A Moment Method for Invariant Ensembles

Jonathan Novak

- A real invariant ensemble is a sequence

$$X^{(N)} = \left[\begin{array}{ccc} & \vdots & \\ \cdots & X_{ij}^{(N)} & \cdots \\ & \vdots & \end{array} \right]_{i,j=1}^{N}, \qquad N = 1, 2, 3, \ldots$$

of random real selfadjoint matrices such that $X^{(N)} \overset{\text{law}}{=} O X^{(N)} O^{-1}$ for any $O \in O(N)$.

- A complex invariant ensemble is a sequence $X^{(N)}$ of random complex selfadjoint matrices such that $X^{(N)} \overset{\text{law}}{=} U X^{(N)} U^{-1}$ for any $U \in U(N)$.

- A quaternionic invariant ensemble is a sequence $X^{(N)}$ of random quaternionic selfadjoint matrices such that $X^{(N)} \overset{\text{law}}{=} S X^{(N)} S^{-1}$ for any $S \in Sp(N)$.

- Dyson Ensembles: distribution of $X^{(N)}$ has density proportional to

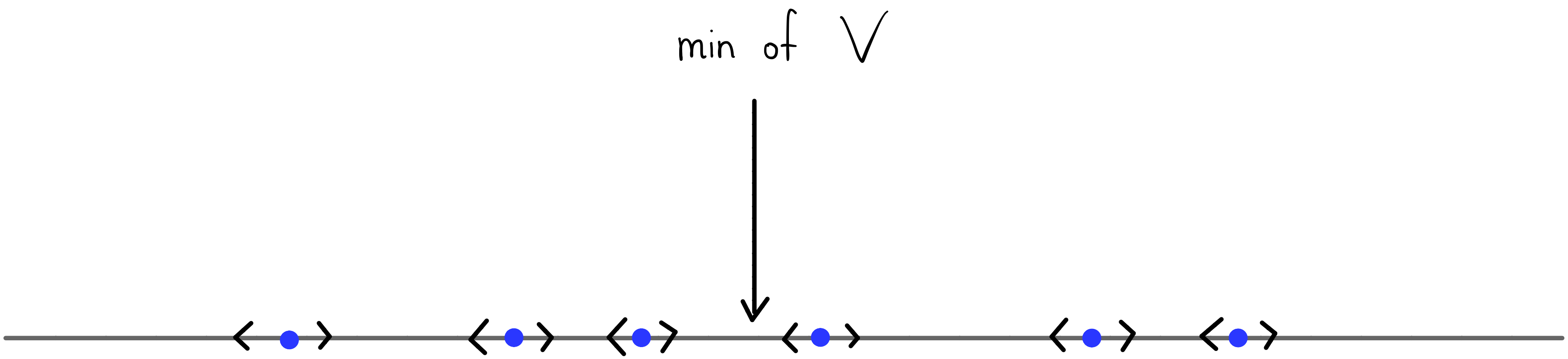
$$e^{-\frac{\beta}{2}\mathrm{Tr}\,V(X)},$$

where $V:\mathbb{R}\to\mathbb{R}$ is the potential and $\beta\in\{1,2,4\}$ is the Dyson index.

- Joint distribution of eigenvalues $E_1^{(N)}\geq\ldots\geq E_N^{(N)}$ proportional to $e^{-\frac{\beta}{2}N^2\mathcal{H}}$, where

$$\mathcal{H}(E_1,\ldots,E_N)=\frac{1}{N}\sum_{i=1}^{N}V(E_i)-\frac{1}{N^2}\sum_{i\neq j}\log|E_i-E_j|.$$

Coulomb Gas



$$\mathcal{H}(E_1,\dots,E_N) = \frac{1}{N}\sum_{i=1}^{N} V(E_i) - \frac{1}{N^2}\sum_{i\neq j} \log|E_i - E_j|$$

Theorem (Johansson): Under mild hypotheses on V, the spectral measure $\mu^{(N)}$ of $X^{(N)}$ converges weakly, in probability, to a nonrandom probability measure $\mu^{(\infty)}$, which is the unique minimizer of

$$\mu \mapsto \int V(x)\,\mu(dx) - \iint \log|x-y|\,\mu(dx)\,\mu(dy).$$

- Dyson ensembles form a small island in the vast sea of invariant ensembles.

- Newton observables:

$$p_d^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \left(E_i^{(N)}\right)^d , \qquad d \in \mathbb{N}.$$

- Moments of $\mu^{(N)}$: $p_d^{(N)} = \int x^d \mu^{(N)}(dx)$.

- Moment method: any technique relating distribution of Newton observables to joint distribution of $X_{ij}^{(N)}$.

- Wigner's moment method: $p_d^{(N)} = \frac{1}{N} \sum_{\phi:\{1,\dots,d\}\to\{1,\dots,N\}} X^{(N)}_{\phi(1)\phi(2)} X^{(N)}_{\phi(2)\phi(3)} \cdots X^{(N)}_{\phi(d)\phi(1)}$.

- Let $X^{(N)}$ be any selfadjoint ensemble. Consider the Fourier transform,

$$A \mapsto \mathbb{E}[e^{i \operatorname{Tr} A X^{(N)}}].$$

- Diagonalize selfadjoint matrix A:

$$\mathbb{E}[e^{i \operatorname{Tr} A X^{(N)}}] = \mathbb{E}[e^{i \operatorname{Tr} U \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U^{-1} X^{(N)}}] = \mathbb{E}[e^{i \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U^{-1} X^{(N)} U}].$$

- If $X^{(N)}$ conjugation invariant, then

$$\mathbb{E}[e^{i \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U^{-1} X^{(N)} U}] = \mathbb{E}[e^{i \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} X^{(N)}}].$$

Proposition: The distribution of $X^{(N)}$ is completely determined by the joint distribution of the real random variables

$$X_{11}^{(N)}, \dots, X_{NN}^{(N)},$$

which are identically distributed and exchangeable.

Proof: For any selfadjoint A,

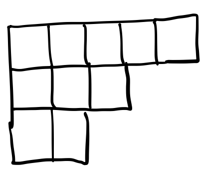
$$\mathbb{E}\left[e^{i \operatorname{Tr} A X^{(N)}}\right] = \mathbb{E}\left[e^{i(a_1,\dots,a_N)\cdot(X_{11}^{(N)},\dots,X_{NN}^{(N)})}\right],$$

where $a_1,\dots,a_N$ is any enumeration of the eigenvalues of A.

- An invariant ensemble $X^{(N)}$ is smooth if $X^{(N)}_{11}, \ldots, X^{(N)}_{NN}$ admit joint moments of all orders.

- By exchangeability, suffices to consider joint moments indexed by Young diagrams: if $|\lambda| = d$ and $\ell(\lambda) = r$, put

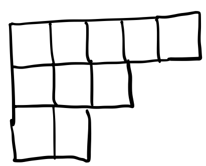
$$m_\lambda^{(N)} = m_d\big(\underbrace{X_{11}^{(N)}, \ldots, X_{11}^{(N)}}_{\lambda_1}, \ldots, \underbrace{X_{rr}^{(N)}, \ldots, X_{rr}^{(N)}}_{\lambda_r}\big) = \mathbb{E}\Big[\prod_{i=1}^{r} \big(X_{ii}^{(N)}\big)^{\lambda_i}\Big].$$

- Example: if $\lambda =$  , then $m_\lambda^{(N)} = \mathbb{E}\big[(X_{11}^{(N)})^5 (X_{22}^{(N)})^3 (X_{33}^{(N)})^2\big]$.

  Equivalently, $m_\lambda^{(N)}$ is coefficient of $\frac{a_1^5}{5!}\frac{a_2^3}{3!}\frac{a_3^2}{2!}$ in $\mathbb{E}\big[e^{i(a_1,a_2,a_3)\cdot(X_{11}^{(N)}, X_{22}^{(N)}, X_{33}^{(N)})}\big]$.

- General principle: cumulants are better than moments.

- Trade $m_\lambda^{(N)}$ for $c_\lambda^{(N)} = c_d\big(\underbrace{X_{11}^{(N)},\dots,X_{11}^{(N)}}_{\lambda_1},\dots,\underbrace{X_{rr}^{(N)},\dots,X_{rr}^{(N)}}_{\lambda_r}\big)$.

- Example: if $\lambda =$ , then

$$c_\lambda^{(N)} = c_{10}\big(\underbrace{X_{11}^{(N)}, X_{11}^{(N)}, X_{11}^{(N)}, X_{11}^{(N)}, X_{11}^{(N)}}_{5}, \underbrace{X_{22}^{(N)}, X_{22}^{(N)}, X_{22}^{(N)}}_{3}, \underbrace{X_{33}^{(N)}, X_{33}^{(N)}}_{2}\big).$$

Coefficient of $\frac{a_1^5}{5!}\frac{a_2^3}{3!}\frac{a_3^2}{2!}$ in Maclaurin series of $\log \mathbb{E}\left[e^{i(a_1,a_2,a_3)\cdot(X_{11}^{(N)}, X_{22}^{(N)}, X_{33}^{(N)})}\right]$.

Theorem (Matsumoto - N.): For a smooth invariant ensemble $X^{(N)}$, the following are equivalent:

1) For each d, $p_d^{(N)}$ converges in probability to a deterministic limit $p_d^{(\infty)}$;

2) For each λ, the limit $c_\lambda^{(\infty)} = \lim_{N\to\infty} N^{|\lambda|-1} c_\lambda^{(N)}$ exists, and vanishes if $\ell(\lambda) > 1$.

- What is the relationship between $p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \dots$ and $c_1^{(\infty)}, c_2^{(\infty)}, c_3^{(\infty)}, \dots$?

Theorem (Matsumoto – N.): Under the same hypotheses, we have

$$R\left(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \dots\right) = \left(\frac{1}{0!} c_1^{(\infty)}, \frac{1}{1!} c_2^{(\infty)}, \frac{1}{2!} c_3^{(\infty)}, \dots\right).$$

- Idea of proof: use $X_{ii}^{(N)} = \sum_{j=1}^{N} U_{ij}^{(N)} E_j^{(N)} \overline{U}_{ij}^{(N)}$, where $U^{(N)} = [U_{ij}^{(N)}]_{i,j=1}^{N}$ is uniformly random in $O(N)$, $U(N)$, or $Sp(N)$ according to whether $\beta = 1, 2,$ or 4.

- "Integrate out" eigenvectors using Weingarten Calculus.

- Olshanski–Vershik: considered the case $N=\infty$.

- Collins: considered the case where eigenvalues of $X^{(N)}$ are deterministic.

- Guionnet–Maïda: considered $Z^{(N)}=X^{(N)}+Y^{(N)}$, where eigenvalues of $X^{(N)}, Y^{(N)}$ deterministic.

- Bufetov–Gorin: related results for discrete particle systems.

Example: Let $X^{(N)}$ be an invariant ensemble such that $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$ are iid Gaussians of mean c_1, variance $c_2 N^{-1}$.

Easy: higher pure cumulants vanish by Gaussianity, mixed cumulants vanish by independence. Thus limits

$$c_\lambda^{(\infty)} = \lim_{N \to \infty} N^{|\lambda|-1} c_\lambda^{(N)} \quad , \quad \lambda \in \mathbb{Y}$$

exist, with $c_1^{(\infty)} = c_1$, $c_2^{(\infty)} = c_2$, and $c_\lambda^{(\infty)} = 0$ otherwise.

Conclusion: each $p_d^{(N)}$ converges in probability to deterministic $p_d^{(\infty)}$, and

$$R(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \ldots) = (c_1, c_2, 0, 0, \ldots).$$

Example: Let $X^{(N)}, Y^{(N)}$ be independent invariant ensembles with

$$p_d(X^{(N)}) \to x_d \quad \text{and} \quad p_d(Y^{(N)}) \to y_d.$$

Form $Z^{(N)} = X^{(N)} + Y^{(N)}$. Then for any $\lambda \in \mathbb{Y}$, $|\lambda| = d$, $\ell(\lambda) = r$, independence yields

$$c_d(\underbrace{Z_{11}^{(N)}, \dots, Z_{11}^{(N)}}_{\lambda_1}, \dots, \underbrace{Z_{rr}^{(N)}, \dots, Z_{rr}^{(N)}}_{\lambda_r})$$

$$= c_d(\underbrace{X_{11}^{(N)}, \dots, X_{11}^{(N)}}_{\lambda_1}, \dots, \underbrace{X_{rr}^{(N)}, \dots, X_{rr}^{(N)}}_{\lambda_r}) + c_d(\underbrace{Y_{11}^{(N)}, \dots, Y_{11}^{(N)}}_{\lambda_1}, \dots, \underbrace{Y_{rr}^{(N)}, \dots, Y_{rr}^{(N)}}_{\lambda_r}).$$

Get that $p_d(Z^{(N)}) \to z_d$, and

$$R(z_1, z_2, z_3, \dots) = R(x_1, x_2, x_3, \dots) + R(y_1, y_2, y_3, \dots).$$

- Two ingredients:

  (1) A tiling of the plane by equilateral triangles;

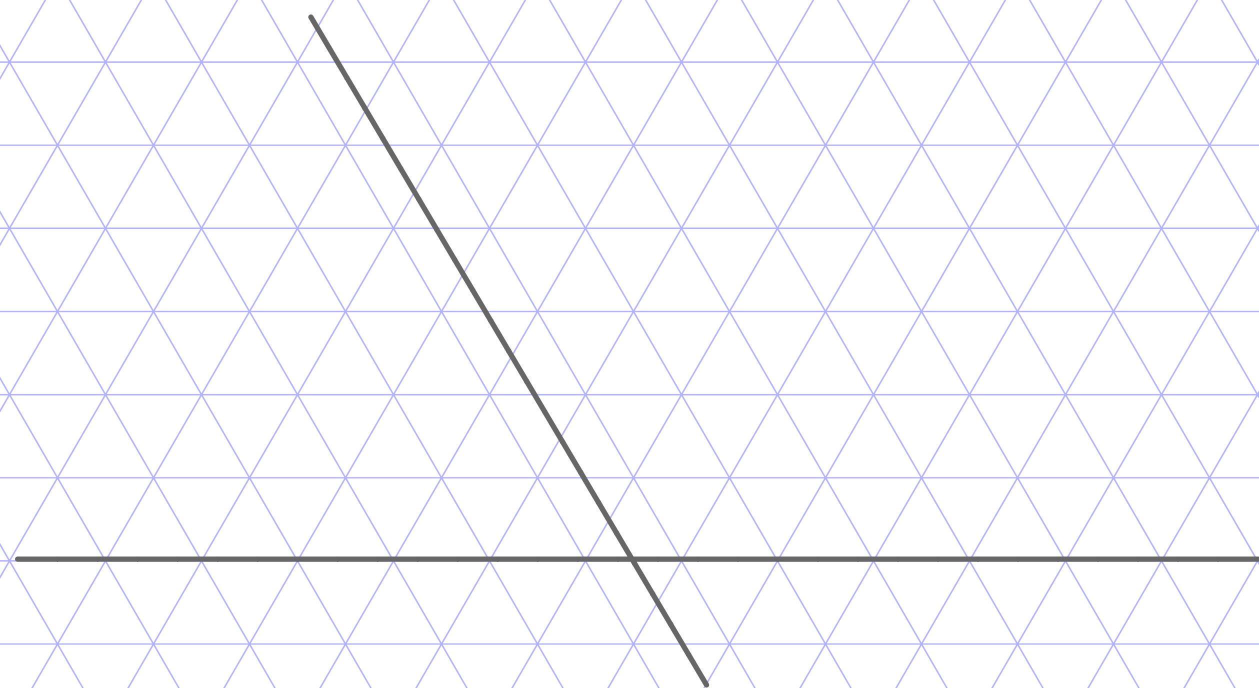
  (2) A triangular array of integers,

$$\begin{array}{ccccc} & & b_1^{(1)} & & \\ & b_1^{(2)} & & b_2^{(2)} & \\ b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\ \begin{matrix}\cdot\\[-1ex]\,\cdot\\[-1ex]\;\cdot\end{matrix} & & & & \ddots \end{array}$$

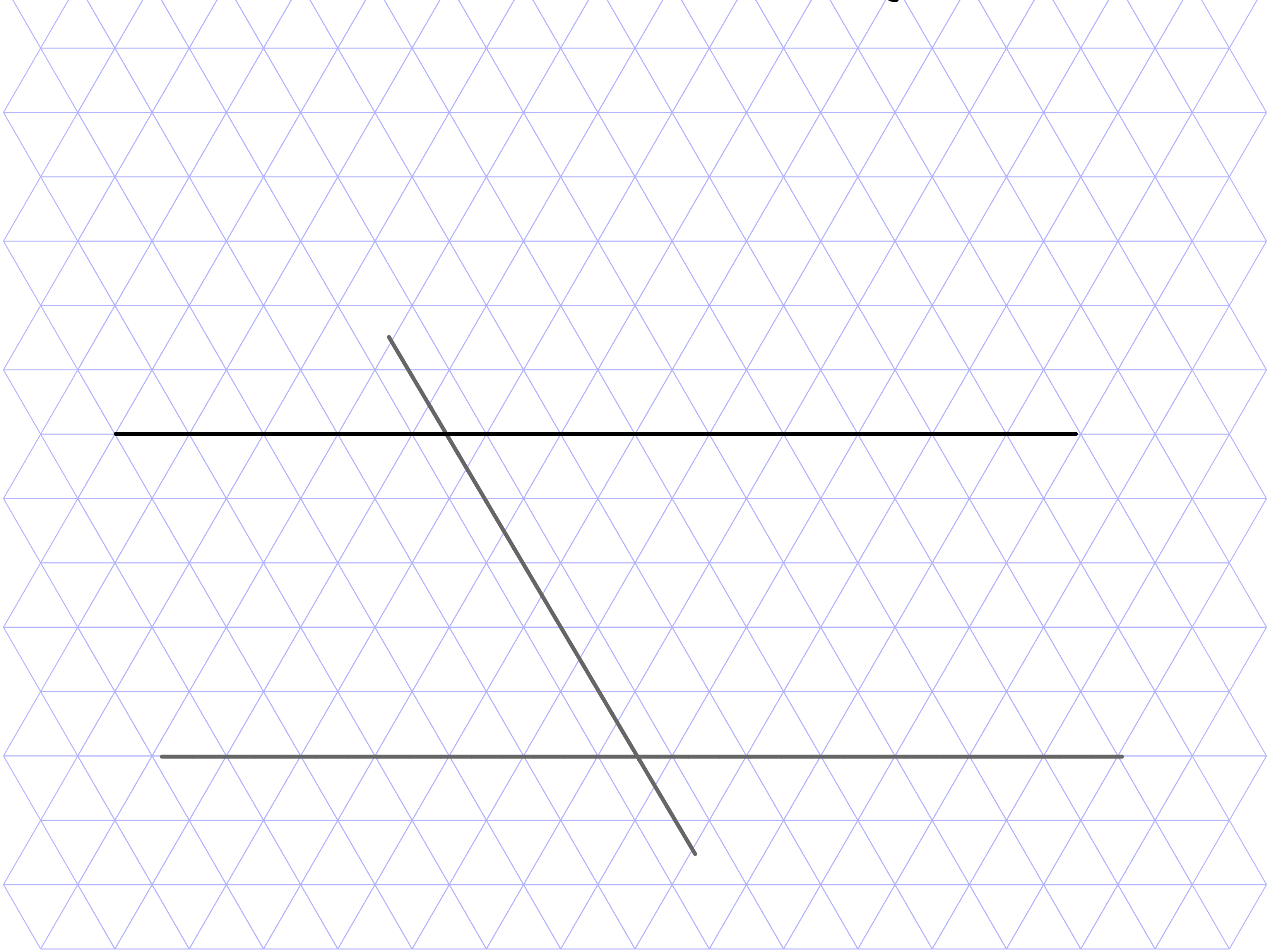
- Using this data, build a sequence of simply-connected planar domains,

$$\Omega^{(1)}, \Omega^{(2)}, \Omega^{(3)}, \ldots$$

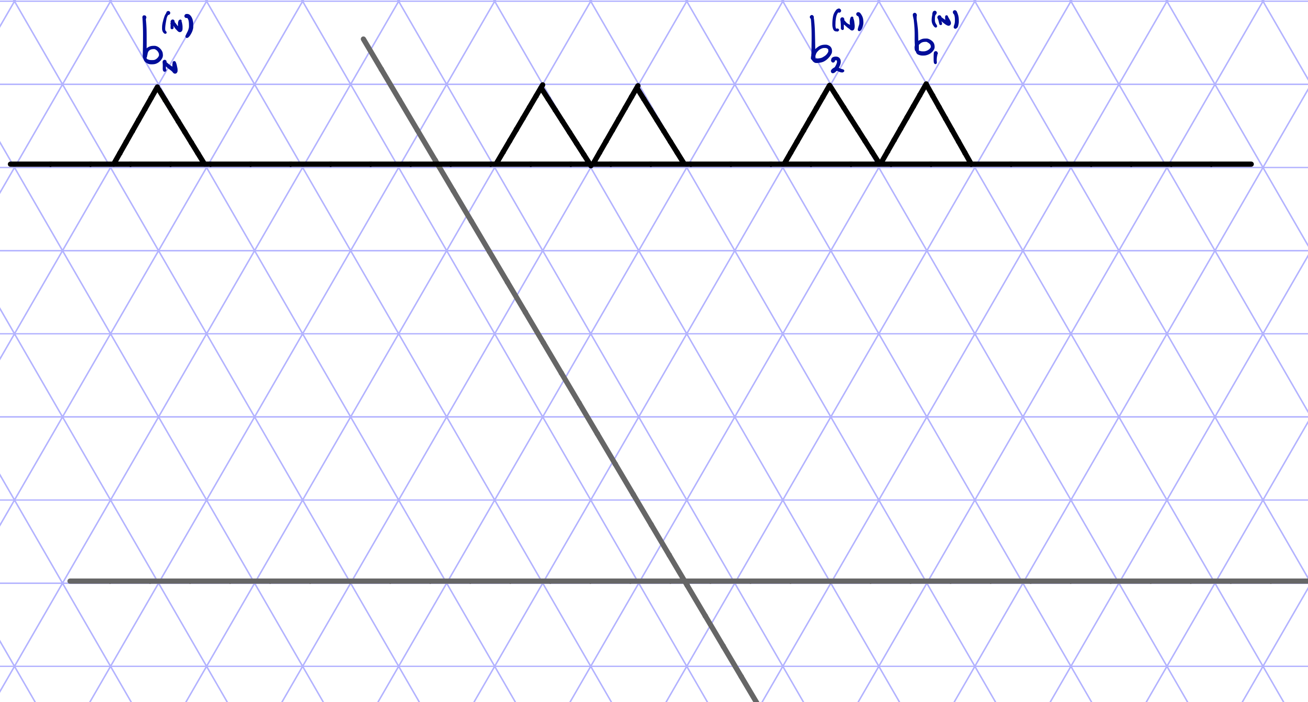
- STEP ZERO: Introduce a coordinate system,



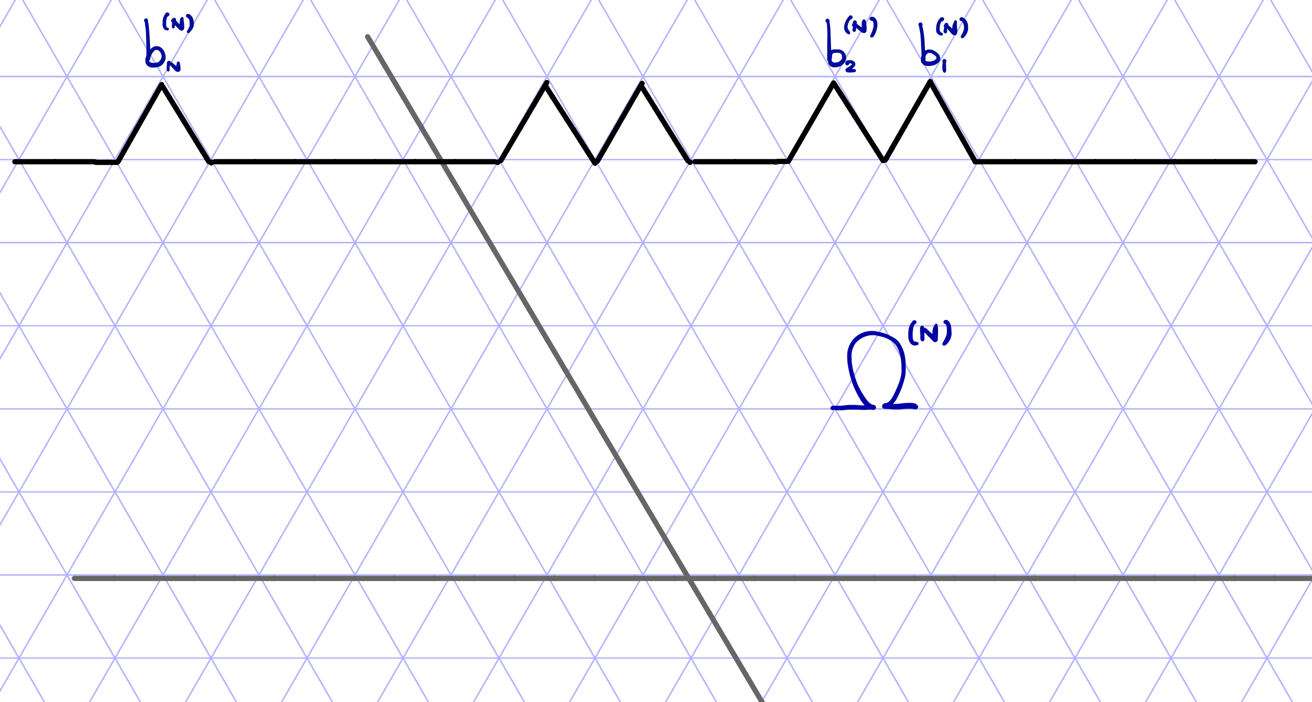
- STEP ONE: Construct the horizontal line through (0,N),



- STEP TWO: Construct N outward facing unit triangles on the line such that the midpoints of their bases have horizontal coordinates $b_1^{(N)} > \dots > b_N^{(N)}$.

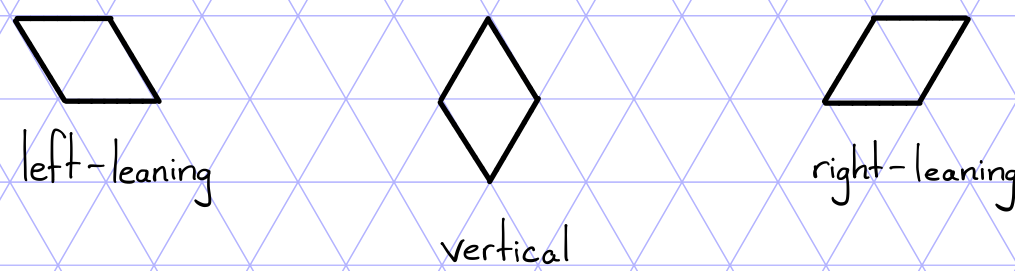


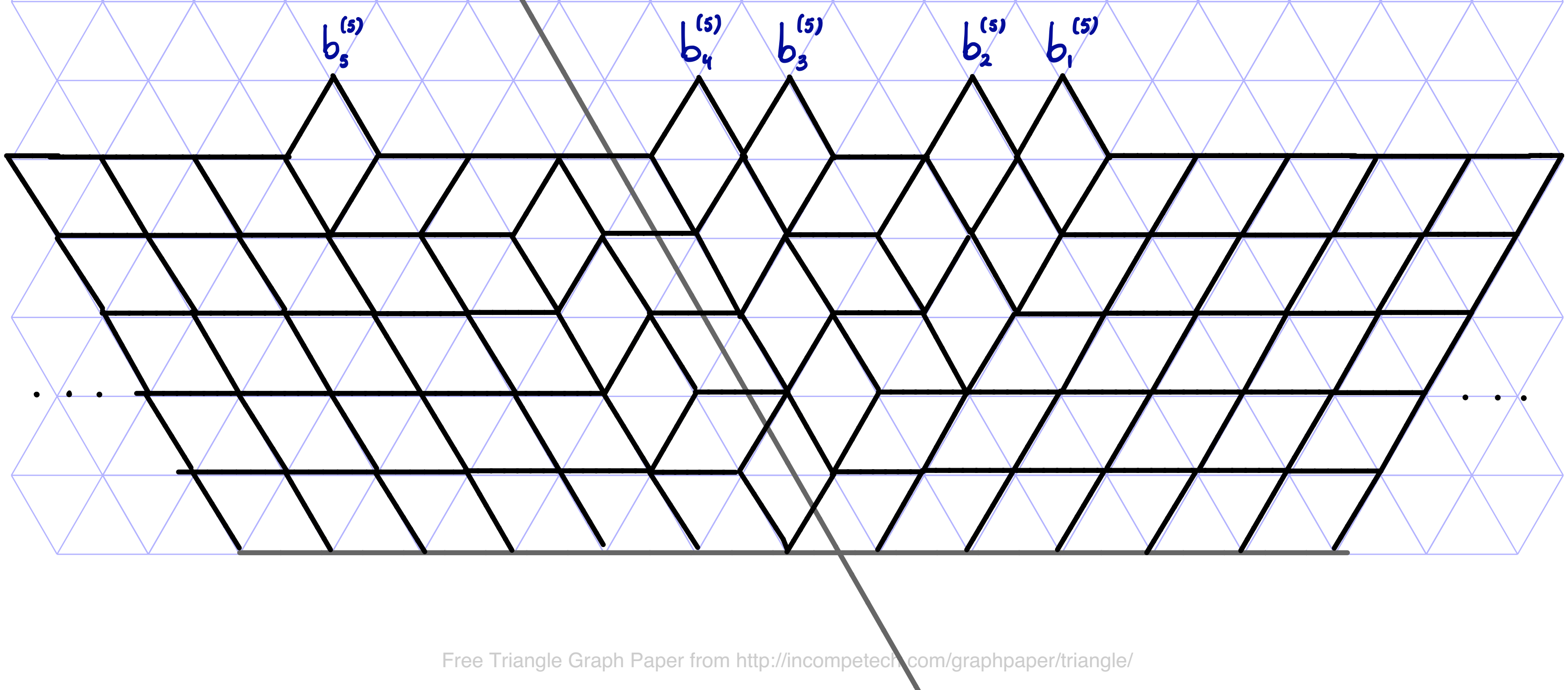
- STEP THREE: Erase the bases of the triangles,



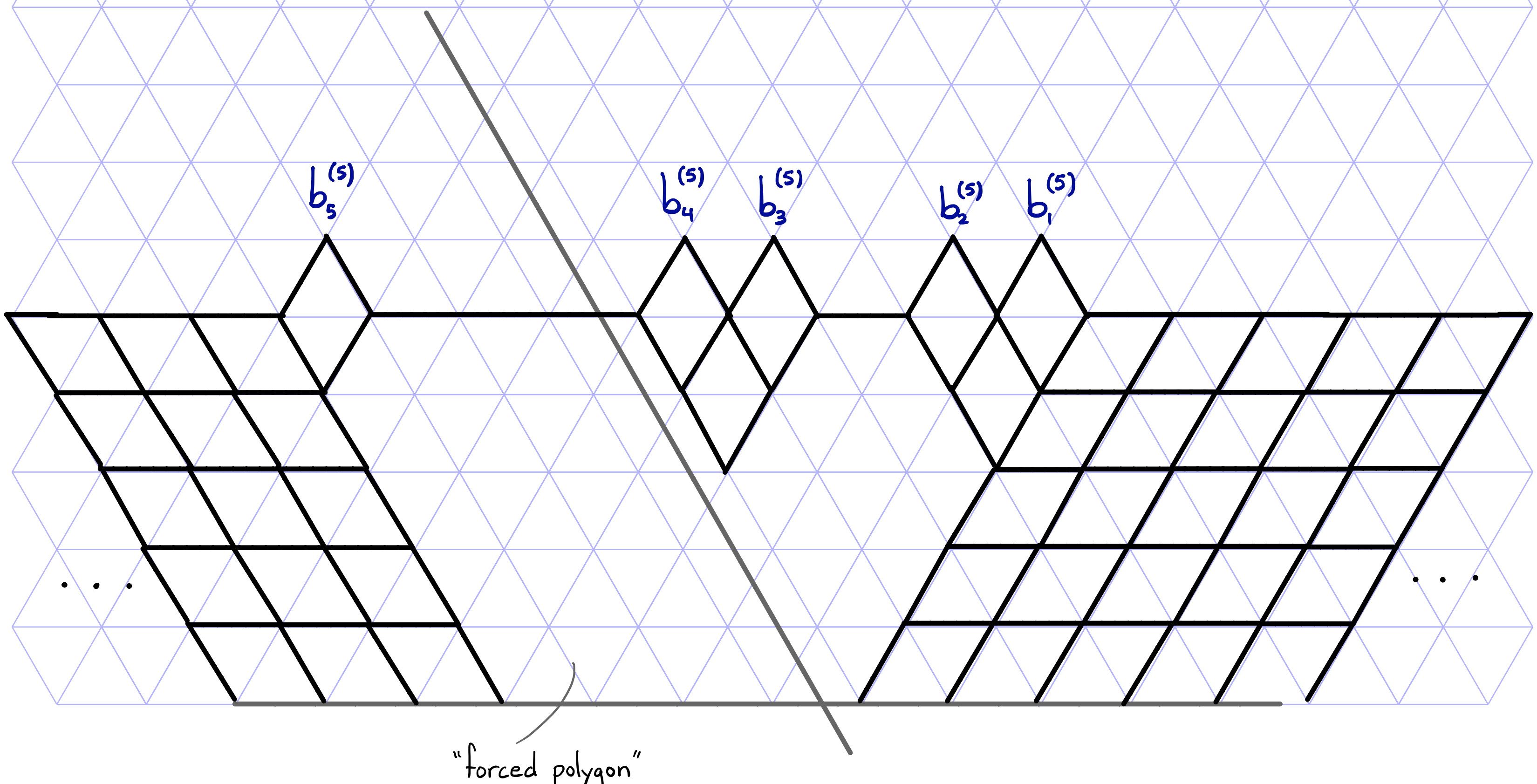
- You have now constructed the sawtooth domain of rank N with boundary conditions $b_1^{(N)} > \ldots > b_N^{(N)}$.

- FACT: $\Omega^{(N)}$ can be tessellated using tiles of three types, called lozenges.

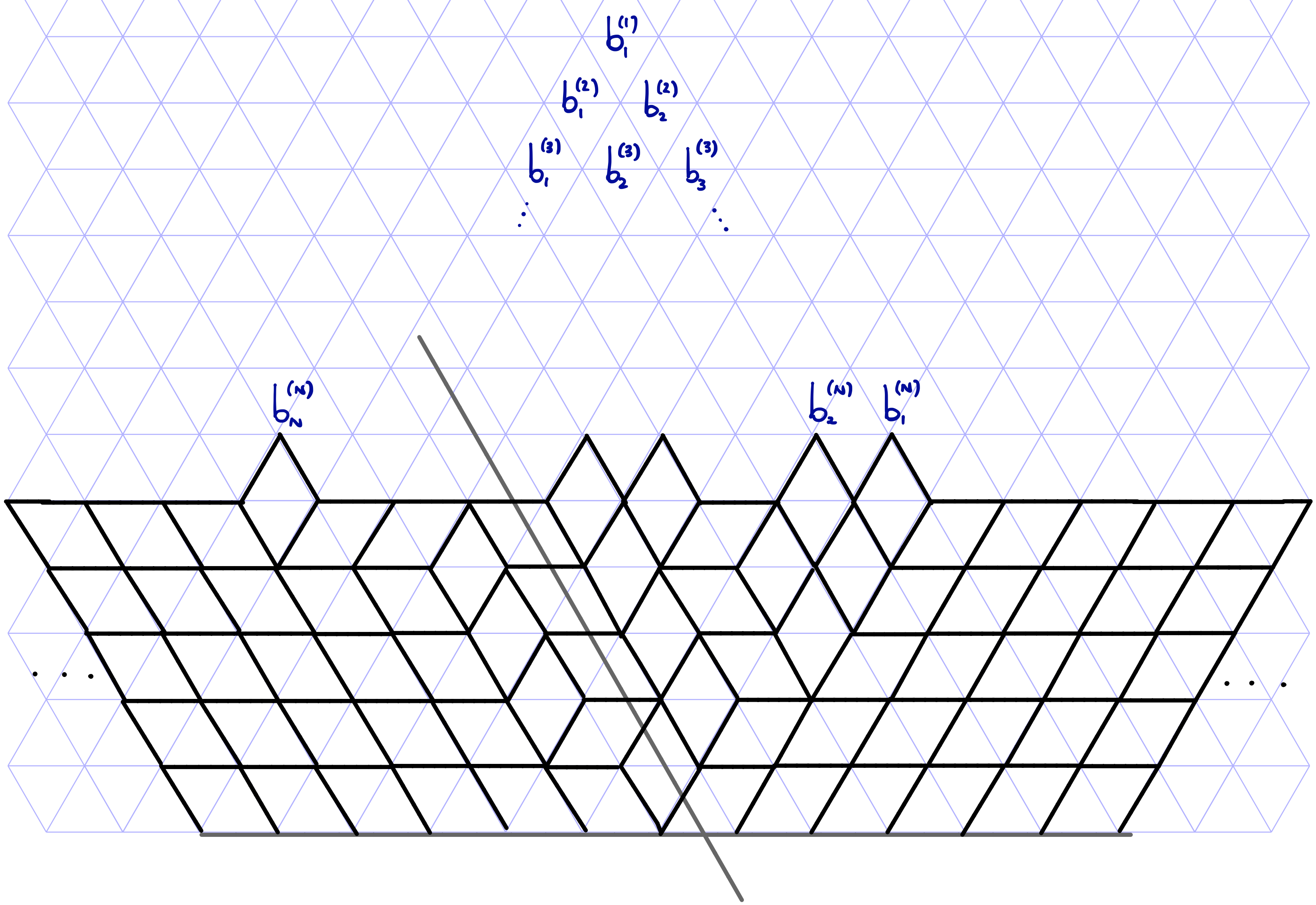




- FACT: $\Omega^{(N)}$ admits finitely many lozenge tilings.



- Ensemble: For each $N \in \mathbb{N}$, $T^{(N)}$ is a uniformly random lozenge tiling of $\Omega^{(N)}$. Ensemble determined by boundary conditions,

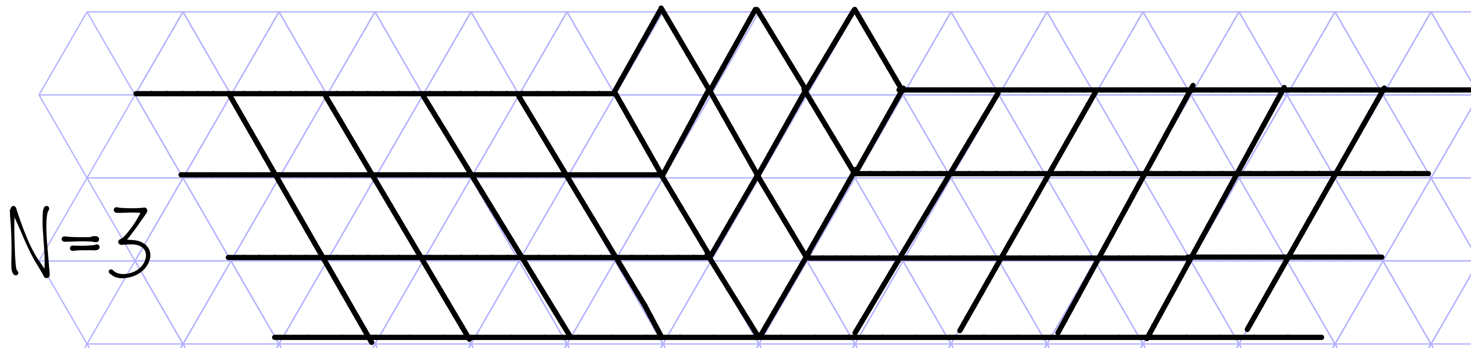


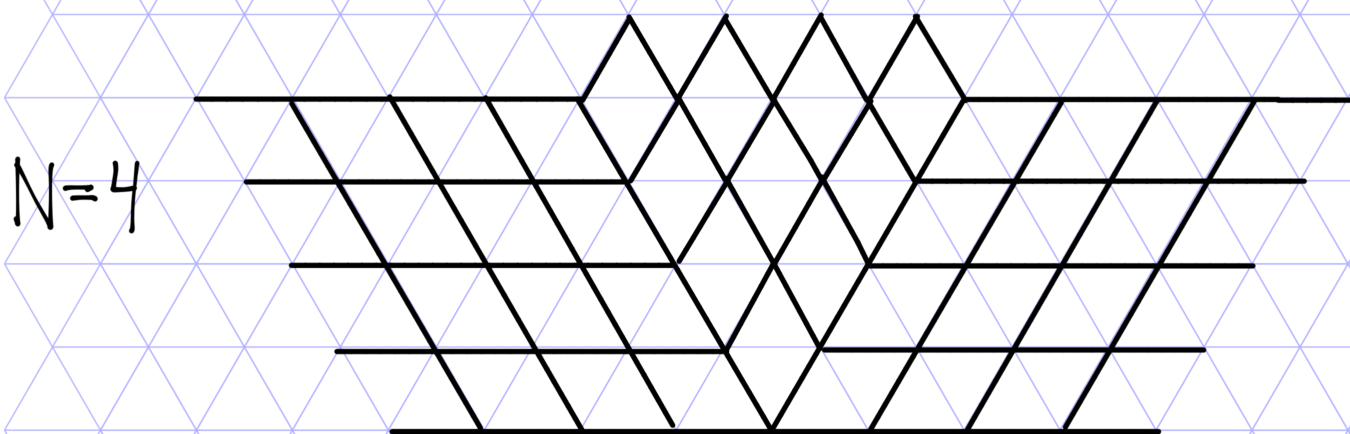
Free Triangle Graph Paper from http://incompetech.com/graphpaper/triangle/

- Example: "fully packed" boundary conditions,

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & & 0 & & \\ & 2 & & 1 & & 0 & \\ 3 & & 2 & & 1 & & 0 \\ \iddots & & & & & & \ddots \end{array}$$

- $\Omega^{(N)}$ has all teeth in one clump, forced polygon is null, no randomness – "trivial" ensemble.


N=3

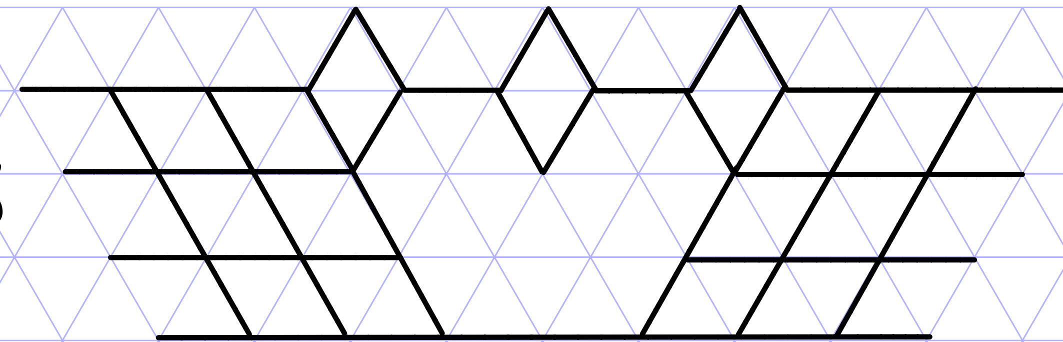

N=4

- Nordenstam - Young ensemble: boundary conditions which maximize number of clumps, conditional on minimizing spread:

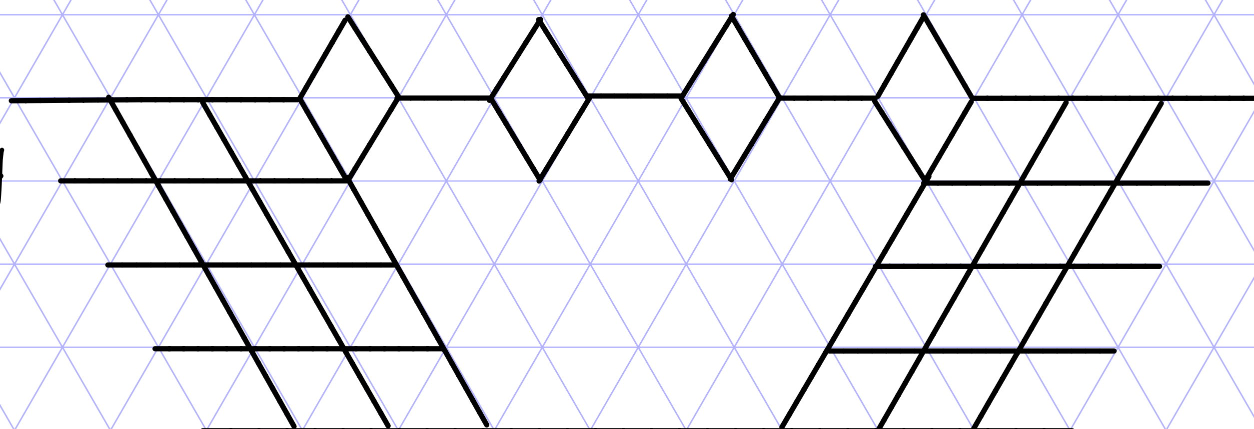
$$\begin{array}{ccccccccc} &&&&0&&&& \\ &&&2&&0&&& \\ &&4&&2&&0&& \\ &6&&4&&2&&0& \\ \cdot^{\cdot^{\cdot}}&&&&&&&&\ddots \end{array}$$

- Closely related to domino tilings of Aztec diamond – partition functions coincide.

N=3



N=4



- Note: number of sides of the forced polygon increases with N.

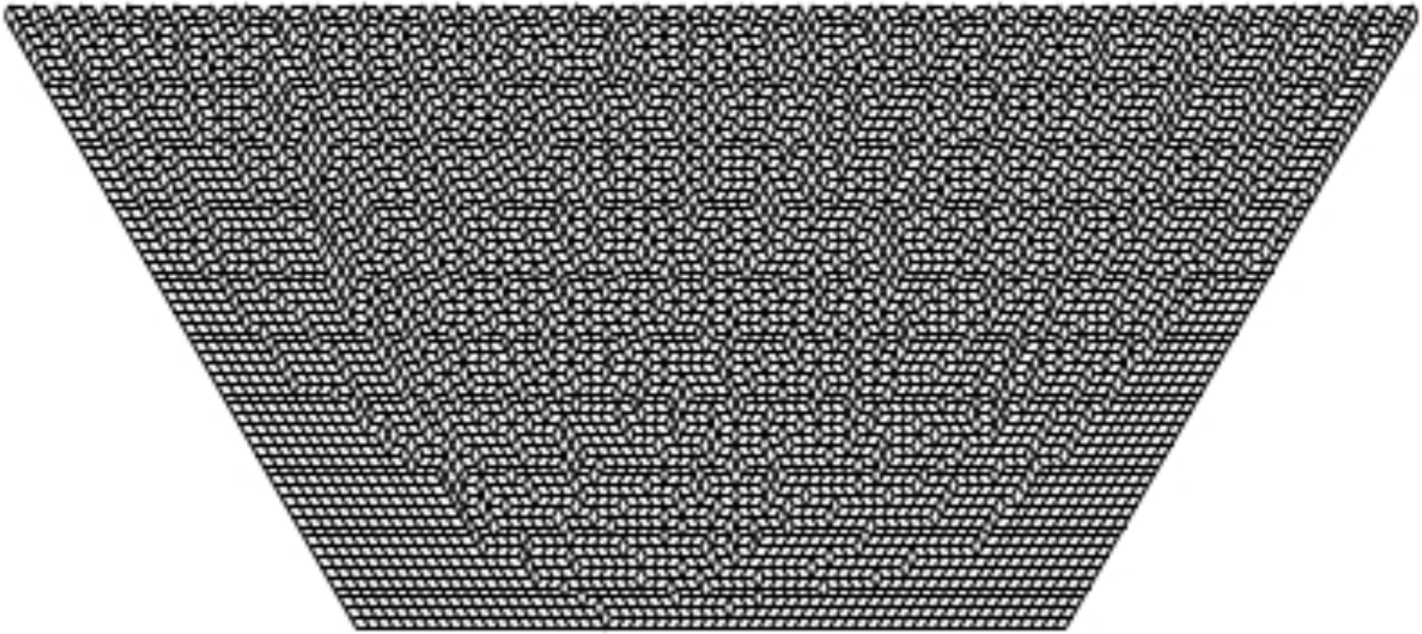


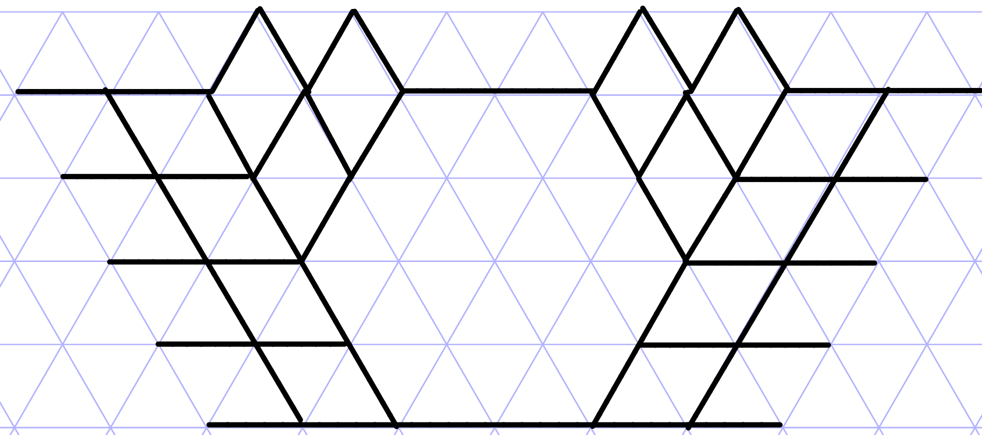
Fig. by Nordenstam and Young

- Cohn-Larsen-Propp ensemble: minimize number of clumps conditional on getting something non-trivial.

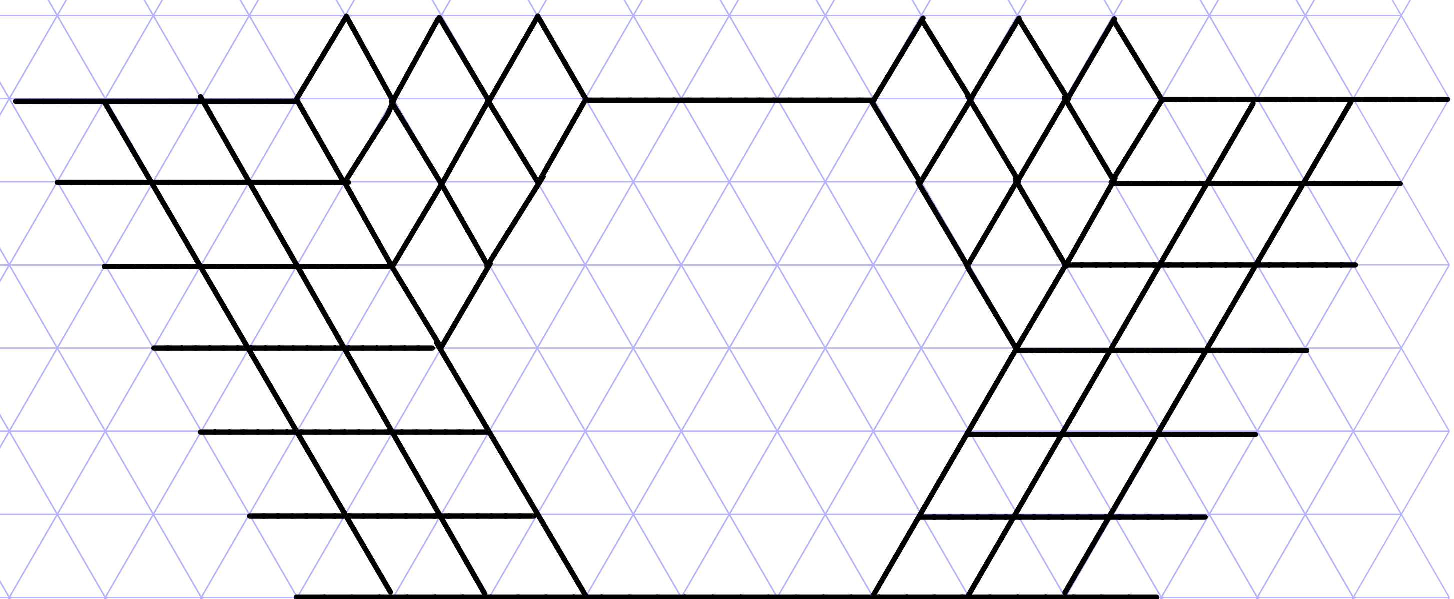
$$\begin{array}{cccccccc}
 & & & 2 & 0 & & & \\
 & & 5 & 4 & 1 & 0 & & \\
 & 8 & 7 & 6 & 2 & 1 & 0 & \\
\iddots & & & & & & & \ddots
\end{array}$$

- Prototypical ensemble in the subject.

N=4



N=6



- Forced polygon is a hexagon, number of sides remains fixed.

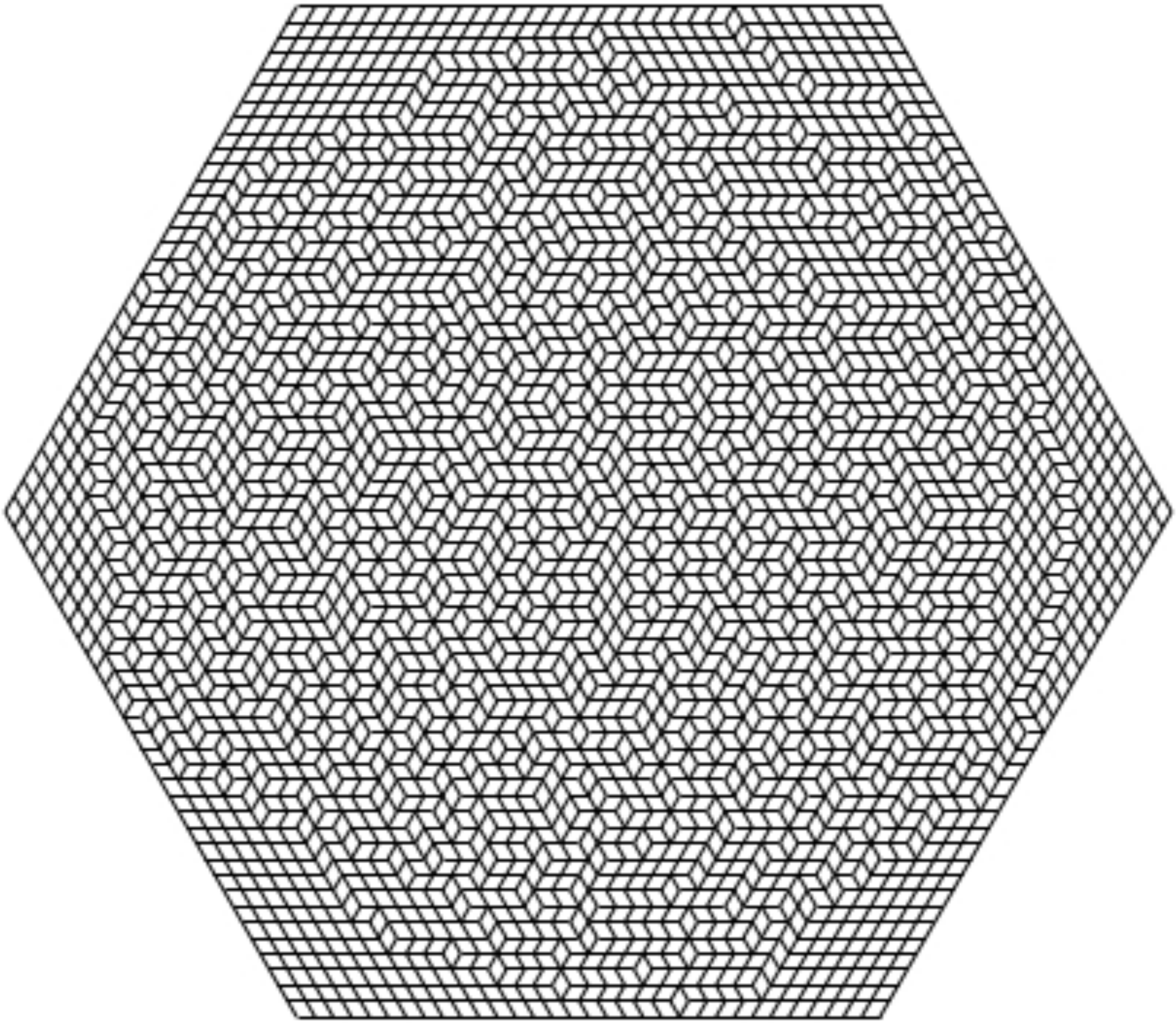
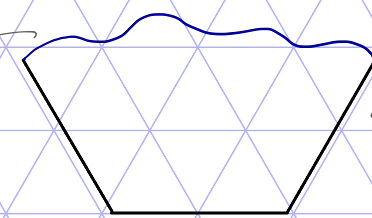


Figure by Cohn, Larsen, Propp

- Suppose the boundary data

$$\begin{array}{ccccc} & & b_1^{(1)} & & \\ & b_1^{(2)} & & b_2^{(2)} & \\ b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\ \iddots & & & & \ddots \end{array}$$

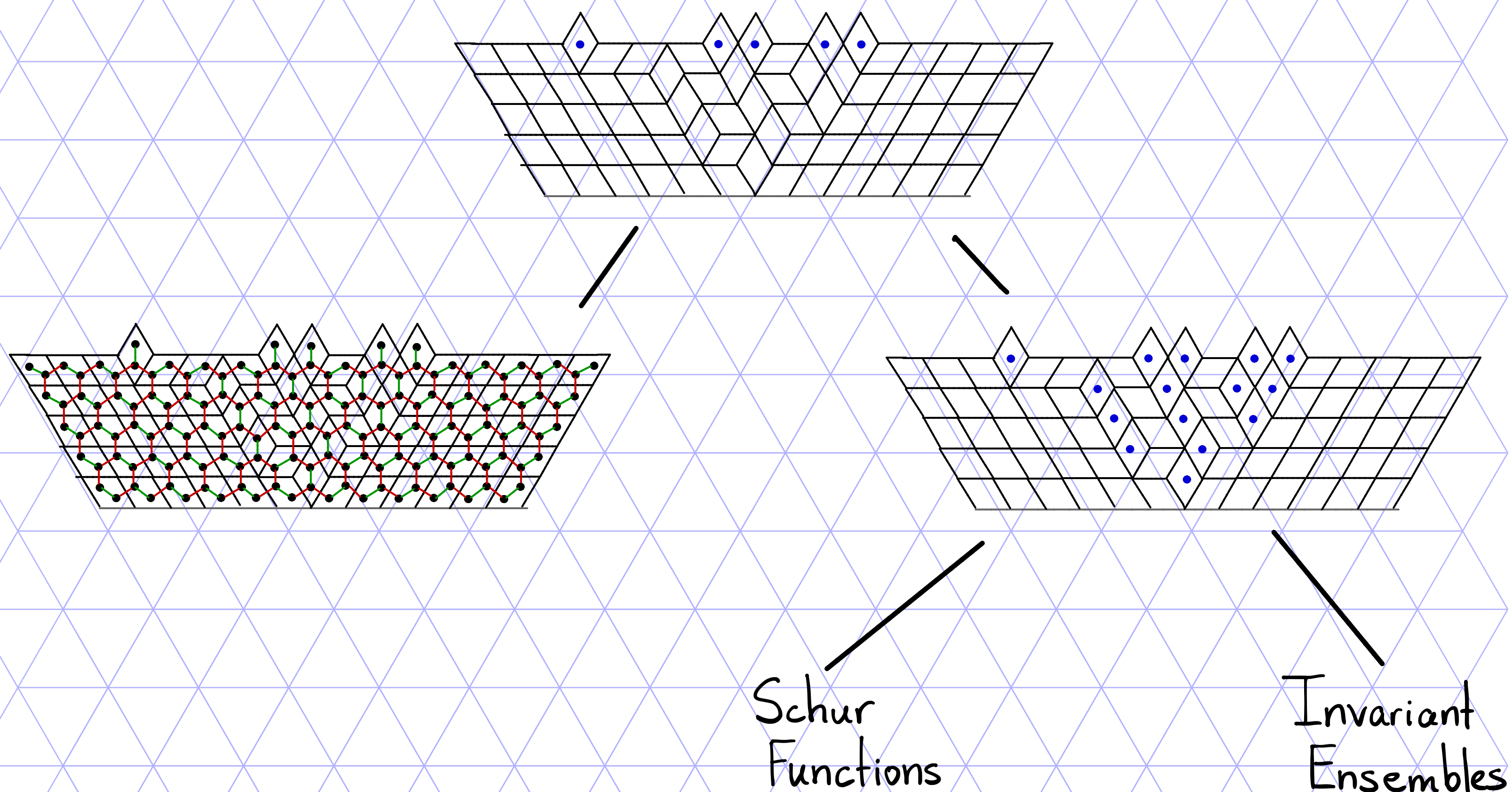


is convergent: have limits

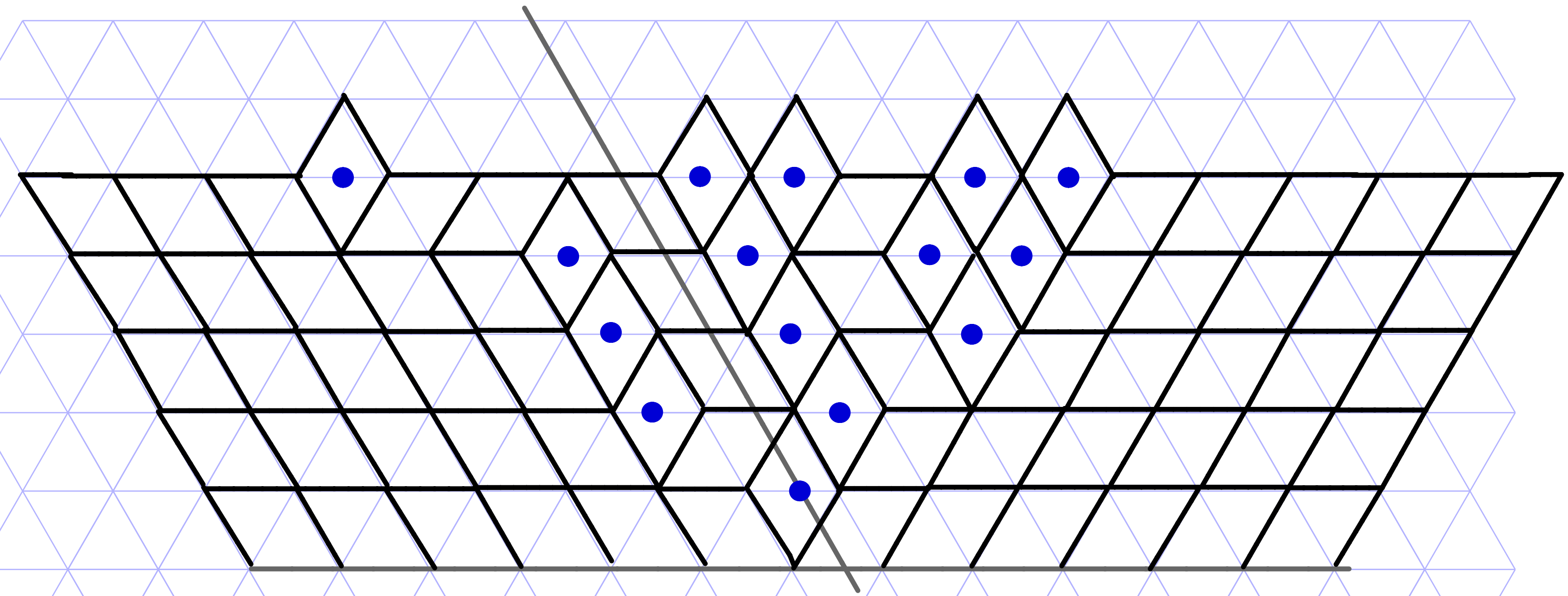
$$p_d^{(\infty)} = \lim_{N\to\infty} p_d\left(\frac{b_1^{(N)}}{N}, \ldots, \frac{b_N^{(N)}}{N}\right).$$

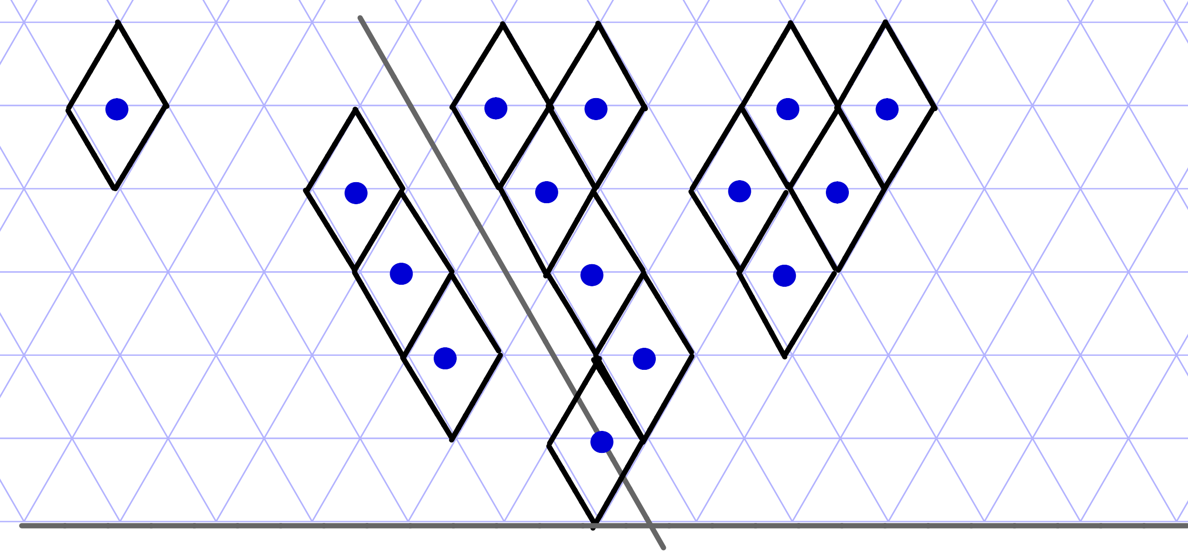
- Does $N^{-1}T^{(N)}$ converge to a deterministic object?

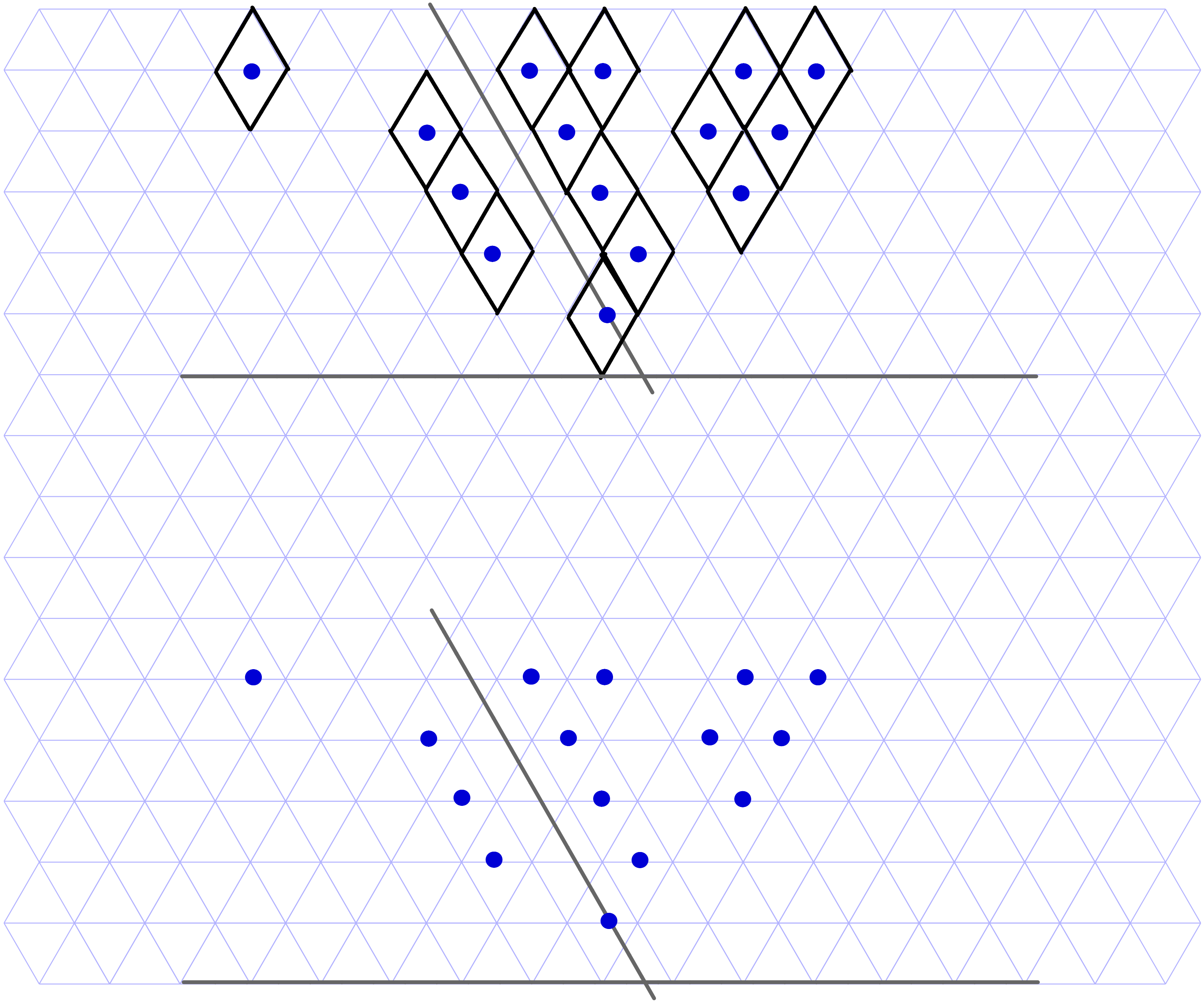
- Want to study lozenge tilings? Good news: you have options.



- We associate to $T^{(N)}$ a "chain" of invariant ensembles.





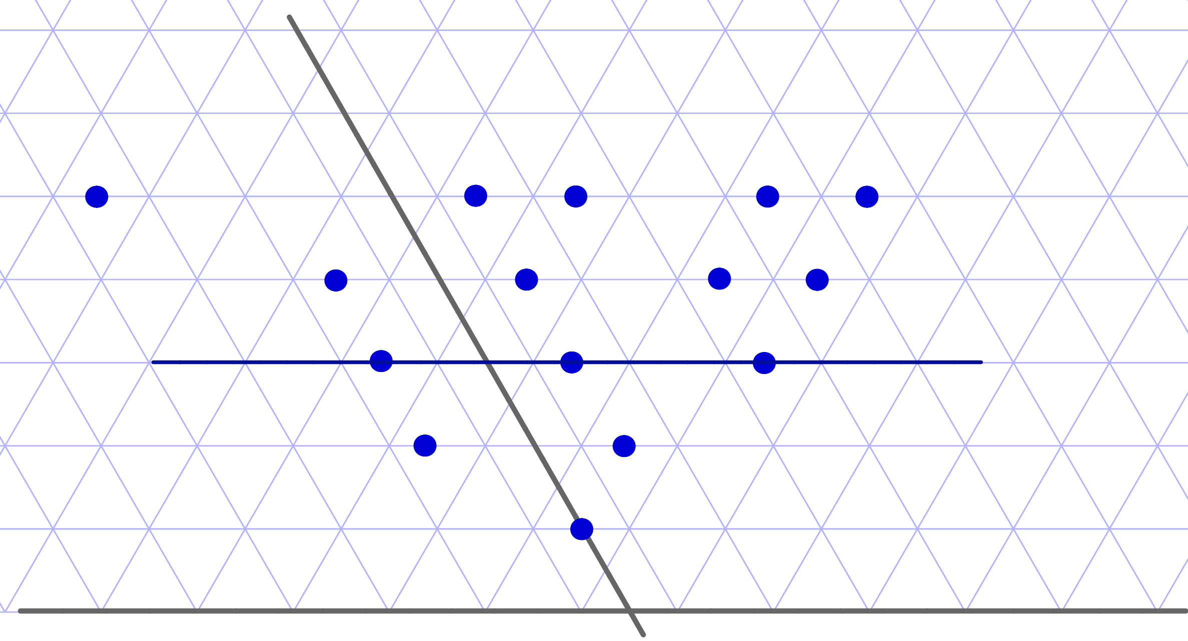


- From particle perspective, $T^{(N)}$ becomes a sequence of N random vectors,

$$\left(b_{k1}^{(N)}, \dots, b_{kk}^{(N)}\right), \quad 1 \le k \le N,$$

where $b_{ki}^{(N)}$ is the i^{th} particle from right on the k^{th} wire from bottom.

- Joint distribution of particles on a given wire?



- Trade the random vector $(b_{k1}^{(N)}, \ldots, b_{kk}^{(N)})$ for the random matrix

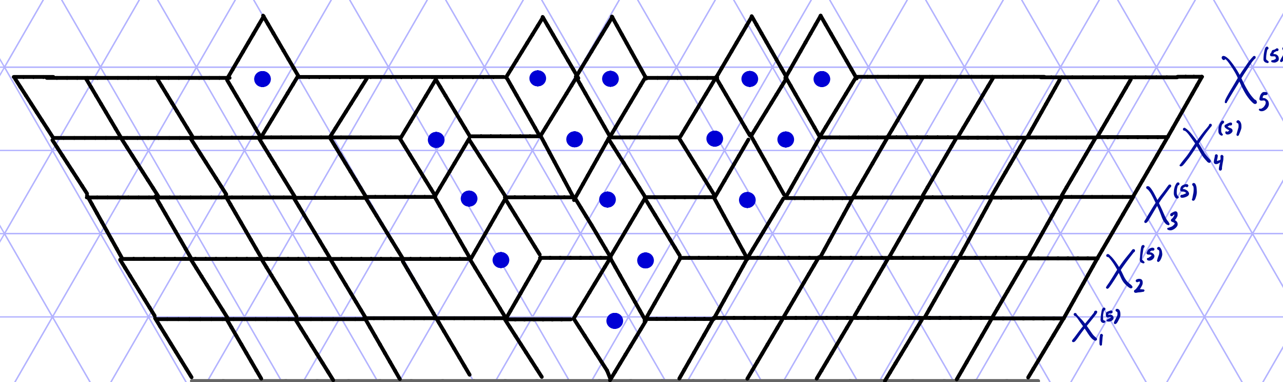
$$X_k^{(N)} = U_k \begin{bmatrix} b_{k1}^{(N)} & & \\ & \ddots & \\ & & b_{kk}^{(N)} \end{bmatrix} U_k^{-1},$$

with U_k a uniformly random $k \times k$ unitary.

- In case $k = N$, just write $X_N^{(N)} = X^{(N)}$; deterministic spectrum of $X^{(N)}$ gives boundary conditions of $\Omega^{(N)}$.

- The joint distribution of $(X_k^{(N)})_{11}, \ldots, (X_k^{(N)})_{kk}$ <u>is</u> known.

- The tiling $T^{(N)}$ becomes a chain of invariant ensembles:



- Want: if (non-random) spectral moments of $N^{-1}X_N^{(N)}$ converge, then spectral moments of $N^{-1}X_{\lfloor tN \rfloor}^{(N)}$ converge, for each $t \in (0,1)$.

- Consider the $(N-k)\times k$ matrix

$$Z = \begin{bmatrix} Z_{11} & \dots & Z_{1k} \\ \vdots & & \vdots \\ Z_{N-k,1} & \dots & Z_{N-k,k} \end{bmatrix}$$

whose entries are independent uniformly random samples from $[0,1]$.

Theorem (N.): We have

$$\left((X_k^{(N)})_{11}, \dots, (X_k^{(N)})_{kk} \right)$$

$$\stackrel{d}{=} (Z_{11}, \dots, Z_{1k}) + \dots + (Z_{N-k,1}, \dots, Z_{N-k,N-k}) + \left((X_N^{(N)})_{11}, \dots, (X_N^{(N)})_{kk} \right).$$

Theorem: Suppose that $p_d^{(\infty)} = \lim_{N\to\infty} N^{-1} X^{(N)}$ exists for each $d \in \mathbb{N}$, and let

$$\left(r_1^{(\infty)}, r_2^{(\infty)}, r_3^{(\infty)}, \ldots\right) = R\left(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \ldots\right).$$

Then, for each $t \in (0,1)$, the random variables

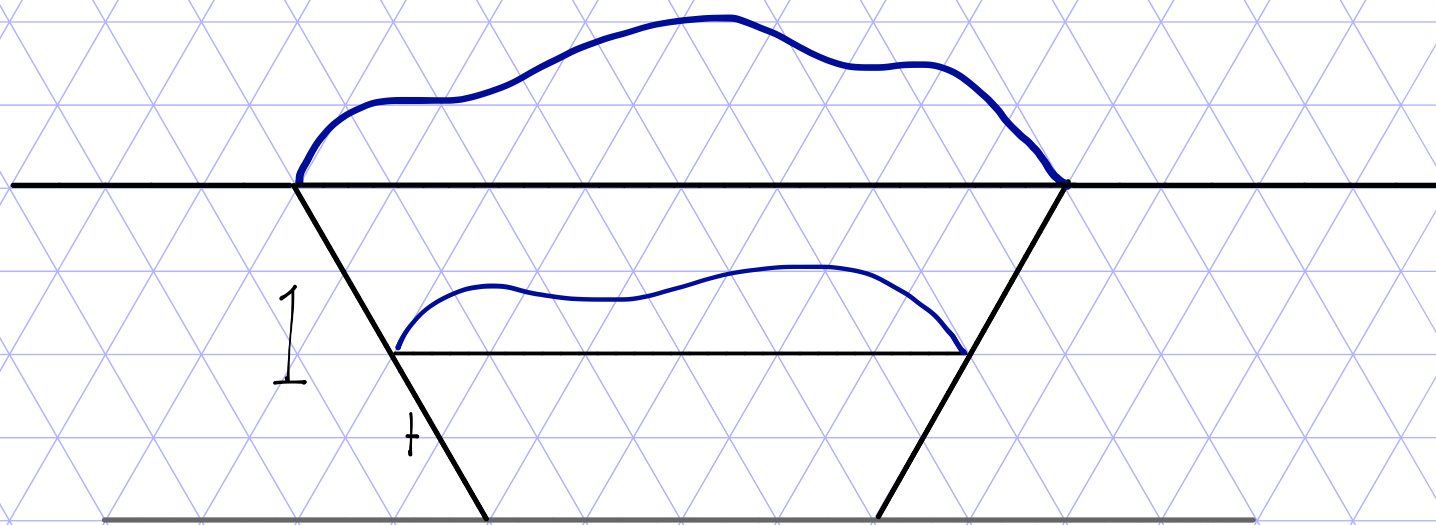
$$p_{d,t}^{(N)} = p_d\left(N^{-1} X_{\lfloor tN \rfloor}^{(N)}\right), \qquad d \in \mathbb{N}$$

converge in probability to deterministic limits $p_{d,t}^{(\infty)}$. Writing

$$\left(r_{1,t}^{(\infty)}, r_{2,t}^{(\infty)}, r_{3,t}^{(\infty)}, \ldots\right) = R\left(p_{1,t}^{(\infty)}, p_{2,t}^{(\infty)}, p_{3,t}^{(\infty)}, \ldots\right),$$

we have

$$r_{d,t}^{(\infty)} = t(1-t)\frac{B_d}{d} + t^{d-1} r_d^{(\infty)}.$$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