# Perturbations of Operators and Commutants mod Normed Ideals

Dan-Virgil Voiculescu
UC Berkeley

D.-V. Voiculescu

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# Normed Ideals of Compact Operators

$\mathcal{H}$ separable $\infty$-dim Hilbert sp. over $\mathbb{C}$

$\mathcal{B}(\mathcal{H})$, $\| \ \|$ bounded operators on $\mathcal{H}$

$\mathcal{K} \subset \mathcal{B}$ compact operators

$(\mathcal{J}, | \ |_{\mathcal{J}})$, $\mathcal{J} \subset \mathcal{K}$ ideal

normed ideals noncommutative $\ell^p$,
noncommutative Lorentz etc

Connes: infinitesimals of
noncommutative geometry

$(\mathcal{J}, |\ |_{\mathcal{J}})$ normed ideal

$s_1 \geq s_2 \geq \cdots$ eigenvalues of $(T^*T)^{1/2}$, $T \in \mathcal{K}$

$(\mathcal{C}_p, |\ |_p)$ p-class $|T|_p = \left(\sum_j s_j^p\right)^{1/p}$

$(\mathcal{C}_p^-, |\ |_p^-)$ Lorentz $(p, 1)$

$$|T|_p^- = \sum_j s_j j^{-1+1/p}$$

$$|T|_\infty^- = \sum_j s_j j^{-1}$$ Macaev norm

perturbation $\tau = (T_j)_{1 \le j \le n}$, $\tau' = (T_j')_{1 \le j \le n}$

$T_j - T_j' \in \mathcal{J}$, $1 \le j \le n$

$\mathcal{J} = \mathcal{K}$ extensions of $C^*$-algebras K-theory

unified perspective

$\mathcal{J} \neq \mathcal{K}$ important patches, but more

patchy image

<u>example</u> $T = T^*$, $T' = T'^*$, $\sigma(T) = \sigma(T') = [0,1]$

$\mathcal{J} \neq \mathcal{C}_1 \Rightarrow \exists\, U$ unitary $UTU^* - T' \in \mathcal{J}$

$\exists U$ unitary $UTU^* - T' \in \mathcal{C}_1 \Leftrightarrow T_{ac}$ unitary equiv $T'_{ac}$

$T_{ac} = T | \mathcal{H}_{ac}(T)$, $\mathcal{H}_{ac}(T)$ Lebesgue abs. cont. subspace

Kuroda - Weyl - v. Neumann, Kato - Rosenblum

Such results can be explained and generalized using the invariant $k_{\mathcal{J}}(\tau)$. Recently $k_{\mathcal{J}}(\tau)$ is becoming related to commutants mod $\mathcal{J}$.

# The Modulus of Quasicentral Approximation

$\tau = (T_j)_{1 \le j \le n}$ , $(\mathcal{J}, |\ |_{\mathcal{J}})$

$k_{\mathcal{J}}(\tau) =$ least $C \in [0, \infty]$ : $\exists\, A_m \uparrow I$, $0 \le A_m \le I$

$A_m$ finite rank, $\max\limits_{1 \le j \le n} |[A_m, T_j]|_{\mathcal{J}} \xrightarrow[m \to \infty]{} C$

$\mathcal{J} = \mathcal{C}_p$ $\quad k_p(\tau)$ , $\mathcal{J} = \mathcal{C}_p^-$ $\quad k_p^-(\tau)$

$p = \infty$ $\quad k_\infty^-(\tau)$

# General Properties

1°. $p \longrightarrow k_p^-(\tau)$

decreasing function of $p \in [1, \infty]$

2°. there is $p_0 \in [1, \infty]$ so that

$p \in [1, p_0) \Rightarrow k_p^-(\tau) = \infty$

$p \in (p_0, \infty] \Rightarrow k_p^-(\tau) = 0$

3°. $\tau - \tau' \in \mathfrak{J} \Rightarrow k_{\mathfrak{J}}(\tau) = k_{\mathfrak{J}}(\tau')$

assuming finite rank ops. dense in $\mathfrak{J}$.

4°. assuming $\tau = \tau^*$ and finite rank ops. dense in $\mathfrak{J}$

$$k_{\mathfrak{J}}(\tau) > 0 \Longleftrightarrow \begin{array}{l} \exists\, Y_j = Y_j^* \in \mathfrak{J}^{dual},\ 1 \leq j \leq n \\ \mathrm{Tr}\ \underbrace{i \sum_j [T_j, Y_j]}_{\mathscr{C}_1 \,\hat{+}\, \mathcal{B}(\mathcal{H})_+} > 0 \end{array}$$

$k_J(\tau)$ "Size-$J$"-dimensional measure of $\tau$

dimension $p$ ~~~~~~~~~~~~ $J = \mathcal{C}_p^-$

$\tau$ n-tuple of commuting Hermitian operators

$$\left(k_n^-(\tau)\right)^n = \gamma_n \int_{\mathbb{R}^n} m(s)\, d\lambda_n(s)$$

n-dim Lebesgue

multiplicity of Lebesgue absolutely continuous part $\tau_{ac}$

<u>Application</u>: $\tau, \tau'$ n-tuples of commuting Hermitian

$\tau - \tau' \in \mathcal{C}_n^- \Rightarrow \tau_{ac}$ and $\tau'_{ac}$ unitarily equiv.

($n=1$ case is Cor. of Kato-Rosenblum Thm)

## $p=\infty$ Macaev Ideal Case

$k_\infty^-(\tau) \leq 2\|\tau\| \log(2n+1)$ ($k_\infty^-$ finite!)

$k_\infty^-(\tau \otimes I_{\mathcal{H}}) = k_\infty^-(\tau)$ (multiplicity plays no role!)

$\mathcal{C}_\infty^- \subsetneqq \mathcal{J} \implies k_\mathcal{J}(\tau) = 0$ (all $\tau$!)

$S_1, \ldots, S_n$ isometries with orthogonal ranges $n \geqslant 2$ (Cuntz isometries) $\implies k_\infty^-(S_1, \ldots, S_n) > 0$

# $\bar{h}_\infty$ and Entropy

1° Kolmogorov-Sinai dynamical entropy

$\theta$ measure-preserving ergodic automorphism of $(\Omega, \Sigma, \mu)$, $\mu(\Omega) = 1$

$U_\theta$ induced unitary operator in $L^2(\Omega, \Sigma, \mu)$

$\underline{\Phi}$ multiplication operators in $L^2$ by measurable functions with finite range

$$\mathcal{H}_\mu(\theta) = \sup_{\substack{\varphi \subset \underline{\Phi} \\ \varphi \text{ finite}}} \bar{h}_\infty(\varphi \cup \{U_\theta\})$$

$$\mathcal{H}_p(\theta) \asymp h(\theta)$$

K-S entropy

$\theta$ Bernoulli shift, then

$$\mathcal{H}_p(\theta) = \gamma\, h(\theta)$$

$\gamma$ universal constant

2°. Avez entropy

G group with generator $g_1, \dots, g_n$

$\mu$ finitary probability measure on G

$$h(G,\mu) > 0 \Rightarrow h_\infty^- (\lambda(g_1), \dots, \lambda(g_n)) > 0$$

Avez entropy of random walk on G defined by $\mu$

$\lambda$ left regular representation of G on $\ell^2(G)$

Further results on $h_\infty^-$ for Gromov hyperbolic G, entropy of subshifts in papers of Rui Okayasu

## G group with finite generator K

$$k_J(\lambda(K)) = \begin{cases} 0 \\ \text{finite} > 0 \\ \infty \end{cases} \quad \text{independent of choice of } K$$

$k_J(\lambda(K)) =$ least $C \in [0,\infty] : \exists\, A_n \uparrow I$
$0 \le A_n \le I$, diagonal in canonical basis of $\ell^2(G)$
finite rank, $\max_{g\in K} |\lambda(g) A_n \lambda(g^{-1}) - A_n|_J \xrightarrow[n\to\infty]{} C$

$k_p(\lambda(K)) > 0 \iff G$ Yamasaki $p$-hyperbolic finite $p$ examples

$G = \mathbb{Z}^n$ $\quad 0 < \bar{k}_n(\lambda(K)) < \infty$

$G =$ discrete Heisenberg $\quad 0 < \bar{k}_4(\lambda(K)) < \infty$

(Bernier)

$p = \infty$

$G$ has subexponential growth $\Rightarrow \bar{k}_\infty(\lambda(K)) = 0$

$\bar{k}_\infty(\lambda(K)) = 0 \Rightarrow G$ supramenable

recent result uses Kellerhals-Monod-Rørdam

G supramenable J. M. Rosenblatt

$\emptyset \neq A \subset G \Rightarrow \exists\, \mu : \mathcal{P}(G) \to [0, \infty]$

$\mu$ finitely additive, G invariant

$\mu(A) = 1$

G subexponential growth $\Rightarrow$ G supramenable

Kellerhals–Monod–Rørdam

G supramenable (countable)

$\Updownarrow$

there is no injective Lipschitz map

$\varphi : F_2 \to G$

**Problem**: $G$ supramenable $\Rightarrow h_\infty^-(\lambda(K)) = 0$ ?

(i.e. supramenable $\Leftrightarrow h_\infty^- = 0$)

**Remark**: it is possible that both

$h_\infty^-(\lambda(K)) = 0$ and $G$ supramenable

are $\Leftrightarrow$ subexponential growth of $G$.

## $\mathcal{E}(\tau;\mathcal{J})$ the Commutant of $\tau$ mod $\mathcal{J}$

$\tau = \tau^* = (T_j)_{1 \le j \le n} \subset \mathcal{B}(\mathcal{H})$, $(\mathcal{J}, |\ |_{\mathcal{J}})$

$\mathcal{E}(\tau;\mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, T_j] \in \mathcal{J},\ 1 \le j \le n\}$

$|||X||| = \|X\| + \max_{1 \le j \le n} |[X, T_j]|_{\mathcal{J}}$

Banach $*$-algebra

$\mathcal{E}(\tau;\mathcal{J})$ <u>not</u> a $C^*$-algebra in general

$C^*$algebras = Banach $*$-algebras isometrically isomorphic to self-adjoint norm-closed algebras of operators on a Hilbert space

[ $\mathcal{E}(\tau;\mathcal{J})$ has no continuous functional calculus on selfadjoint elements in general ]

$\mathcal{K}(\tau; \mathcal{J}) = \mathcal{E}(\tau; \mathcal{J}) \cap \mathcal{K}$ compact ideal of $\mathcal{E}(\tau; \mathcal{J})$

$\mathcal{E}/\mathcal{K}(\tau; \mathcal{J}) = \mathcal{E}(\tau; \mathcal{J}) / \mathcal{K}(\tau; \mathcal{J})$ Calkin algebra of $\mathcal{E}(\tau; \mathcal{J})$

if $\mathcal{J} = \mathcal{K}$ and $C^*(\tau) \cap \mathcal{K} = 0$

$\mathcal{E}/\mathcal{K}(\tau; \mathcal{K})$ = Paschke Dual of $C^*(\tau)$

Paschke Dual duality construction in K-theory of $C^*$-algebras

<u>in general</u> $\mathcal{E}/\mathcal{K}(\tau; \mathcal{J})$ <u>is not a smooth subalgebra of</u> $\mathcal{E}/\mathcal{K}(\tau; \mathcal{K})$ : $\mathcal{E}/\mathcal{K}(\tau; \mathcal{J})$ <u>may have much richer K-theory</u>

# $K_0(\mathcal{E}(\tau; \mathcal{J}))$ Simple Examples

$\tau_n$ $n$-tuple of multiplication operators in $L^2([0,1]^n, d\lambda_n)$ by coordinate functions.

$K_0(\mathcal{E}(\tau_n; \mathcal{K})) = 0$ since $\sigma(\tau_n) = [0,1]^n$ contractable.

$\mathcal{F}_n$ ordered group of Lebesgue measurable functions $f : [0,1]^n \to \mathbb{Z}$ which are in $L^\infty([0,1]^n, d\lambda_n)$ with a.e. equivalence.

$\mathcal{F}_n = K_0((\tau_n)')$.

1°. $n=1$, $\mathcal{J}=\mathcal{C}_1$

$K_0(\mathcal{E}(\tau_1;\mathcal{C}_1))\simeq \mathcal{F}_1$

$P$ projection in $M_n(\mathcal{E}(\tau_1,\mathcal{C}_1))$

$[P]_0 \rightsquigarrow$ multiplicity of Lebesgue abs. cont. part of $P(T\otimes I_n)P$

Kato–Rosenblum thm. corollary used

2°. $n=1$, $\mathcal{J}\neq\mathcal{C}_1$ (i.e. $\mathcal{C}_1 \subsetneq \mathcal{J}$)

$K_0(\mathcal{E}(\tau_1;\mathcal{J}))=0$.

3° $n \geq 3$, $\mathcal{J} = \mathcal{C}_n^-$

$$K_0(\mathfrak{E}(\tau_n; \mathcal{C}_n^-)) \simeq \mathcal{F}_n \oplus \underset{\text{unknown}}{\mathcal{X}_n}$$

uses perturbation results based on $k_n^-$

4° $n = 2$, $\mathcal{J} = \mathcal{C}_2$

$$K_0(\mathfrak{E}(\tau_2; \mathcal{C}_2)) \longrightarrow L^1_{real}([0,1]^2, d\lambda)$$

$$[P]_0 \rightsquigarrow \mathfrak{g}_{P(T_1+iT_2)P}$$ Pincus principal function

nontrivial homomorphism with ~~oo~~ rank range

$X = P(T_1 + iT_2)P$ almost normal operator

self-commutator $[X^*, X] \in \mathcal{C}_1$.

# $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$

many similarities between

$\mathcal{K}$ $\qquad$ $\mathcal{B}$ $\qquad$ $\mathcal{B}/\mathcal{K}$ and

$\mathcal{K}(\tau;\mathcal{J})$ $\qquad$ $\mathcal{E}(\tau;\mathcal{J})$ $\qquad$ $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$

notation: $\mathcal{R}$ finite rank operators
$p : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{K}$

$$\mathcal{B}/\mathcal{K} \supset p(\mathcal{E}(\tau;\mathcal{J})) \simeq \mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$$

algebraic isomorphism

I. assuming $R$ dense in $\mathcal{J}$

a) if $k_{\mathcal{J}}(\tau) = 0$
$\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ and $p(\mathcal{E}(\tau;\mathcal{J}))$ are $C^*$-algebras and isometrically isomorphic

b) if $k_{\mathcal{J}}(\tau) < \infty$
$p(\mathcal{E}(\tau;\mathcal{J}))$ is a $C^*$-algebra and is isomorphic to $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ (not isometrically)

<u>Cor.</u> $k_{\infty}^{-}(\tau) = 0 \implies \mathcal{E}/\mathcal{K}(\tau;\mathcal{C}_{\infty}^{-})$ $C^*$-algebra
In general $\mathcal{E}/\mathcal{K}(\tau;\mathcal{C}_{\infty}^{-})$ isomorphic to a $C^*$-algebra (all $\tau$ !)

# II. Duality

a) assume $\mathcal{R}$ dense in $\mathcal{J}$ and $\mathcal{J}^d$
and $k_{\mathcal{J}}(\tau) = 0$

$$\mathcal{E}(\tau;\mathcal{J}) \simeq \text{bidual of } \mathcal{K}(\tau;\mathcal{J})$$

b) assume $\mathcal{J}$ reflexive and $k_{\mathcal{J}}(\tau) = 0$

$\mathcal{E}(\tau;\mathcal{J})$ has unique predual

III assume $\mathcal{R}$ dense in $\mathcal{J}$ and $k_{\mathcal{J}}(\tau)=0$, then

$\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ is countably degree-1 saturated $C^*$-algebra

(in the model theory sense of Farah-Hart)

Cor. assume $\mathcal{R}$ dense in $\mathcal{J}$ and $k_{\mathcal{J}}(\tau)=0$

$\Gamma$ countable amenable group

$\rho : \Gamma \longrightarrow \mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ bounded representation

then: $\rho$ unitarizable

(i.e. $\exists$ $s$ invertible $s\rho(\cdot)s^{-1}$ unitary rep.)

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