The quantum Gromov-Hausdorff distance and the Gromov-Hausdorff propinquity

Konrad Aguilar
(This talk includes joint work with *Stephan Ramon Garcia* and *Elena Kim* and *Frédéric Latrémolière* and *Timothy Rainone*)

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Background

- In the late 90's and early 2000's, Rieffel introduced compact quantum metric spaces and L-seminorms/Lip-norms and the *quantum Gromov-Hausdorff distance* (dist_Q) to establish convergence results arising from the high-energy physics literature such as matrices $M_n(\mathbb{C})$ (fuzzy spheres) converge to the sphere, $C(S^2)$.

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- However, there are many other ways to consider when spaces converge like *continuous fields of C*-algebras* and *inductive/direct limits of C*-algebras*.

- Our work has focused on showing cases when an *inductive sequence of C*-algebras* converges to its *inductive limit* in the quantum Gromov-Hausdorff distance as well as Latrémolière's Gromov-Hausdorff propinquity.

We say that unital C*-algebra A is an *inductive limit of C*-algebras* if $A = \overline{\cup_{n \in \mathbb{N}} A_n}^{\|\cdot\|_A}$ and for each $n \in \mathbb{N}$

$$A_n \text{ is a unital C*-subalgebra and } A_n \subseteq A_{n+1}.$$

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Let $(\beta(n))_{n\in\mathbb{N}}$ be a summable sequence of positive real numbers. If for each $n \in \mathbb{N}$, we have

1. L_n is a *Leibniz* L-seminorm on A_n (Leibniz=product rule),

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then there exists a *Leibniz* L-seminorm L on A such that

$$\lim_{n\to\infty} \Lambda^*((A,L),(A_n,L_n)) = 0,$$

where Λ^* is the Gromov-Hausdorff propinquity.

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The proof of this goes as follows.

- (2) and (3) provide that $((A_n, L_n))_{n\in\mathbb{N}}$ is *Cauchy*, thus providing a Leibniz L-seminorm L_F and a unital C*-algebra F for which this sequence converges to since Λ^* is *complete*.

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- We are then able to show that F is *-isomorphic to A using *Latrémolière's completeness argument* and borrow the Leibniz L-seminorm on F for A.

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What if the L_n do not have the *Leibniz* property? (This arose in our work on the *Bunce-Deddens algebras*, which is are inductive limits of circle algebras). Two problems arise.

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- The limit space F might not be closed under multiplication, and thus not a C*-algebra (but can be shown to be an *order unit space*).
- It is not immediate that F and A are even order isomorphic.

Our solution (A-Latrémolière-Rainone, 2021).

- Define an *order-unit* version of the *Gromov-Hausdorff propinquity* that recovers *Rieffel's quantum Gromov-Hausdorff distance*, which establishes it as a metric.

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- For instance, the quotient of an order unit space by an order ideal might not have the *Archimedean property*, and even if it does, the induced order unit norm might not be the *quotient norm*.

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- In another direction, what if we add more properties to L_n like the *strongly Leibniz property* (strongly Leibniz=quotient rule)?

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- In another direction, what if we add more properties to L_n like the *strongly Leibniz property* (strongly Leibniz=quotient rule)?
- In joint work with Garcia, Kim, and Latrémolière(2022), we have been able to show that the above L has the strongly Leibniz property following a similar approach.

Compact Quantum Metric Spaces

Let (X, d) be a compact metric space. The *topological* structure of X can be captured by the state space. Indeed

$$x \in X \longmapsto \delta_x \in S(C(X))$$

is a *homeomorphism* onto its image, where $\delta_x(f) = f(x)$ for all $f \in C(X)$. If we can prove that this map is an *isometry*, then we will have captured the metric structure in this C*-algebraic structure. But, to do this, we need a metric on $S(C(X))$.

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Recall the Lipschitz seminorm L_d, for $f \in C(X)$

$$\mathsf{L}_\mathsf{d}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} \mid x, y \in X \right\}.$$

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The *Monge-Kantorovich metric* on $S(C(X))$ is defined by

$$\mathsf{mk}_{\mathsf{L}_\mathsf{d}}(\phi, \psi) = \sup\{|\phi(f) - \psi(f)| : f \in C(X), \mathsf{L}_\mathsf{d}(f) \leqslant 1\}$$

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Motivated by work of *Connes*, Rieffel introduced...

Definition (Rieffel, 1998)

A pair (A, L) of an order unit space A and a lower semicontinuous seminorm $\mathsf{L} : A \to [0, \infty]$ such that $\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(A) : \mathsf{L}(a) < \infty\}$ is dense is a *compact quantum metric space* if:

1. $\{a \in A : \mathsf{L}(a) = 0\} = \mathbb{R}1_A$,
2. the associated *Monge-Kantorovich metric* on $S(A)$, defined for all states $\varphi, \psi \in S(A)$ by:

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We call the seminorm, L, an L-seminorm, and mk_{L}, the quantum metric. Rieffel showed that for all $a \in \mathrm{dom}(\mathsf{L})$, it holds that

$$\mathsf{L}(a) = \mathsf{L}_{\mathsf{mk}_{\mathsf{L}}}(\hat{a}) = \sup_{\phi,\psi \in S(A), \phi \neq \psi} \frac{|\hat{a}(\phi) - \hat{a}(\psi)|}{\mathsf{mk}_{\mathsf{L}}(\phi,\psi)},$$

where $\hat{a} \in C(S(A))$ is defined by $\hat{a}(\phi) = \phi(a)$ for all $\phi \in S(A)$.

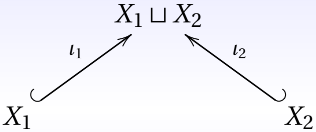
Examples of compact quantum metric space

Examples of compact quantum metric spaces include but are not limited to:

1. noncommutative tori (*Rieffel, 1998*),
2. curved noncommutative tori (*Latrémolière, 2015*),
3. various classes of group C*-algebras including Hyperbolic group C*-algebras (*Rieffel, 2002; Ozawa-Rieffel, 2005*), and
4. AF-algebras (*Antonescu-Christensen, 2004; A-Latrémolière, 2015, A-2016, 2018*), and
5. noncommutative solenoids (*Latrémolière-Packer, 2016*).
6. The standard quantum Podleś spheres (*A-Kaad, 2018*) (where the quantum metric is given by the *Connes metric* associated to the *Dabrowski-Sitarz* spectral triple)

Gromov-Hausdorff distance

Let (X_1, d_1), (X_2, d_2) be compact metric spaces. There exist many metrics on the *disjoint union* $X_1 \sqcup X_2$ such that the inclusion mappings ι_1, ι_2 are *isometries* (called an admissible metric).



The *Gromov-Hausdorff distance* between (X_1, d_1), (X_2, d_2) is

$$\mathrm{GH}(X_1, X_2) = \inf\{ \mathsf{Haus}_{\mathsf{d}}(\iota_1(X_1), \iota_2(X_2)) \mid \mathsf{d} \text{ is an admissible semi-metric}\},$$

where $\mathsf{Haus}_{\mathsf{d}}$ is the Hausdorff distance.

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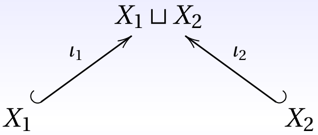
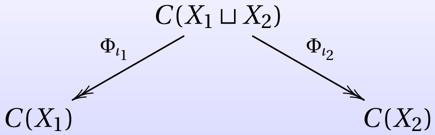
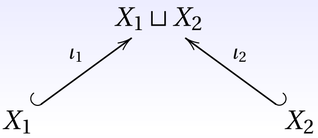


Define $\Phi_{\iota_1} : f \in C(X_1 \sqcup X_2) \to f \circ \iota_1 \in C(X_1)$

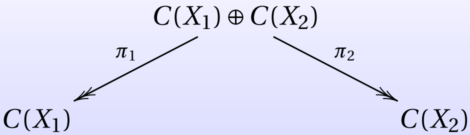


Gromov-Hausdorff distance

Let (X_1, d_1), (X_2, d_2) be compact metric spaces. There exist many metrics on the *disjoint union* $X_1 \sqcup X_2$ such that the inclusion mappings ι_1, ι_2 are *isometries* (called an admissible metric).



Now $C(X_1 \sqcup X_2) \cong C(X_1) \oplus C(X_2)$



where $\pi_1(f \oplus g) = f$.

$$\begin{array}{ccc} & C(X_1) \oplus C(X_2) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ C(X_1) & & C(X_2) \end{array}$$

where $\pi_1(f \oplus g) = f$.

$$\begin{array}{ccc} & S(C(X_1) \oplus C(X_2)) & \\ \nearrow^{\pi_1^*} & & \nwarrow^{\pi_2^*} \\ S(C(X_1)) & & S(C(X_2)) \end{array}$$

where $\pi_1^*(\varphi) = \varphi \circ \pi_1$. It holds that π_1^* and π_2^* are *isometries* with respect to the associated Monge-Kantorovich metrics given by the *Lipschitz constants*.

Tunnels

Definition (Rieffel 00)

Let (A_1, L_1) and (A_2, L_2) be two Compact Quantum Metric Spaces. An *R-tunnel* $(A_1 \oplus A_2, \mathsf{L})$ is a compact quantum metric space such that:

$$\mathsf{L}_j(a) = \inf\left\{\mathsf{L}(d) \middle| \pi_j(d) = a\right\}$$

for all $j \in \{1,2\}$ and $a \in A_j$, where $\pi_j : A_1 \oplus A_2 \to A_j$ is the canonical surjection.

Due to Rieffel, this is equivalent to the dual maps

$$\pi_j^* : \phi \in S(A_j) \to \phi \circ \pi_j \in S(A_1 \oplus A_2)$$

being isometries with respect to the associated *Monge-Kantorovich metrics*.

Rieffel's quantum Gromov-Hausdorff distance

Let $(A_1, L_1), (A_2, L_2)$ be compact quantum metric spaces. The *quantum Gromov-Hausdorff distance* between (A_1, L_1) and (A_2, L_2) is defined as

$$\begin{aligned} &\mathrm{dist}_q((A_1, L_1), (A_2, L_2)) \\ &\quad = \inf\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}}}(\pi_1^*(S(A_1)), \pi_2^*(S(A_2))) : (A_1 \oplus A_2, \mathsf{L}) \text{ is an } R\text{-tunnel}\} \end{aligned}$$

Rieffel's quantum Gromov-Hausdorff distance

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This is a *complete metric* up to order unit isomorphisms whose dual maps are isometries of the state spaces. Moreover, the map

$$(X, d) \longmapsto (C(X), L_d)$$

is a homeomorphism onto its image with respect to the *Gromov-Hausdorff distance*.

Compact quantum metric spaces, again

Definition (Rieffel, 1998)

A pair (A, L) of a unital C*-algebra A and a lower semicontinuous seminorm $\mathsf{L} : A \to [0, \infty]$ such that $\mathrm{dom}(\mathsf{L}) = \{a \in A : \mathsf{L}(a) < \infty\}$ is dense is a *compact quantum metric space* if:

1. $\{a \in A : \mathsf{L}(a) = 0\} = \mathbb{C}1_A$,
2. the associated *Monge-Kantorovich metric* on $S(A)$, defined for all states $\varphi, \psi \in S(A)$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in A, \mathsf{L}(a) \leqslant 1\right\}$$

   metrizes the weak* topology.
3. For all $a, b \in A$,

$$\mathsf{L}(ab) \leqslant \mathsf{L}(a)\|b\|_A + \mathsf{L}(b)\|a\|_A.$$

We call the seminorm, L, an Leibniz L-seminorm, and mk_{L}, the quantum metric.

Tunnels

Definition (Rieffel 00, Latrémolière, 2013)

Let (A_1, L_1) and (A_2, L_2) be two Compact Quantum Metric Spaces. An *L-tunnel* $\tau = (E, \mathsf{L}_E, \pi_1, \pi_2)$ is a Compact quantum metric space (E, L_E) together with two surjective unital *-homomorphisms $\pi_1 : E \to A_1$ and $\pi_2 : E \to A_2$ such that:

$$\mathsf{L}_j(a) = \inf\left\{\mathsf{L}_E(d) \middle| \pi_j(d) = a\right\}$$

for all $j \in \{1, 2\}$ and $a \in A_j$.

Note that this allows for E to be different than $A_1 \oplus A_2$, but we still get that the dual maps are *isometries* between the associated *Monge-Kantorovich metrics*.

Latrémolière's Gromov-Hausdorff propinquity

However, we can't simply define the Gromov-Hausdorff propinquity in the same way as the Gromov-Hausdorff distance since it won't be a metric (for instance, the triangle inequality fails and distance zero may not provide a *-isomorphism). Thus, Latrémolière introduced:

Latrémolière's Gromov-Hausdorff propinquity

However, we can't simply define the Gromov-Hausdorff propinquity in the same way as the Gromov-Hausdorff distance since it won't be a metric (for instance, the triangle inequality fails and distance zero may not provide a *-isomorphism). Thus, Latrémolière introduced:

Definition (Latrémolière, 2014)

Let $\tau = (E, \mathsf{L}_E, \pi_A, \pi_B)$ be an L-tunnel between two Compact Quantum Metric Spaces (A, L_A) and (B, L_B), where A, B are unital C*-algebras. The *extent* $\chi(\tau)$ *of* τ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_E}}\left(S(E), \pi_A^*(S(A))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_E}}\left(S(E), \pi_B^*(S(B))\right)\right\}.$$

The Gromov-Hausdorff Propinquity

Theorem-Definition (Latrémolière, 2013, 2014)

The *Gromov-Hausdorff propinquity* between two Compact Quantum Metric Spaces (A, L_A) and (B, L_B), where A, B are unital C*-algebras, is:

$$\Lambda^*((A,\mathsf{L}_A),(B,\mathsf{L}_B)) = \inf\left\{\chi(\tau) \middle| \tau \text{ is an } L\text{-tunnel from } (A,\mathsf{L}_A) \text{ and } (B,\mathsf{L}_B)\right\}.$$

Λ^* is a complete metric up to *-isomorphisms whose dual maps are isometries of the state spaces.
Moreover, the map

$$(X,d) \longmapsto (C(X), L_d)$$

is a homeomorphism onto its image with respect to the *Gromov-Hausdorff distance*.

Recall that we would like to have a theorem similar to the following theorem, where one does not need the *Leibniz* property.

Theorem (A-2019)

Let $(\beta(n))_{n\in\mathbb{N}}$ be a summable sequence of positive real numbers. If for each $n \in \mathbb{N}$, we have

1. L_n is a *Leibniz* L-seminorm on A_n,
2. $L_{n+1}(a) \leqslant L_n(a)$ for all $a \in A_n$, and
3. for each $a \in A_{n+1}$, there exists $b \in A_n$ such that $\|a-b\|_A \leqslant \beta(n)$ and $L_n(b) \leqslant L_{n+1}(a)$,

then there exists a *Leibniz* L-seminorm L on A such that

$$\lim_{n\to\infty} \Lambda^*((A,L),(A_n,L_n)) = 0.$$

Order unit Gromov-Hausdorff propinquity

Definition (A-Latrémolière-Rainone,2021)

The *order unit Gromov-Hausdorff propinquity* between two compact quantum metric spaces (A, L_A) and (B, L_B), where A, B are order unit spaces, is

$$\Lambda^*_{ou}((A, \mathsf{L}_A), (B, \mathsf{L}_B)) \\ = \inf\left\{\chi(\tau) \middle| \tau \text{ is an } L\text{-tunnel from } (A, \mathsf{L}_A) \text{ and } (B, \mathsf{L}_B)\right\},$$

where the surjections defining L-tunnels are now order homomorphisms rather than *-homomorphisms.

Order unit Gromov-Hausdorff propinquity

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The *order unit Gromov-Hausdorff propinquity* between two compact quantum metric spaces (A, L_A) and (B, L_B), where A, B are order unit spaces, is

$$\begin{aligned}&\Lambda^*_{ou}((A,\mathsf{L}_A),(B,\mathsf{L}_B))\\&\quad= \inf\left\{\chi(\tau) \,\middle|\, \tau \text{ is an } L\text{-tunnel from } (A,\mathsf{L}_A) \text{ and } (B,\mathsf{L}_B)\right\},\end{aligned}$$

where the surjections defining L-tunnels are now order homomorphisms rather than *-homomorphisms.

Theorem (A-Latrémolière-Rainone,2021)

It holds that

$$\mathrm{dist}_q((A,\mathsf{L}_A),(B,\mathsf{L}_B)) = \Lambda^*_{ou}((A,\mathsf{L}_A),(B,\mathsf{L}_B)).$$

$A = \overline{\cup_{n\in\mathbb{N}} A_n}^{\|\cdot\|_A}$

Theorem (A-Latrémolière-Rainone,2021)

Let $(\beta(n))_{n\in\mathbb{N}}$ be a summable sequence of positive real numbers. If for each $n \in \mathbb{N}$, we have

1. A_n is an order unit space and L_n is a L-seminorm on A_n,
2. $L_{n+1}(a) \leqslant L_n(a)$ for all $a \in A_n$, and
3. for each $a \in A_{n+1}$, there exists $b \in A_n$ such that $\|a - b\|_A \leqslant \beta(n)$ and $L_n(b) \leqslant L_{n+1}(a)$,

then there exists an L-seminorm L on A such that

$$\lim_{n\to\infty} \Lambda^*_{ou}((A,L),(A_n,L_n)) = 0.$$

As a consequence, we were able to place quantum metrics on Bunce-Deddens algebras for which their canonical *circle algebras* converge to the associated Bunce-Deddens algebra.

Strongly Leibniz Gromov-Hausdorff propinquity

Definition

Let A be a unital C*-algebra, we say that an L-seminorm is *strongly Leibniz* if it is Leibniz and for all $a \in \mathrm{dom}(L)$ such that a is invertible, it holds that $L(a^{-1}) \leqslant \|a^{-1}\|_A^2 L(a)$.

Theorem-Definition (A-Kim-Garcia-Latrémolière, 2022)

The *strongly Leibniz Gromov-Hausdorff propinquity* between two compact quantum metric spaces (A, L_A) and (B, L_B), where A, B are unital C*-algebras and $\mathsf{L}_A, \mathsf{L}_B$ are *strongly Leibniz*, is

$$\Lambda^*_{SL}((A, \mathsf{L}_A), (B, \mathsf{L}_B)) = \inf\left\{\chi(\tau) \middle| \tau \text{ is an } L\text{-tunnel from } (A, \mathsf{L}_A) \text{ and } (B, \mathsf{L}_B)\right\},$$

where the L-seminorms defining L-tunnels are now *strongly Leibniz*. Λ^*_{SL} is a complete metric up to *-isomorphisms whose dual maps are isometries of the state spaces.

$A = \overline{\cup_{n\in\mathbb{N}} A_n}^{\|\cdot\|_A}$

Theorem (A-Garcia-Kim-Latrémolière,2021)

Let $(\beta(n))_{n\in\mathbb{N}}$ be a summable sequence of positive real numbers. If for each $n \in \mathbb{N}$, we have

1. A_n is a unital C*-algebra and L_n is a *strongly Leibniz* L-seminorm on A_n,
2. $L_{n+1}(a) \leqslant L_n(a)$ for all $a \in A_n$, and
3. for each $a \in A_{n+1}$, there exists $b \in A_n$ such that $\|a - b\|_A \leqslant \beta(n)$ and $L_n(b) \leqslant L_{n+1}(a)$,

then there exists a *strongly Leibniz* L-seminorm L on A such that

$$\lim_{n\to\infty} \Lambda^*_{SL}((A,L),(A_n,L_n)) = 0.$$

strongly Leibniz L-seminorms on AF algebras

Let $A = \overline{\cup_{n\in\mathbb{N}} A_n}^{\|\cdot\|_A}$ be a unital AF algebra equipped with a faithful tracial state. For each $n \in \mathbb{N}$, let

$$E_n : A \to A_n$$

be the unique trace-preserving conditional expectation, and define for all $a \in A$

$$\|a\|_{E_n} = \sqrt{\|E_n(a^*a)\|_A}$$

the associated *Frobenius-Rieffel norm*. Let $k_n > 0$ such that

$$k_n \|a\|_A \leqslant \|a\|_{E_n}$$

for all $a \in A_{n+1}$.

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Let $A = \overline{\cup_{n\in\mathbb{N}} A_n}^{\|\cdot\|_A}$ be a unital AF algebra equipped with a faithful tracial state. For each $n \in \mathbb{N}$, let

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$$k_n \|a\|_A \leqslant \|a\|_{E_n}$$

for all $a \in A_{n+1}$.

Following a suggestion from Rieffel in 2016 and our previous work, we define for all $a \in A_n$,

$$L_n(a) = \max\left\{ \frac{\max\{\|a - E_m(a)\|_{E_m}, \|a^* - E_m(a^*)\|_{E_m}\}}{k_m \beta(m)} : m = 1, \ldots, n-1 \right\}$$

which is strongly Leibniz due to (Rieffel, 2014).

strongly Leibniz L-seminorms on AF algebras

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We have

Theorem (A-Garcia-Kim-Latrémolière, 2022)

For each $n \in \mathbb{N}$, it holds that L_n is a *strongly Leibniz* seminorm on A_n such that

1. $L_{n+1}(a) \leqslant L_n(a)$ for all $a \in A_n$, and
2. for each $a \in A_{n+1}$, there exists $b \in A_n$ such that $\|a - b\|_A \leqslant \beta(n)$ and $L_n(b) \leqslant L_{n+1}(a)$,

and thus there exists a *strongly Leibniz* L-seminorm L on A such that

$$\lim_{n\to\infty} \Lambda^*_{SL}((A, L), (A_n, L_n)) = 0.$$

Thank you!

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