Spectral gap characterizations of property (T) for II_1 Factors

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Goal: characterization of Property (T) by spectral gaps in inclusions into tracial von Neumann algebras for separable II_1 factors.

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(A question asked in [Goldbring, 2020])

Bimodules (Correspondences) and Connes' Tensor Product

Hilbert bimodules

- For two tracial von Neumann algebras (M, τ_M) and (N, τ_N), a *M-N-bimodule* is a Hilbert space $\mathcal{H}$ equipped with a normal unital homomorphism $\pi_l : M \to \mathcal{B}(\mathcal{H})$ and a normal unital homomorphism $\pi_r : N^{op} \to \mathcal{B}(\mathcal{H})$ such that π_l and π_r commute.

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Example

Given a tracial von Neumann algebra (M, τ_M), let $L^2(M, \tau)$ be the completion of M with the inner product $\langle x, y \rangle = \tau(x^* y)$. Then $L^2(M, \tau)$ is an M-M-bimodule.

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- For an M-N-bimodule ${}_M\mathcal{H}_N$ and an N-P-bimodule ${}_N K_P$, where M, N and P are three tracial von Neumann algebras, the *Connes fusion tensor product* $\mathcal{H} \otimes_N \mathcal{K}$ is a M-P-bimodule .

Kazhdan Property (T)

For groups

Let Γ be a discrete group. Then Γ has *Property (T)* ([Kazhdan, 1967]), if for any unitary representation $(\pi, \mathcal{H})$ of Γ with almost invariant unit vectors ξ_i's:

$$\xi_i \in \mathcal{H} \text{ such that } ||\pi(\gamma)\xi_i - \xi_i|| \to 0 \text{ for every } \gamma \in \Gamma,$$

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A II_1 factor M has Property (T) ([Connes, 1982]), if for any M-M-bimodule $\mathcal{H}$ with almost central unit vectors ξ_i's,

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- The group von Neumann algebra $L(\Gamma)$ has Property (T) iff Γ has Property (T).

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Fix a free ultrafilter ω on $\mathbb{N}$, the *ultrapower* of M is $M^\omega = \Pi_n M / I_\omega$, where $\Pi_n M = \{(x_n) | x_n \in M, \sup_n ||x_n|| < \infty\}$ and $I_\omega = \{(x_n) | \lim_\omega \tau(x_n^* x_n) = 0\}$.

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Fix a free ultrafilter ω on $\mathbb{N}$, the *ultrapower* of M is $M^\omega = \Pi_n M / I_\omega$, where $\Pi_n M = \{(x_n) | x_n \in M, \sup_n ||x_n|| < \infty\}$ and $I_\omega = \{(x_n) | \lim_\omega \tau(x_n^* x_n) = 0\}$.

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Need to show $2 \Rightarrow 1$.

Suppose an M-M-bimodule $\mathcal{H}$ has almost central, unit vectors ξ_i but no non-zero central vectors. Construct from $\mathcal{H}$ an inclusion $M \subseteq \tilde{M}$ such that $(M' \cap \tilde{M})^\omega \subsetneq M' \cap \tilde{M}^\omega$.

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- For $\xi \in \mathcal{H}^0$, define the *left creation operator* $l(\xi) \in \mathcal{B}(\tilde{\mathcal{H}})$

$$l(\xi)(x) = \xi x \text{ for } x \in L^2(M),$$
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- $\tilde{\mathcal{H}} \cong L^2(\tilde{M}, \tau_{\tilde{M}})$ as M-M-bimodules.

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So $(M' \cap \tilde{M})^\omega \subsetneq M' \cap \tilde{M}^\omega$, 2$\Leftrightarrow$ 1.

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Extra step. We need $\mathcal{H}$ to satisfy $\mathcal{H}^{\bigotimes_M^n}$ not to have non-zero M-M-central vectors, so that $M' \cap L^2(\tilde{M}) = \mathbb{C}1$.

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if M does not have Property (T) then we have an M-M-bimodule $\mathcal{H}$ such that $\mathcal{H}$ has almost central, unit vectors ξ_n and $\mathcal{H}^{\bigotimes_M^n}$ has no non-zero M-central vectors

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Weak Mixing of Bimodules [Peterson and Sinclair, 2012]

The following are equivalent definitions for $\mathcal{H}$ being a (left) weakly mixing M-M-bimodule:

1. the M-M-bimodule $\mathcal{H} \otimes_M \overline{\mathcal{H}}$ contains no non-zero central vector;
2. $\mathcal{H}$ has no non-zero right M-finite dimensional subbimodule;
3. there exists a sequence of unitaries $(u_n) \subset \mathcal{U}(M)$ such that $\lim_n \sup_{b \in (N)_1} |\langle u_n \xi b, \eta \rangle| = 0$ for any ξ and η in $\mathcal{H}$.

Characterization of Property (T) with non weak mixing

We need to show if M does not have Property (T) then there is an M-M-bimodule $\mathcal{H}$ such that $\mathcal{H}$ has almost central, unit vectors ξ_n and $\mathcal{H}$ is weakly mixing.

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In the group case:

Theorem ([Bekka and Valette, 1993])

Let G be a group. Then the following are equivalent:

1. *G has Property (T);*
2. *any unitary representation π of G on a Hilbert space which has almost invariant unit vectors has a non-zero finite dimensional subrepresentation.*

Characterization of Property (T) with non weak mixing

Theorem ([Tan, 2022])

For a separable II_1 factor M, the following are equivalent:

1. M has Property (T);
4. for any M-M-bimodule $\mathcal{H}$ with almost central unit vectors, $\mathcal{H}$ has a subbimodule $\mathcal{K}$ which is left or right finite M-dimensional (*not* both left and right weakly mixing);

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It suffices to show Condition 4 implies the following:

- there exists a finite subset $F \subset \mathcal{U}(M)$ and $\varepsilon > 0$, such that for any M-M-bimodule $\mathcal{K} \cong {}_M(\bigoplus_1^n L^2(M))p_{\theta(M)}$ where $\theta(M)' \cap pM_n(M)p = \mathbb{C}p$, if $\mathcal{K}$ has a (F, ε)-almost central unit vector, then $\mathcal{K}$ has a non-zero central vector.

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Intermediate steps

- M is non-Gamma.
- cp maps close to the identity are uniformly non-weakly mixing.

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