Operator-Valued Chordal Loewner Chains and Non-Commutative Probability

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Introduction

Chordal Loewner Chains

Definition

A **normalized chordal Loewner chain** on $[0, T]$ is a family of analytic functions $F_t : \mathbb{H} \to \mathbb{H}$ such that

- $F_0(z) = z$.
- The F_t's are analytic in a neighborhood of ∞.
- If $F_t(z) = z + t/z + O(1/z^2)$.
- For $s < t$, we have $F_t = F_s \circ F_{s,t}$ for some $F_{s,t} : \mathbb{H} \to \mathbb{H}$.

Fact

The F_t's are conformal maps from $\mathbb{H}$ onto $\mathbb{H} \setminus K_t$, where K_t is a growing compact region touching the real line, e.g. a growing slit.

Chordal Loewner Chains

Theorem (Bauer 2005)

Every normalized Loewner chain satisfies the generalized Loewner equation

$$\partial_t F_t(z) = D_z F_t(z) \cdot V(z,t)$$

where $V(z,t)$ is some vector field of the form $V(z,t) = -G_{\nu_t}(z)$. Conversely, given such a vector field, the Loewner equation has a unique solution.

History

Loewner chains in the disk were studied by Loewner in 1923 in the case F_t maps $\mathbb{D}$ onto $\mathbb{D}$ minus a slit. Kufarev and Pommerenke considered more general Loewner chains in the disk. Loewner chains in the half-plane were studied by Schramm in the case $V(z,t) = -1/(z - B_t)$ where B_t is a Brownian motion (SLE).

Loewner Chains and Free Probability

Theorem (Voiculescu, Biane)

If X and Y are freely independent, then $G_{X+Y} = G_X \circ F$ for some analytic $F : \mathbb{H} \to \mathbb{H}$.

Observation (Bauer 2004, Schleißinger 2017)

If X_t is a process with freely independent increments, and if $E(X_t) = 0$ and $E(X_t^2) = t$, then $F_{X_t}(z) = 1/G_{X_t}(z)$ is a normalized chordal Loewner chain.

Remark

The converse is not true. In fact, if σ is the semicircle law and if $F_\mu = F_\sigma \circ F_\sigma$, then μ cannot be written as $\sigma \boxplus \nu$.

Loewner Chains and Monotone Probability

Theorem (Muraki 2000-2001)

If X and Y are monotone independent, then $F_{X+Y} = F_X \circ F_Y$.

Observation (Schleißinger 2017)

If X_t is a process with monotone independent increments, and if $E(X_t) = 0$ and $E(X_t^2) = t$, then $F_t(z) = 1/G_{X_t}(z)$ is a normalized chordal Loewner chain. Every normalized Loewner chain arises in this way.

History

The differential equation $\partial_t F_t(z) = DF_t(z)[V(z)]$ was studied earlier by Muraki and Hasebe, and Schleißinger connected it with the Loewner equation.

Overview

Goal

Adapt the theory of Loewner chains to the non-commutative upper half-plane $\mathbb{H}(\mathcal{A})$ for a C^* algebra $\mathcal{A}$.

Overview:

1. Background on operator-valued laws.
2. Loewner chains $F_t = F_{\mu_t}$ and the Loewner equation.
3. Combinatorial computation of moments for μ_t.
4. Central limit theorem describing behavior for large t.

Operator-valued Laws and Cauchy Transforms

$\mathcal{A}$-valued Probability Spaces

Definition

Let $\mathcal{A}$ be a C^*-algebra. An $\mathcal{A}$-**valued probability space** $(\mathcal{B}, E)$ is a C^* algebra $\mathcal{B} \supseteq \mathcal{A}$ together with a bounded, completely positive, unital, $\mathcal{A}$-bimodule map $E : \mathcal{B} \to \mathcal{A}$, called the **expectation**.

Definition

$\mathcal{A}\langle X \rangle$ denotes the $*$-algebra generated by $\mathcal{A}$ and a non-commutating self-adjoint indeterminate X.

$\mathcal{A}$-valued Laws

Definition

A linear map $\mu : \mathcal{A}\langle X \rangle \to \mathcal{A}$ is called a **(bounded) law** if

1. μ is a unital $\mathcal{A}$-bimodule map.
2. μ is completely positive.
3. There exist $C > 0$ and $M > 0$ such that

$$||\mu(a_0 X a_1 X \dots a_{n-1} X a_n)|| \leq C M^n ||a_0|| \dots ||a_n||.$$

Definition

We call μ a **(bounded) generalized law** if it satisfies (2) and (3) but not necessarily (1).

$\mathcal{A}$-valued Laws

Definition

For a generalized law μ, we define

$$\text{rad}(\mu) = \inf\{M > 0 : \exists C > 0 \text{ s.t. condition (3) is satisfied}\}.$$

Theorem (Popa-Vinnikov 2013, Williams 2013)

For a generalized law μ, there exists a C^*-algebra $\mathcal{B}$, a $*$-homomorphism $\pi : \mathcal{A}\langle X \rangle \to \mathcal{B}$ which is bounded on $\mathcal{A}$, and a completely positive $\tilde{\mu} : \mathcal{B} \to \mathcal{A}$ such that $\mu = \tilde{\mu} \circ \pi$ and $\|\pi(X)\| = \text{rad}(\mu)$.

In particular, every law μ is realized as the law of a self-adjoint $\pi(X)$ in a probability space $(\mathcal{B}, \tilde{\mu})$.

Matricial Upper Half-Plane

Definition

The matricial upper half-plane is defined by

$$\mathbb{H}^{(n)}(\mathcal{A}) = \bigcup_{\epsilon > 0} \{z \in M_n(\mathcal{A}) : \operatorname{Im} z \geq \epsilon\}$$

$$\mathbb{H}(\mathcal{A}) = \{\mathbb{H}^{(n)}(\mathcal{A})\}_{n \geq 1}.$$

Definition

A **matricial analytic function** on $\mathbb{H}(\mathcal{A})$ is a sequence of analytic functions $F^{(n)}(z)$ defined on $\mathbb{H}^{(n)}(\mathcal{A})$ such that F preserves direct sums of matrices and conjugation by scalar matrices.

Cauchy Transforms

Definition (Voiculescu)

The Cauchy transform of a generalized law μ is defined by
$G_\mu^{(n)}(z) = \mu \otimes \mathrm{id}_{M_n(\mathbb{C})}[(z - X \otimes 1_{M_n(\mathbb{C})})^{-1}]$.

Theorem (Williams 2013, Williams-Anshelevich 2015)

A matricial analytic function $G : \mathbb{H}(\mathcal{A}) \to -\mathbb{H}(\mathcal{A})$ is the Cauchy transform of a generalized law μ with $\mathrm{rad}(\mu) \leq M$ if and only if

1. *G is matricial analytic.*
2. *$\tilde{G}(z) := G(z^{-1})$ extends to be matricial analytic on $\{\|z\| < 1/M\}$.*
3. *$\|G^{(n)}(z)\| \leq C_\epsilon$ for $\|z\| < 1/(M+\epsilon)$, where C_ϵ is independent of n.*
4. *$\tilde{G}(z^*) = \tilde{G}(z)^*$.*
5. *$\tilde{G}(0) = 0$.*

Also, μ is a generalized law if and only if $\lim_{z \to 0} z^{-1} \tilde{G}^{(n)}(z) = 1$ for each n.

$\mathcal{A}$-valued Chordal Loewner Chains

Definition

Definition

An $\mathcal{A}$-**valued chordal Loewner chain** on $[0, T]$ is a family of matricial analytic functions $F_t(z) = F(z, t)$ on $\mathbb{H}(\mathcal{A})$ such that

- $F_0 = \text{id}$
- F_t is the recriprocal Cauchy transform of an $\mathcal{A}$-valued law μ_t.
- If $s < t$, then $F_t = F_s \circ F_{s,t}$ for some matricial analytic $F_{s,t} : \mathbb{H}(\mathcal{A}) \to \mathbb{H}(\mathcal{A})$.
- $\mu_t(X)$ and $\mu_t(X^2)$ are continuous functions of t.

Basic Properties

Remark

Loewner chains relate to free and monotone independence over $\mathcal{A}$ just as in the scalar case.

Lemma

- $F_{s,t}$ *is unique.*
- $F_{0,t} = F_t$.
- $F_{s,t} \circ F_{t,u} = F_{s,u}$.
- $F_{s,t}$ *is the* F*-transform of a law* $\mu_{s,t}$.
- $\sup_{s,t} \operatorname{rad}(\mu_{s,t}) \leq C \operatorname{rad}(\mu_T) + C \sup_t \|\mu_t(X)\|$.

Basic Properties

Lemma

There exists a generalized law $\sigma_{s,t}$ *such that*

$$F_{s,t}(z) = z - \mu_{s,t}(X) - G_{\sigma_{s,t}}(z).$$

We have $\operatorname{rad}(\sigma_{s,t}) \leq 2 \operatorname{rad}(\mu_{s,t})$ *and* $\sigma_{s,t}(1) = \mu_{s,t}(X^2) = \mu_t(X^2) - \mu_s(X^2)$.

Biholomorphicity

Theorem

Each $F_{s,t}$ is a biholomorphic map onto a matricial domain and the inverse is matricial analytic. Moreover, given $\epsilon > 0$, there exists $\delta > 0$ depending only on ϵ and the modulus of continuity of $t \mapsto \mu_t(X^2)$, such that

1. $\operatorname{Im} z \geq \epsilon \implies \|DF_{s,t}(z)^{-1}\| \leq 1/\delta$.
2. $\operatorname{Im} z, \operatorname{Im} z' \geq \epsilon \implies \|F_{s,t}(z) - F_{s,t}(z')\| \geq \delta \|z - z'\|$.

Proof:

- By the inverse function theorem, it suffices to prove the estimates (1) and (2).
- Renormalize so that μ_t has mean zero.

Biholomorphicity

- Fix $\epsilon > 0$.
- If $t - s$ is small, then $F_{s,t}(z) \approx z$ because

$$F_{s,t}(z) - z = G_{\sigma_{s,t}}(z) = O(\gamma),$$

  where

$$\gamma = \epsilon^{-1} \| \sigma_{s,t}(1) \| = \epsilon^{-1} \| \mu_t(X^2) - \mu_s(X^2) \|,$$

  which goes to zero as $t - s \to 0$.
- Similar estimates show that $DF_{s,t}(z) = \mathrm{id} + O(\gamma)$ and $F_{s,t}(z) - F_{s,t}(z') = z - z' + O(\gamma \| z - z' \|)$.

Biholomorphicity

- Hence, the claims hold when $t - s$ is sufficiently small.
- The claims hold for arbitrary $s < t$ using iteration:

$$F_{s,t} = F_{s,t_1} \circ F_{t_1,t_2} \circ \cdots \circ F_{t_{n-1},t}$$

and each function maps $\{\operatorname{Im} z \geq \epsilon\}$ into $\{\operatorname{Im} z \geq \epsilon\}$.

The Loewner Equation

The Loewner Equation

- The operator-valued version of the Loewner equation is

$$\partial_t F(z,t) = DF(z,t)[V(z,t)],$$

  where $DF(z,t)$ is the Fréchet derivative with respect to z, and $V(z,t)$ is a vector field of the form $V(z,t) = -G_{\nu_t}(z)$ for a generalized law ν_t.
- We want to show that the Loewner equation defines a bijection between Loewner chains $F(z,t)$ and Herglotz vector fields $V(z,t)$ on $[0,T]$.

Problems with Pointwise Differentiation

- We should allow Loewner chains which are Lipschitz in t, so we need to differentiate Lipschitz functions $[0, T] \to M_n(\mathcal{A})$.
- A C^*-algebra $\mathcal{A}$ is a bad Banach space for differentiation.
- It would not be enough to differentiate for a.e. t for each fixed z; we would also need to have the *same* exceptional set of times for every z in an open set in our huge Banach space.
- Pointwise differentiation won't work.
- So consider $\partial_t F(z, \cdot)$ as an $M_n(\mathcal{A})$-valued distribution on $[0, T]$.

Distributional Differentiation

- But we need to manipulate $\partial_t F(z, \cdot)$ like a pointwise defined function, e.g. we want:

$$\partial_t[F(G(z,t),t)] = \partial_t F(G(z,t),t) + DF(G(z,t),t)[\partial_t G(z,t)].$$

- Luckily, since $F(z, \cdot)$ is Lipschitz, it makes sense to pair $\partial_t F(z, \cdot)$ with an L^1 function $\phi : [0, T] \to \mathbb{C}$.
- Thus, $\partial_t F(z,t)$ is an element of $\mathcal{L}(L^1[0,T], M_n(\mathcal{A}))$, which is "almost as nice" as an L^∞ function $[0,T] \to M_n(\mathcal{A})$.

Distributional Differentiation

- A family of Banach-valued analytic functions $F(z,t)$ for $t \in [0,T]$ is a called a **locally Lipschitz family** if it is Lipschitz in t with uniform Lipschitz constants for z in a neighborhood of each z_0 in the domain.
- If $F(z,t)$ and $G(z,t)$ are locally Lipschitz families, then we can define

$$\partial_t F(G(z,t),t) \in \mathcal{L}(L^1[0,T],\mathcal{X})$$

  by approximating $G(z,t)$ with step-functions of t.
- We can define $DF(G(z,t),t)[\partial_t G(z,t)]$ similarly.
- The chain rule computation above is correct in $\mathcal{L}(L^1[0,T],\mathcal{X})$.

The Loewner Equation: Setup

Definition

A **Lipschitz, normalized Loewner chain** is a Loewner chain such that $\mu_t(X) = 0$ and $\mu_t(X^2)$ is a Lipschitz function of t.

Definition

A **Herglotz vector field** $V(z,t)$ to be a matricial analytic function $\mathbb{H}(\mathcal{A}) \to \mathcal{L}(L^1[0,T], M_n(\mathcal{A}))$ such that for each nonnegative $\phi \in L^1[0,T]$, the function $-\int V(z,t)\phi(t)\,dt$ is the Cauchy transform of a generalized law $\nu[\phi]$ with $\sup_\phi \operatorname{rad}(\nu[\phi]) < +\infty$.

Definition

In this case, we call the map $\nu : L^1[0,T] \times \mathcal{A}\langle X \rangle \to \mathcal{A}$ a **distributional generalized law** and denote $\operatorname{rad}(\nu) = \sup_{\phi \geq 0} \operatorname{rad}(\nu[\phi])$.

The Loewner Equation: Main Theorem

Theorem

On an interval $[0, T]$*, the Loewner equation* $\partial_t F(z,t) = DF(z,t)[V(z,t)]$ *defines a bijection between Lipschitz, normalized* $\mathcal{A}$*-valued Loewner chains and Herglotz vector fields (and hence distributional generalized laws).*

We sketch of the proof in two parts:

- Differentiation of Loewner chains $F(z,t) \rightsquigarrow V(z,t)$.
- Integration of the Loewner equation $V(z,t) \rightsquigarrow F(z,t)$.

Differentiation of Loewner Chains

- We write $F_{s,t}(z) = z - G_{\sigma_{s,t}}(z)$. Note $\sigma_{s,t}(1) = \mu_t(X^2) - \mu_s(X^2)$.
- A priori estimates on Cauchy transforms show that $F(z,t)$ is a locally Lipschitz family.
- We know $DF(z,t)$ is invertible for $\operatorname{Im} z \geq \epsilon$, so we can define

$$V(z,t) = DF(z,t)^{-1}[\partial_t F(z,t)].$$

- To check that $V(z,t)$ is Herglotz vector field, we approximate $V(z,t)$ by the step-function Herglotz vector field

$$V_m(z,t) = -\sum_{j=1}^{m} m\chi_{[t_{j-1},t_j]}(t) G_{\sigma_{t_{j-1},t_j}}(z), \text{ where } t_j = \frac{jT}{m}.$$

Integration of the Loewner Equation

- The proof proceeds the same way as in the scalar case (Bauer 2005).
- Using a chain rule argument, it is sufficient to solve the ODE

$$-\partial_s F_{s,t}(z) = V(F_{s,t}(z), s) \text{ for } s \in [0,t], \qquad F_{t,t}(z) = z.$$

- We use Picard iteration and make explicit estimates to show that the Picard iterates converge uniformly on $\operatorname{Im} z \geq \epsilon$.
- We verify analytically that the iterates and the limit are reciprocal Cauchy transforms.

The Moments of μ_t

Combinatorial Formula

Definition

$NC_{\leq 2}(n)$ is the set of non-crossing partitions of $[n]$ where each block is a pair or a singleton. $NC^0_{\leq 2}(n)$ is the subset consisting of partitions where each singleton block is "inside" some pair block.

Definition

Let $\mathcal{C} = C([0,T],\mathcal{A})$. For a distributional generalized law ν, define $I = I_\nu : \mathcal{C}\langle X \rangle \to \mathcal{C}$ by

$$I_\nu[f(X,t)](t) = \int_t^T \nu_s[f(X,s)]\,ds.$$

Combinatorial Formula

Definition

For $\pi \in NC_{\leq 2}$, we define $Q_\pi(z, t)$ by replacing each singleton by X and each pair by $I_\nu(\dots)$ and inserting z between any two consecutive elements of $\{1, \dots, n\}$. For example, with $n = 5$,

$$\pi = \{\{1,5\},\{2,3\},\{4\}\} \implies Q_\pi(z) = I_\nu(zI_\nu(z)zXz).$$

Theorem

Let $F(z,t) = G_{\mu_t}(z)^{-1}$ be the Loewner chain corresponding to $V(z,t) = -G_{\nu_t}(z)$. Then

$$G_{\mu_t}(z^{-1}) = \sum_{\pi \in NC^0_{\leq 2}} zQ_\pi(z)z.$$

Fock Space Construction

Goal

Realize μ_t by creating self-adjoint operators on a Fock space with the correct moments.

We define a Fock space $\mathcal{H}_\nu = \bigoplus_{n=0}^\infty \mathcal{H}_n$, where

$$\mathcal{H}_n = \mathcal{C}\langle X \rangle \otimes \cdots \otimes \mathcal{C}\langle X \rangle \otimes \mathcal{C}$$

with the $\mathcal{C}$-valued inner product

$$\langle f_n \otimes \cdots \otimes f_0, g_n \otimes \cdots \otimes g_0 \rangle = f_0^* I_\nu(f_1^* \dots I_\nu(f_n^* g_n) \dots g_1) g_0.$$

We denote the creation and annihilation operators by $\ell(f)$ and $\ell(f)^*$. Every $f(X,t) \in \mathcal{C}\langle X \rangle$ defines a multiplication operator acting on the left-most coordinate, where the action on $\mathcal{H}_0 = \mathcal{C}$ is defined to be multiplication by $f(0,t)$.

Fock Space Construction

Theorem

Let $Y_{t_1,t_2} = \ell(\chi_{[t_1,t_2)}) + \ell(\chi_{[t_1,t_2)})^* + \chi_{[t_1,t_2)}(t)X$. *Define an expectation by*

$$E(T) = \langle \Omega, T\Omega \rangle_{\mathcal{H}_\nu}|_{t=0}.$$

Then

1. $Y_{t_1,t_3} = Y_{t_1,t_2} + Y_{t_2,t_3}$.
2. Y_{t_1,t_2} *and* Y_{t_2,t_3} *are monotone independent over* $\mathcal{A}$ *with respect to* E.
3. Y_{t_1,t_2} *has the law* μ_{t_1,t_2} *with respect to* E.

Central Limit Theorem for Loewner Chains

Background for CLT

- Muraki showed that the central limit object for monotone independence is the arcsine law.
- The arcsine law of variance t has reciprocal Cauchy transform $F_t(z) = \sqrt{z^2 - 2t}$ which maps $\mathbb{H}$ onto $\mathbb{H}$ minus a vertical slit.
- F_t solves the Loewner equation with $V(z,t) = -1/z$.

Definition

Let $\eta : \mathcal{A} \times L^1[0,T] \to \mathcal{A}$ be a distributional completely positive map. We define the corresponding $\mathcal{A}$-valued generalized arcsine law μ_η as the law obtained by running the Loewner equation up to time T with $V(z,t) = -\eta_t(z^{-1})$.

CLT via Coupling

Let ν be a distributional generalized law and let $\eta_t = \nu_t|_{\mathcal{A}}$. Using the Fock space $\mathcal{H}_\nu$, define

- $Y_{t_1,t_2} = \ell(\chi_{[t_1,t_2)}) + \ell(\chi_{[t_1,t_2)})^* + \chi_{[t_1,t_2)}(t)X$.
- $Z_{t_1,t_2} = \ell(\chi_{[t_1,t_2)}) + \ell(\chi_{[t_1,t_2)})^*$.

Let $F_t = F_{\mu_t}$ be the solution to the Loewner equation for $-G_{\nu_t}(z)$.

Theorem

Y_{t_1,t_2} has the law μ_{t_1,t_2} and Z_{t_1,t_2} has the generalized arcsine law for $\eta|_{[t_1,t_2]}$. Moreover, we have

$$\|Y_{t_1,t_2} - Z_{t_1,t_2}\| \leq \operatorname{rad}(\nu).$$

As a consequence, for $\operatorname{Im} z \geq \epsilon$,

$$\|T^{1/2} G_{Y_{0,T}}(T^{1/2} z) - T^{1/2} G_{Z_{0,T}}(T^{1/2} z)\| \leq T^{-1/2} \epsilon^{-2} \operatorname{rad}(\nu).$$

CLT via Loewner Equation

Another proof is a "continuous-time Lindeberg exchange" where we interpolate between $Y_{0,T}$ and $Z_{0,T}$ using $Y_{0,t} + Z_{t,T}$. In other words, we write

$$G_{Y_{0,T}} - G_{Z_{0,T}} = \int_0^T \partial_t [G_{Y_{0,t}} \circ F_{Z_{t,T}}]\, dt.$$

Evaluate this using the chain rule and the Loewner equation and make some straightforward estimates ...

Theorem

For $\operatorname{Im} z \geq \epsilon$, *we have*

$$\|T^{1/2} G_{Y_{0,T}}(T^{1/2} z) - T^{1/2} G_{Z_{0,T}}(T^{1/2} z)\| \\ \leq T^{-1/2} \epsilon^{-4} \operatorname{rad}(\nu) \|\nu(1)\|_{\mathcal{L}(L^1[0,T],\mathcal{A})}.$$

Concluding Remarks

Concluding Remarks

- For tensor, free, and boolean independence, there is a similar (but simpler!) differential equation for the analytic transforms of processes with independent increments, which can be analyzed using the same techniques.
- The Fock space construction and coupling argument for the CLT work for other types of independence as well.
- Processes for each type of independence are in bijective correspondence with distributional generalized laws ν, and hence we get a generalization of the Bercovici-Pata bijection for $\mathcal{A}$-valued processes with non-stationary increments.

Concluding Remarks

Conjecture

The same theory will work for multiplicative convolution of unitaries and positive operators.

Warning

For tensor independence, we need to assume $\mathcal{A}$ is commutative (as far as we know), and we must analyze unbounded laws.

Warning

The coupling is produced on a probability space $(\mathcal{B}, E)$, where E is extremely not faithful and not tracial!

Concluding Questions

Question

How well do these techniques adapt to operator-valued laws with unbounded support?

Question

Can every reciprocal $\mathcal{A}$-valued Cauchy transform which is matricially biholomorphic be embedded into a Loewner chain? (Yes in scalar case, Bauer 2005.)

Question

Is there a version of the Riemann mapping theorem for matricial domains?

References

This is based on arXiv:1711.02611, which contains complete citations.
For further reading,

1. R. O. Bauer, Löwner's equation from a noncommutative probability perspective, Journal of Theoretical Probability, 17 (2004), pp. 435457.
2. R.O. Bauer, Chordal Loewner families and univalent Cauchy transforms, Journal of Mathematical Analysis and Applications, 302 (2005), pp. 484 501.
3. S. Schleißinger, The chordal Loewner equation and monotone probability theory, Infinite-dimensional Analysis, Quantum Probability, and Related Topics, 20 (2017).
4. S. T. Belinschi, M. Popa, and V. Vinnikov, On the operator-valued analogues of the semicircle, arcsine and Bernoulli laws, Journal of Operator Theory, 70 (2013), pp. 239258.