A noncommutative version of the Julia-Caratheodory Theorem

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Self-maps of the upper half-plane

We let $\mathbb{C}^+ = \{z \in \mathbb{C}\colon \Im z > 0\}$ and $f\colon \mathbb{C}^+ \to \mathbb{C}^+$ be analytic.

Theorem (The Julia-Carathéodory Theorem)

If $\alpha \in \mathbb{R}$ is such that

$$\liminf_{z\to\alpha} \frac{\Im f(z)}{\Im z} = c < \infty,$$

then

- $\displaystyle\lim_{\substack{z\longrightarrow\alpha \\ \sphericalangle}} f(z) = f(\alpha) \in \mathbb{R}$, *and*
- $\displaystyle\lim_{\substack{z\longrightarrow\alpha \\ \sphericalangle}} \frac{f(z)-f(\alpha)}{z-\alpha} = \lim_{\substack{z\longrightarrow\alpha \\ \sphericalangle}} f'(z) = c.$

(Guarantees identification of a Fatou point - P. Mellon)

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Noncommutative (nc) functions

Let M, N be operator spaces. An nc set is a family $\Omega = (\Omega_n)_{n\in\mathbb{N}}$ such that $\Omega_n \subseteq M_n(M)$ and $\Omega_m \oplus \Omega_n \subseteq \Omega_{m+n}$.

Definition (J. L. Taylor - after Kaliuzhnyi-Verbovetskii & Vinnikov)

An nc function defined on an nc set Ω is a family $f = (f_n)_{n\in\mathbb{N}}$ such that $f_n\colon \Omega_n \to M_n(N)$ and whenever $m, n \in \mathbb{N}$,

1. $f_{m+n}(a \oplus c) = f_m(a) \oplus f_n(c)$ for all $a \in \Omega_m, c \in \Omega_n$, and
2. $Tf_n(c)T^{-1} = f_n(TcT^{-1})$ for all $c \in \Omega_n$, $T \in GL_n(\mathbb{C})$ such that $TcT^{-1} \in \Omega_n$.

We restrict ourselves to $M = N = \mathcal{A}$ - von Neumann algebra. We let $\Omega = H^+(\mathcal{A})$, $\Omega_n = H_n^+(\mathcal{A}) = \{a \in M_n(\mathcal{A})\colon \Im a := (a - a^*)/2i > 0\}$. Fix

$$f = (f_n)_{n\in\mathbb{N}}, \quad f_n\colon H_n^+(\mathcal{A}) \to H_n^+(\mathcal{A}).$$

Derivatives

For any $a \in H_m^+(\mathcal{A}), c \in H_n^+(\mathcal{A})$, there exists a linear operator

$$\Delta f_{m,n}(a,c)\colon M_{m\times n}(\mathcal{A}) \to M_{m\times n}(\mathcal{A})$$

such that

$$f_{m+n}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f_m(a) & \Delta f_{m,n}(a,c)(b) \\ 0 & f_n(c) \end{bmatrix}, \quad b \in M_{m\times n}(\mathcal{A}).$$

If $m = n$, then

- $\Delta f_{n,n}(a,a) = f_n'(a)$, the Fréchet derivative of f_n at a, and
- $\Delta f_{n,n}(a,c)(a-c) = f_n(a) - f_n(c)$.

With these notions, we can state:

The Julia-Carathéodory Theorem for nc functions

Theorem (2015)

Let $f\colon H^+(\mathcal{A}) \to H^+(\mathcal{A})$ be an nc analytic function and let $\alpha = \alpha^* \in \mathcal{A}$. Assume that for any $v \in \mathcal{A}$, $v > 0$ and any state $\varphi\colon \mathcal{A} \to \mathbb{C}$, we have

$$\liminf_{z \to 0, z \in \mathbb{C}^+} \frac{\varphi(\Im f_1(\alpha + zv))}{\Im z} < \infty.$$

Then

(i) $\lim_{z \underset{\sphericalangle}{\longrightarrow} 0} f_n(\alpha \otimes 1_n + zv) = f_1(\alpha) \otimes 1_n \in \mathcal{A}$ *exists in norm and is selfadjoint for any* $n \in \mathbb{N}$, $v \in M_n(\mathcal{A})$, $v > 0$, *and*

(ii) $\lim_{z \underset{\sphericalangle}{\longrightarrow} 0} \Delta f_{n,n}(\alpha \otimes 1_n + zv, \alpha \otimes 1_n + zv')(b)$ *exists in the weak operator topology for any fixed* $v, v' > 0$, $b \in M_n(\mathcal{A})$.

Moreover, if $v = v' = b > 0$, *then the above limit equals the* so*-limit* $\lim_{y \to 0} \Im f_n(\alpha \otimes 1_n + iyv)/y$.

The Julia-Carathéodory Theorem for nc functions

Important: statement (ii) of the main theorem does NOT mean that $f'(\alpha) = \lim_{y\to 0} f'(\alpha + iyv)$ exists, in the sense that the limit operator would not depend on v. (Counterexamples from Rudin, Abate, Agler - Tully-Doyle - Young.) However, IF the limit is independent of v, then it is *completely positive.*

There are many results generalizing the Julia-Carathéodory Theorem for

1. functions of several complex variables (Rudin, Abate, Agler - Tully-Doyle - Young);
2. functions on $\mathbb{C}^+$ with values in spaces of operators (Ky Fan);
3. functions between domains in Banach spaces, operator spaces, operator algebras (Jafari, Włodarczyk, Mackey - Mellon), etc.

Beyond its noncommutative nature, the result above seems to be new in the sense that it guarantees the existence of the limits of operators evaluated in *any* direction b, and it requires, as hypothesis, only a very weak initial condition.

The Julia-Carathéodory Theorem for nc functions

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Using the definition of the domain

Let $a, c \in H_n^+(\mathcal{A})$. Then

$$\Im \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} > 0 \iff 4\Im a > b(\Im c)^{-1} b^* \iff 4\Im c > b^*(\Im a)^{-1} b$$

$$\iff \left\| (\Im a)^{-1/2} b (\Im c)^{-1/2} \right\| < 2.$$

So given $b \in M_n(\mathcal{A})$, $\Im \begin{bmatrix} a & \epsilon b \\ 0 & c \end{bmatrix} > 0$ for any $0 < \epsilon < \frac{2}{\left\| (\Im a)^{-1/2} b (\Im c)^{-1/2} \right\|}$.
Since f maps $H^+(\mathcal{A})$ into itself and $\Delta f(a, c)$ is linear, $\epsilon \left\| (\Im f(a))^{-1/2} \Delta f(a, c)(b) (\Im f(c))^{-1/2} \right\| < 2$ for any such ϵ. Get

$$\left\| (\Im f(a))^{-1/2} \Delta f(a, c)(b) (\Im f(c))^{-1/2} \right\| \le \left\| (\Im a)^{-1/2} b (\Im c)^{-1/2} \right\|, \text{ or}$$

$$\Delta f(a, c)(b) (\Im f(c))^{-1} \Delta f(a, c)(b)^* \le \left\| (\Im a)^{-\frac{1}{2}} b (\Im c)^{-\frac{1}{2}} \right\|^2 \cdot \Im f(a).$$

Aside (not used in this proof)

If $\mathcal{A} = \mathbb{C}$, $a = c = z$, get $|f'(z)| \leq \Im f(z)/\Im z$, the Schwarz-Pick ineq.
It is natural to define

$$B_n^+(c, r) = \left\{ a \in H_n^+(\mathcal{A}) \colon \left\| (\Im a)^{-1/2}(a - c)(\Im c)^{-1/2} \right\| \leq r \right\}.$$

- $B_n^+(c, r)$ is convex, norm-closed, noncommutative;
- If $f(c) = c$, then $f_n(B_n^+(c, r)) \subseteq B_n^+(c, r)$;
- If $a \in B_n^+(c, r)$, then

$$\|a\| \leq \|\Re c\| + \|\Im c\| \left[\frac{r^2 + 2 + r\sqrt{r^2 + 4}}{2} + r\sqrt{\frac{r^2 + 2 + r\sqrt{r^2 + 4}}{2}} \right],$$

$$\Im a \geq \frac{1}{2 + r^2} \Im c.$$

Aside (not used in this proof)

Note similarity with [Agler, *Operator theory and the Carathéodory metric*] - description of pseudo-Carathéodory metric on $U \subset \mathbb{C}^d$ as $d(z, w) = \inf \sin \theta_M$, θ_M being the angle between the eigenvectors of a d-tuple M of commuting 2×2 matrices for which the joint spectrum is in U and U is a spectral domain for M. (Thanks to V. Paulsen)

Pseudo-Carathéodory metric: if $z, w \in U$, then

$$d(z, w) = \sup\{|f(z) - f(w)|/|1 - \overline{f(w)}f(z)| : f : U \to \mathbb{D} \text{ holo}\}.$$

Spectral domain: set containing the joint spectrum of M s.t. $\Pi : H^\infty(U) \to \mathcal{B}(\mathbb{C}^2)$, $\Pi(h) = h(M)$ is a contraction.

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Some steps in the proof

$$\Delta f(a,c)(b)(\Im f(c))^{-1}\Delta f(a,c)(b)^* \leq \left\|(\Im a)^{-1/2}\, b\,(\Im c)^{-1/2}\right\|^2 \cdot \Im f(a).$$

- $\liminf \frac{\varphi(\Im f(\alpha+zv))}{\Im z} < \infty \implies c(v) = \lim \frac{\Im f(\alpha+iyv)}{y} > 0$ and the family is unif. bounded in y;
- Then

$$\|f(\alpha+iyv)-f(\alpha+iy'1)\|^2 \leq \left\|v^{-1}\right\| \|yv-y'\|^2 \left\|\frac{\Im f(\alpha+iyv)}{y}\right\| \left\|\frac{\Im f(\alpha+iy'1)}{y'}\right\|$$

  providing norm-convergence to $f(\alpha)$.
- $\|\Delta f(\alpha+iyv,\alpha+iyv')(w)\|$ bdd, unif. in $y\in(0,1)$, $w\in\mathcal{A}$, $\|w\|<1$;
- $$\liminf_{y\to 0}\frac{1}{y}\left\|\Im f\left(\begin{bmatrix}\alpha+iyv_1 & \frac{iyb}{2}\\ \frac{iyb^*}{2} & \alpha+iyv_2\end{bmatrix}\right)\right\|<\infty;$$
- Finally, for any $\epsilon>0$, there exists a $d_\epsilon\in\mathcal{A}$ such that any wo cluster point of $\Delta f(\alpha+iyv,\alpha+iyv')(b)$ is at norm-distance $\sim\sqrt{\epsilon}$ from d_ϵ.

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Example

Consider an nc map $h\colon H^+(\mathcal{A}) \to \overline{H^+(\mathcal{A})}$ and the functional equation

$$\omega(a) = a + h(\omega(a)), \quad \omega\colon H^+(\mathcal{A}) \to H^+(\mathcal{A}) \text{ nc map.}$$

Equivalently, $\omega(a)$ is the unique fixed point of $f_a\colon H^+(\mathcal{A}) \to H^+(\mathcal{A})$, $f_a(w) = a + h(w)$. We have $f_a(B_n^+(\omega(a), r)) \subsetneq B_n^+(\omega(a), r)\ \forall r > 0$.
If $\alpha = \alpha^* \in \mathcal{A}$, $\{y_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ and $v > 0$ in $\mathcal{A}$ are such that $\lim_{n\to\infty} \frac{\omega(\alpha+iy_nv)}{\|\omega(\alpha+iy_nv)\|} = \ell > 0$ and $\lim_{n\to\infty} \omega(\alpha + iy_nv) = \omega(\alpha) \in \mathcal{A}$, then automatically

$$h(H_1(\omega(\alpha), \ell)) \subseteq \bar{H}_1(\omega(\alpha) - \alpha, \ell),$$

where

$$H_1(\omega(\alpha), \ell) = \left\{ w \in \mathbb{H}_1^+(\mathcal{A})\colon (w - \omega(\alpha))^*(\Im w)^{-1}(w - \omega(\alpha)) < \ell \right\}.$$

In particular,

$$\liminf_{z\to 0} \frac{\varphi(h(\omega(\alpha) + zv))}{\Im z} < \infty,$$

for all $v > 0$ in $\mathcal{A}$.
Result applies to operator valued free convolution semigroups.

Thank you!