangle|^2 = \|x\|_2^2, \quad x \in \mathbb{C}^N.$$

$f_1, \ldots, f_N$ are i.i.d. real random variables with mean 0, variance 1 and uniformly bounded moments.

$$\mathbb{E}\left|\sum_{k=1}^{N} f_k\right|^2 = N \qquad \mathbb{E}\left|\sum_{k=1}^{N} \zeta^k\right|^2 = N$$

$$\mathbb{E}\left|\sum_{k=1}^{N} f_k\right|^4 \sim 3N^2 \qquad \mathbb{E}\left|\sum_{k=1}^{N} \zeta^k\right|^4 \sim \frac{2}{3}N^3$$

$$\mathbb{E}\left|\sum_{k=1}^{N} f_k\right|^{2p} = O(N^p) \qquad \mathbb{E}\left|\sum_{k=1}^{N} \zeta^k\right|^{2p} = O(N^{2p-1})$$

To compute

$$\mathbb{E}\left|\sum_{k=1}^{N} f_k\right|^4,$$

we expand it as

$$\sum_{k_1,k_2,k_3,k_4=1}^{N} \mathbb{E} f_{k_1} f_{k_2} f_{k_3} f_{k_4}.$$

We consider all partitions on $\{1,2,3,4\}$. Only pair partitions contribute. There are 3 pair partitions so

$$\mathbb{E}\left|\sum_{k=1}^{N} f_k\right|^4 \sim 3N^2.$$

To compute

$$\mathbb{E}\left|\sum_{k=1}^{N}\zeta^{k}\right|^{4},$$

we expand it as

$$\sum_{k_1,k_2,k_3,k_4=1}^{N}\mathbb{E}\zeta^{k_1}\zeta^{-k_2}\zeta^{k_3}\zeta^{-k_4}=\sum_{k_1,k_2,k_3,k_4=1}^{N}\mathbb{E}\zeta^{k_1-k_2+k_3-k_4}.$$

$$\mathbb{E}\zeta^{k_1-k_2+k_3-k_4}=\begin{cases}1, & k_1-k_2+k_3-k_4=0\\ 0, & \text{Otherwise}\end{cases}.$$

So

$$\sum_{k_1,k_2,k_3,k_4=1}^{N} \mathbb{E}\zeta^{k_1-k_2+k_3-k_4} = |\{(k_1,k_2,k_3,k_4) \in \{1,\dots,N\}^4 : k_1 - k_2 + k_3 - k_4 = 0\}|.$$

The limit

$$\lim_{N\to\infty} \frac{1}{N^3} |\{(k_1,k_2,k_3,k_4) \in \{1,\dots,N\}^4 : k_1 - k_2 + k_3 - k_4 = 0\}|$$

is given by

$$\mathrm{Vol}_3\{(t_1,t_2,t_3,t_4) \in [0,1]^4 : t_1 - t_2 + t_3 - t_4 = 0\} = \frac{2}{3}.$$

Therefore,

$$\sum_{k_1,k_2,k_3,k_4=1}^{N} \mathbb{E}\zeta^{k_1-k_2+k_3-k_4} \sim \frac{2}{3}N^3.$$

So

$$\mathbb{E}\left|\sum_{k=1}^{N}\zeta^k\right|^4 \sim \frac{2}{3}N^3$$

Random Vandermonde matrix

$$X_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \zeta_1 & \zeta_1^2 & \dots & \zeta_1^{N-1} \\ 1 & \zeta_2 & \zeta_2^2 & \dots & \zeta_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_N & \zeta_N^2 & \dots & \zeta_N^{N-1} \end{bmatrix},$$

where $\zeta_1, \dots, \zeta_N$ are i.i.d. random variables uniformly distributed on the unit circle.

First moment

$$(X_N)_{i,j} = \frac{1}{\sqrt{N}} \zeta_i^j.$$

$$(X_N^*)_{i,j} = \frac{1}{\sqrt{N}} \zeta_j^{-i}$$

Thus

$$\begin{aligned} \mathbb{E} \circ \mathrm{tr} X_N^* X_N &= \frac{1}{N} \sum_{i(1),i(2)=1}^{N} \mathbb{E}(X_N^*)_{i(1),i(2)} (X_N)_{i(2),i(1)} \\ &= \frac{1}{N^2} \sum_{i(1),i(2)=1}^{N} \mathbb{E} \zeta_{i(2)}^{-i(1)} \zeta_{i(2)}^{i(1)} = 1. \end{aligned}$$

Here tr means normalized trace.

Second moment

$$\mathbb{E} \circ \operatorname{tr}(X_N^* X_N)^2 = \frac{1}{N^3} \sum_{i(1),i(2),i(3),i(4)=1}^{N} \mathbb{E}\zeta_{i(2)}^{-i(1)} \zeta_{i(2)}^{i(3)} \zeta_{i(4)}^{-i(3)} \zeta_{i(4)}^{i(1)}$$

$$= \frac{1}{N^3} \sum_{i(1),i(2),i(3),i(4)=1}^{N} \mathbb{E}\zeta_{i(2)}^{i(3)-i(1)} \zeta_{i(4)}^{i(1)-i(3)}.$$

Summing over $i(2) = i(4)$, we get 1.

If $i(2) \neq i(4)$ then $i(1) = i(3)$. Summing over $i(2) \neq i(4)$, we get

$$\frac{N(N-1)N}{N^3} = 1 - \frac{1}{N}.$$

Third moment

So

$$\mathbb{E} \circ \mathrm{tr}(X_N^* X_N)^2 = 2 - \frac{1}{N}.$$

So

$$\mathbb{E} \circ \mathrm{tr}(X_N^* X_N)^2 \to 2$$

Using the same method, we obtain

$$\mathbb{E} \circ \mathrm{tr}(X_N^* X_N)^3 \to 5.$$

because there are 5 partitions on $\{2, 4, 6\}$.

Fourth moment

$$\mathbb{E} \circ \operatorname{tr}(X_N^* X_N)^4$$

$$= \frac{1}{N^5} \sum_{i(1),\dots,i(8)=1}^{N} \mathbb{E} \zeta_{i(2)}^{i(3)-i(1)} \zeta_{i(4)}^{i(5)-i(3)} \zeta_{i(6)}^{i(7)-i(5)} \zeta_{i(8)}^{i(1)-i(7)}.$$

For each noncrossing partition π on $\{2, 4, 6, 8\}$, summing over $i(2), i(4), i(6), i(8)$ that respect π, we get 1.

There are 14 noncrossing partitions on $\{2, 4, 6, 8\}$. So noncrossing partitions give 14.

For the crossing partition $i(2) = i(6) \neq i(4) = i(8)$,

$$\begin{aligned}
&\mathbb{E}\zeta_{i(2)}^{i(3)-i(1)}\zeta_{i(4)}^{i(5)-i(3)}\zeta_{i(6)}^{i(7)-i(5)}\zeta_{i(8)}^{i(1)-i(7)} \\
=&\mathbb{E}\zeta_{i(2)}^{i(7)-i(5)+i(3)-i(1)}\mathbb{E}\zeta_{i(4)}^{i(1)-i(7)+i(5)-i(3)} \\
=&\begin{cases} 1, & i(7)-i(5)+i(3)-i(1)=0 \\ 0, & \text{Otherwise} \end{cases}.
\end{aligned}$$

Same as before: this gives $\frac{2}{3}$.

Therefore, $\mathbb{E} \circ \mathrm{tr}(X_N^* X_N)^4 \to 14 + \frac{2}{3}$.

General moments

Let

$$m_p = \lim_{N\to\infty} \mathbb{E} \circ \mathrm{tr}(X_N^* X_N)^p.$$

1. $m_1 = 1,\ m_2 = 2,\ m_3 = 5,\ m_4 = 14 + \frac{2}{3}$ (Ryan, Debbah 09)
2. $c_p \leq m_p \leq B_p$ (Ryan, Debbah 09)
3. $\exists$ measure μ on $[0, \infty)$ of unbounded support such that (Tucci, Whiting 11)

$$m_p = \int x^p \, d\mu(x), \quad p \geq 0.$$

$*$-moments

Question

Compute

$$\lim_{N\to\infty} \mathbb{E} \circ \mathrm{tr} P(X_N, X_N^*)$$

for all polynomial P in two noncommuting variables.

R-diagonality

Let $(\mathcal{A}, \phi)$ be a tracial $*$-probability space.

Definition (Nica and Speicher 97)

$a \in \mathcal{A}$ is *R-diagonal* if a has the same $*$-distribution as up where

1. u and p are $*$-free in some $*$-probability space $(\mathcal{A}', \tau)$,
2. u is a Haar unitary, i.e., $\tau(u^n) = \delta_{n=0}$.

Maximal alternating interval partition

Definition

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_p) \in \{1, *\}^p$. Then $\sigma(\epsilon)$ is the interval partition on $\{1, \ldots, p\}$ determined by

$$j \overset{\sigma(\epsilon)}{\sim} j+1 \iff \epsilon_j \neq \epsilon_{j+1}.$$

If $\epsilon = (1, 1, *, 1, *, *)$ then

$$\sigma(\epsilon) = \{\{1\}, \{2, 3, 4, 5\}, \{6\}\}.$$

Equivalent definition

Lemma (Nica, Shlyakhtenko, Speicher 01)

a is R-diagonal if and only if

1. $\phi(aa^*aa^*\dots aa^*a) = 0$ *and*
2. $\forall \epsilon_1, \dots, \epsilon_p \in \{1, *\}$,

$$\phi\left(\prod_{B\in\sigma(\epsilon)}\left(\prod_{k\in B} a^{\epsilon_k} - \phi\left(\prod_{k\in B} a^{\epsilon_k}\right)\right)\right) = 0.$$

Observations

1. R-diagonality completely determines the $*$-distribution of a in terms of the distribution of a^*a.
2. a^*a and aa^* are free.

Is Vandermonde R-diagonal

Question (Tucci)

Is the asymptotic $*$-distribution of X_N R-diagonal?

Affirmative reason: X_N has i.i.d. rows so

$$X_N \sim H_N X_N$$

where H_N is the $N \times N$ random permutation matrix independent of X_N.

Question

Are $X_N^* X_N$ and $X_N X_N^*$ asymptotically free?

By hand computation:

Low moments of $X_N^* X_N$ and $X_N X_N^*$ coincide as if they were asymptotically free.

By Matlab:

When $N = 10,000$,

$$|\mathbb{E} \circ \mathrm{tr}(X_N^* X_N)^4 (X_N X_N^*)^2 (X_N^* X_N)^4 (X_N X_N^*)^2$$
$$- \text{Value computed as if they were asymptotically free}| < 0.05.$$

$X_N^* X_N$ and $X_N X_N^*$ are not asymptotically free

$$\lim_{N\to\infty} \mathbb{E} \circ \mathrm{tr}(X_N^* X_N)^4 (X_N X_N^*)^2 (X_N^* X_N)^4 (X_N X_N^*)^2$$

$$-\text{Value computed as if they were asymptotically free} = \frac{1}{270}.$$

R-diagonality with amalgamation

Let $\mathcal{A}$ be a unital $*$-algebra.

Let $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ be a conditional expectation onto a $*$-subalgebra $\mathcal{B}$.

Definition (Śniady and Speicher 01)

$a \in \mathcal{A}$ is *R-diagonal with amalgamation over $\mathcal{B}$* if

1. $$\mathcal{E}(ab_1a^*b_2ab_3a^* \dots b_{2p}a) = 0,$$
2. $\forall \epsilon_1, \dots, \epsilon_p \in \{1, *\}$,

$$\mathcal{E}\left(\prod_{B \in \sigma(\epsilon)} \left(\prod_{k \in B} b_k a^{\epsilon_k} - \mathcal{E}\left(\prod_{k \in B} b_k a^{\epsilon_k}\right)\right)\right) = 0,$$

Main result

Theorem (B. and Dykema)

$\exists$ $*$-algebra $\mathcal{A}$ containing $C[0,1]$, a conditional expectation $\mathcal{E} : \mathcal{A} \to C[0,1]$ and $X \in \mathcal{A}$ such that

1. X is R-diagonal with amalgamation over $C[0,1]$ and
2. $\forall b_1, \dots, b_p \in C[0,1]$ and $\epsilon_1, \dots, \epsilon_p \in \{1, *\}$

$$\lim_{N\to\infty} \mathbb{E} \circ \mathrm{tr} \left(\prod_{k=1}^{p} b_k^{(N)} X_N^{\epsilon_k} \right) = \int_0^1 \mathcal{E} \left(\prod_{k=1}^{p} b_k X^{\epsilon_k} \right) \, d\lambda,$$

where λ is the Lebesgue measure on $[0,1]$ and

$$b_k^{(N)} = \mathrm{diag}(b_k(\frac{1}{N}), b_k(\frac{2}{N}), \dots, b_k(\frac{N}{N})).$$

Some $C[0,1]$-valued moments

$$\mathcal{E}(XX^*) = 1,\ \mathcal{E}(XX^*)^2 = 2,\ \mathcal{E}(XX^*)^3 = 5$$

$$\mathcal{E}(XX^*)^4 = 14 + \frac{2}{3}$$

$$\mathcal{E}(X^*X) = 1,\ \mathcal{E}(X^*X)^2 = 2,\ \mathcal{E}(X^*X)^3 = 5$$

$$\mathcal{E}(X^*X)^4 = 14 + \frac{3}{4} - (t - \frac{1}{2})^2, \quad t \in [0,1].$$

THANK YOU