Property (T) and strong 1- boundedness

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Defn: G a group &

$\pi: G \to \mathcal{U}(\mathcal{H})$ repn.

Say π has almost invariant vectors if $\exists\, (\xi_n)_{n=1}^{\infty}$ in $\mathcal{H}$ s.t.

$\|\xi_n\| = 1$ & $\|\pi(g)\xi_n - \xi_n\| \xrightarrow[n\to\infty]{} 0 \quad \forall g \in G.$

Say G has (T) if whenever π has a.i. vectors $\exists\, \eta \neq 0,\ \eta \in \mathcal{H}$

$\pi(g)\eta = \eta \quad \forall g \in G.$

Fact $\Longleftrightarrow$ $\exists\, S \subseteq G$ finite & $C \geq 0$ s.t. if $\pi: G \to \mathcal{U}(\mathcal{H})$ repn & $\xi \in \mathcal{H}$, then

$$\|\xi - P_{\mathrm{Fix}(\pi)}(\xi)\| \leq C\left(\frac{1}{|S|}\sum_{s\in S} \|\pi(s)\xi - \xi\|^2\right)^{1/2}.$$

Examples: $SL_n(\mathbb{Z}),\ n \geq 3$

- $Sp(n,1)$, $n \geq 2$
- $SO(p,q) = \{A \in SL_n(\mathbb{R}) : A$ preserves
  $(x_1, \dots, x_n) \mapsto x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2\}$
  If $p > q \geq 2$.
- $Aut(\mathbb{F}_n)$, $n \geq 5$ (Kaluba-Nowak-Ozawa '20
  Kaluba-Kielak-Nowak '21)

__Defn__ (Tracial vNa).

__Example__: $(L(G), \tau_G)$

1-Bounded Entropy: Modification of
Voiculescu's MFED Theory.

Goal of MFED: Understand
vNa's in terms of matrix models

vNa's in terms of matrix models

e.g. $L(\mathbb{F}_n)$ has no Cartan '96

$L(\mathbb{F}_n)$ prime (Ge '98)

non-Gamma (Voiculescu '76)

— —

<u>Defn</u>: $\mathrm{Char}(\mathbb{F}_d)$ for $d \in \mathbb{N}$

& wk*-topology.

For $d=1$ characterizes moments.

<u>Thm</u> [VAF]: Fix $r \in \mathbb{N}$ & $\mathcal{O}$ wk*-nbhd of $\delta_{\{1\}}$. If $U^{(k)} \in \mathcal{U}(M_k(\mathbb{C}))^r$ is chosen randomly wrt Haar

$$\Longrightarrow \mathbb{P}(\phi_{U^{(k)}} \in \mathcal{O}) \xrightarrow[k\to\infty]{} 1.$$

$\rightsquigarrow$

. $h(u) = h(v)$ or $\mathrm{wt}(u) = \mathrm{wt}(v)$

(Dong '07. H. '18)

- $h(M_1 \vee M_2) \le h(M_1) + h(M_2)$, if $M_1 \cap M_2$ diffuse
- $h(M) \le 0$, if M hyperfinite
- $h(W^*(\mathcal{R}_M(N))) \le h(N)$
- $h(L(\mathbb{F}_r)) = +\infty$, if $r \ge 2$.

Examples: Recovers absence of Cartan, primeness, non-Gamma.

Thm [H. - Jekel - Kunnawalkam Elayavalli]

$h(L(G)) < +\infty$ if G has (T)

(More generally $h(M) < +\infty$ if M Property (T) II_1-factor in sense of Connes-Jones '85, Popa '06

(M, τ) tracial vNa s.t. $\exists$

$$\pi: G \to \mathcal{U}(\mathcal{H})/S^1$$

with $M = W^*(\pi(G))$.

Previous Results:

- $\delta_0(X) = 1$ for natural generators of $L(SL_n(\mathbb{Z}))$ (Voiculescu '99)
- $\delta_0(X) = 1$ $\forall$ generators of $L(SL_n(\mathbb{Z}))$ (Ge-Shen '02)
- $\delta_0(X) = 1$ $\forall$ generators of M has Property (T) & is finitely gen. Jung-Shlyakhtenko '07
- $h(L(G)) \leq 0$ if G is sofic, fin. presented & $\beta^{(2)}_1(G) = 0$.

Jung, Shlyakhtenko '21

Idea of Proof.

$S = (g_1, \ldots, g_r) \in G^r$ s.t. $G = \langle S \rangle$.

Let $\Phi_S = \tau_{L(G)} \circ \pi_S$ $\quad \pi_S : \mathbb{F}_r \to G$

$\pi_S(e_j) = g_j$

(i.e. $\Phi_S = 1_{\ker(\pi_S)}$)

<u>Lem</u>: (S, C) Kazhdan pair

$\forall \varepsilon, \delta > 0$ $\exists$ nbhd $\mathcal{O}$ of $1_{\ker(\pi_S)}$ & $n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$

$\forall\, U, V \in \Gamma^{(n)}(\mathcal{O})$ s.t.
$\|U - V\|_2 \leq \varepsilon \Rightarrow \exists\ w \in \mathcal{U}(M_n(\mathbb{C}))$
& $P \in \mathrm{Proj}(M_k(\mathbb{C}))$ s.t.
$\|(w^* U w - V)(1 - P)\|_2 < \delta$

& $\mathrm{tr}_\mu(P) \leq C\varepsilon(1 + C\varepsilon) + \delta$.

<u>Lem</u>: If $\varepsilon < C/2$

$$\Rightarrow h_\eta(x) \leq h_\varepsilon(x) + 12\, C(r+1)\varepsilon \log\left(\frac{C\sqrt{r}}{\eta}\right).$$

For $\eta = \varepsilon^2$

Set $\eta = \varepsilon^2$

$$h_{\varepsilon^2}(x) \le h_\varepsilon(x) + 12\,C(r+1)\,\varepsilon \log\left(\frac{C\sqrt{r}}{\varepsilon^2}\right)$$

Iterating:

$$h_{\varepsilon^{2^k}}(x) \le h_\varepsilon(x) + 12(r+1)C \sum_{j=0}^{k-1} \varepsilon^{2^j} \log\left(\frac{C\sqrt{r}}{\varepsilon^{2^{j+1}}}\right)$$

Comments:

Thm [H-Jekel - Kunnawalkam Elayavalli]

TFAE

(a) $h(M,\tau) \le 0$ $\forall\ (M,\tau)$ Property (T)

(b) $h(M,\tau) < +\infty$ $\forall (M,\tau)$ Property (T)

(c) $h(M) \le 0$ $\forall M$ II_1-factor

Further, if (a)-(c) fail $\Rightarrow \exists\ M$ II_1-factor with (T) & $0 < h(M) < +\infty$.

Such an M has $\mathcal{F}(M) = \{1\}$.

Connection to Popa rigidity. Conjecture:

References: Connes' 80 (T) $\Rightarrow$ $\mathcal{F}(M)$ countable

Popa '06 Conjecture (T) $\Rightarrow$ $\mathcal{F}(M) = \{1\}$

Chifan - Das - Houdayer - Khan $\exists$

Γ (T) s.t. $\mathcal{F}(L(\Gamma)) = \{1\}$