Free transport for interpolated free group factors

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Joint with Brent Nelson

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  - Trace vector: $\mathrm{tr}(x) = \langle \Omega | x\Omega \rangle$, $x \in W^*(X_1, X_2, \cdots)$.
  - Key property:

$$\mathrm{tr}(X_{i_1} X_{i_2} \cdots X_{i_n}) = \sum_{i_k = i_1} \mathrm{tr}(X_{i_2} \cdots X_{i_{k-1}})\, \mathrm{tr}(X_{i_{k+1}} \cdots X_{i_n})$$

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  3. $pL(\mathbb{F}_t)p = L(\mathbb{F}(1 + \frac{t-1}{\mathrm{tr}(p)^2}))$ for $p \in \mathcal{P}(L(\mathbb{F}_t))$

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    - vN$(R, p_s X_s p_s)$ depends only on $\sum_{s \in S} \operatorname{tr}(p_s)^2$. Get $L(\mathbb{F}_t)$ with $t = 1 + \sum_{s \in S} \operatorname{tr}(p_s)^2$

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  3. $x_{\epsilon^{op}} = x_\epsilon^*$
  4. $x_{\epsilon_1} \cdots x_{\epsilon_n} = 0$ unless $\epsilon_1 \cdots \epsilon_n$ is a path.
  5. $\mathrm{tr}(x_{\epsilon_1} \cdots x_{\epsilon_n}) = \frac{1}{\sqrt{\mu(s(\epsilon_1))\mu(t(\epsilon_1))}} \sum_{\epsilon_j = \epsilon_1^{op}} \mathrm{tr}(x_{\epsilon_2} \cdots x_{\epsilon_{j-1}}) \,\mathrm{tr}(x_{\epsilon_{j+1}} \cdots x_{\epsilon_n})$

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    6. If $|V| = 1$, this gives Voiculescu's free semicircular system.

von Neumann algebras

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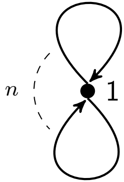
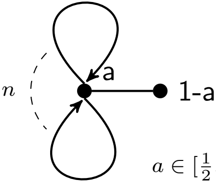
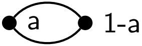
- Set $\mathcal{M}(\Gamma,\mu) = (\mathcal{S}(\Gamma,\mu),\mathrm{tr})''$
- Theorem (H. '13): Let $V_>$ be the set of vertices, β satisfying $\mu(\beta) > \sum_{\alpha\sim\beta} n_{\alpha,\beta}\mu(\alpha)$. We have

$$\mathcal{M}(\Gamma,\mu) \cong L(\mathbb{F}_t) \oplus \bigoplus_{\gamma\in V_>}^{r_\gamma} \mathbb{C}$$

where $r_\gamma \leq p_\gamma$ and $\tau(r_\gamma) = \mu(\gamma) - \sum_{\alpha\sim\gamma} n_{\alpha,\beta}\mu(\alpha)$. If $\mathcal{M}(\Gamma,\mu)$ is a factor, then

$$t = 1 - \sum_{v\in V}\mu(v)^2 + \sum_{v\in V}\mu(v)\sum_{w\sim v} n_{v,w}\mu(w)$$

Examples

Γ	$\mathcal{M}(\Gamma, \mu)$
	$L(\mathbb{F}_n)$
 $a \in [\frac{1}{2}, 1)$	$L(\mathbb{F}_t)$ $t = (n-4)a^2 + 4a$
 $a \in [\frac{1}{3}, \frac{2}{3}]$	$L(\mathbb{F}_t)$ $t = 6(a - a^2)$

C*-algebras

- Theorem (H '16) Let $V_=$ be the set of vertices β satisfying $\mu(\beta) = \sum_{\alpha\sim\beta} n_{\alpha,\beta}\mu(\alpha)$, and let $V_\geq = V_> \cup V_=$. Let I be the norm-closed ideal generated by generated by $(x_e)_{e\in E}$. Then:
  - I is minimal, simple, has unique trace, and has stable rank 1.
  - I is unital if and only if $V_=$ is empty. If $V_=$ is empty, then

    $$\mathcal{S}(\Gamma,\mu) = I \oplus \bigoplus_{\gamma\in V_>} \overset{r_\gamma}{\mathbb{C}}$$

    with $r_\gamma \leq p_\gamma$ and $\tau(r_\gamma) = \mu(\gamma) - \sum_{\alpha\sim\gamma} n_{\alpha,\beta}\mu(\alpha)$. If $V_=$ is not empty, then

    $$\mathcal{S}(\Gamma,\mu) = \mathcal{I} \oplus \bigoplus_{\gamma\in V_>} \overset{r_\gamma}{\mathbb{C}}$$

    where $\mathcal{I}$ is unital, and the strong operator closures of I and $\mathcal{I}$ coincide in $L^2(\mathcal{S}(\Gamma,\mu),\tau)$, and $\mathcal{I}/I \cong \bigoplus_{\beta\in V_=} \mathbb{C}$.
  - $K_0(I) \cong \mathbb{Z}\{[p_\beta] | \beta \in V \setminus V_\geq\}$ and $K_1(I) = \{0\}$ where the first group is the free abelian group on the classes of projections $[p_\beta]$. Furthermore, $K_0(I)^+ = \{x \in K_0(I) | \mathrm{tr}(x) > 0\} \cup \{0\}$.

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$$\left\| \sum_{v \in V} a_v p_v + \sum_{\epsilon_1 \cdots \epsilon_n} a_{\epsilon_1, \ldots, \epsilon_n} x_{\epsilon_1} \cdots x_{\epsilon_n} \right\|_R$$

$$= (\sup_{v \in V} |a_v|) + \sum_{\epsilon_1 \ldots \epsilon_n} |a_{\epsilon_1 \ldots \epsilon_n}| R^n$$

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$$\mathscr{J}^*\left(\frac{P}{1 + \mathscr{J} f}\right) = M\#x + (\mathscr{D}W)(x + f)$$

We further assume $f = \mathscr{D}g$

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  1. Choose R so that $R\min_{\epsilon\in\vec{E}}\mu(\epsilon) > 4$
  2. Choose $S > R + \frac{1}{R}$.
  3. Assume $W \in A^S$ with
     - $\|W\|_S \leq \frac{1}{2}\min_{\epsilon\in\vec{E}}\mu(\epsilon)$
     - $\|W\|_S \leq 2e\left(R+\frac{1}{R}\right)\log\left(\frac{S}{R+\frac{1}{R}}\right)$.

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- With even smaller norm conditions on W, one can express each x_ϵ as a power series in the y_ϕ via an inverse function theorem.
- This establishes the following theorem: If W is of sufficiently small analytic norm, then there exists a linear functional ϕ on $B = \mathsf{Alg}((p_v)_{v \in V}, (y_\epsilon)_{e \in \vec{E}})$ satisfying Schwinger-Dyson with potential $V_\mu + W$. Furthermore, $C^*(B, \phi) \cong \mathcal{S}(\Gamma, \mu)$ and $W^*(B, \phi) \cong \mathcal{M}(\Gamma, \mu)$

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$$\eta_{e,e'}(p_v) = \begin{cases} \delta_{e,e'} \sqrt{\frac{\mu(v)}{\mu(w)}} p_w & \text{if } v \sim_e w \\ \delta_{e,e'} p_v & \text{if } e \text{ is a loop} \end{cases}$$

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