Chapter 14 Summary

Notation

- A scalar valued function of $n$ variables $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is denoted by $f(x_1, x_2, \ldots, x_n) = w$ and $w$ is said to be the dependent variable and $x_1, x_2, \ldots, x_N$ are said to be the independent variables.
- We have to notations for partial derivative. $f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0,y_0)$ and similarly for $y$ and $z$.

Definitions

**Definition 1:** The open disk of radius $r$ centered at $(x_0, y_0)$ is the set of points in the plane

$$\{(x,y) | \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\}$$

**Definition 2:** The open ball of radius $r$ centered at $(x_0, y_0, z_0)$ is the set of points in $\mathbb{R}^3$

$$\{(x,y,z) | \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < r\}$$

**Definition 3:** A region $R$ in $\mathbb{R}^2$ is called bounded if $R$ is contained in some disk with positive finite radius. A region $R$ in $\mathbb{R}^3$ is called bounded if $R$ is contained in some open ball of finite positive radius.

**Definition 4:** A region is called unbounded if it is not bounded.

**Definition 5:** A point $x \in R$ in a region $R$ in $\mathbb{R}^2$ ($\mathbb{R}^3$) is called an interior point of $R$ if there is some open disk (open ball in $\mathbb{R}^3$) of positive radius centered at $x$ that is contained completely within $R$. A point $x \in R$ is said to be a boundary point of $R$ if every open disk (open ball in $\mathbb{R}^3$) centered at $x$ contains a point in $R$ and a point not in $R$.

**Definition 6:** A region $R$ is called open if every point of $R$ is an interior point of $R$. It is called closed if it contains all of its boundary points.

**Definition 7:** A level set of a function $f$ is the set of points $(x, y, z)$ in the domain of $f$ such that $f(x, y, z) = c$ for some fixed constant $c$. In particular, a level set of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called a level curve, and a level set of a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is called a level surface.

**Definition 8:** We say that a function $f(x, y)$ approaches the limit $L$ as $\vec{x} = (x, y)$ approaches $\vec{x}_0 = (x_0, y_0)$, written

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

if for every number $\epsilon > 0$, there exists $\delta > 0$ such that for all $\vec{x}$ in the domain of $f$ we have if $0 < ||\vec{x} - \vec{x}_0|| < \delta$, then we must have $|f(x, y) - L| < \epsilon$. 

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**Definition 9:** A function \( f(x, y) \) is **continuous at** \((x_0, y_0)\) if

1. \( f \) is defined at \((x_0, y_0)\)
2. \( \lim_{(x,y)\to(x_0,y_0)} f(x, y) \) exists.
3. \( \lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0) \)

\( f \) is said to be **continuous** if it is continuous at each point of its domain.

**Definition 10:** The partial derivative of \( f(x, y) \) with respect to \( x \) at a point \((x_0, y_0)\) is

\[
\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}
\]

With respect to \( y \) we have

\[
\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}
\]

**Definition 11:** The tangent plane to the graph of \( f(x, y) \) at a point \((x_0, y_0, f(x_0, y_0))\) is the unique plane passing through \((x_0, y_0, f(x_0, y_0))\) that is tangent to the surface. One equation for it is given by

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0
\]

**Definition 12:** A function \( z = f(x, y) \) is **differentiable** at \((x_0, y_0)\) if \( f_x(x_0, y_0) \) and \( f_y(x_0, y_0) \) exist and \( f(x, y) \) satisfies an equation

\[
f(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0) + \varepsilon(x, y)
\]

where \( \varepsilon(x, y) \) is a function such that

\[
\lim_{x \to x_0} \frac{\varepsilon(x, y)}{\| x - x_0 \|} = 0
\]

**Definition 13:** The derivative of \( f(x, y) \) at \( \vec{p}_0 \) in the direction of \( \vec{u} \), or directional derivative is

\[
(D_{\vec{u}}(f))_{\vec{p}_0} = \left. \frac{df}{dt} \right|_{t=0} \vec{p}_0 = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3) - f(x_0, y_0)}{t} = \nabla f|_{\vec{p}_0} \cdot \vec{u}
\]

**Definition 14:** The gradient vector of \( f(x, y) \) at a point \( \vec{p}_0 = (x_0, y_0) \) is

\[
\nabla f|_{(x_0, y_0)} = \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} \hat{i} + \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} \hat{j}
\]

**Definition 15:** An interior point \((x_0, y_0)\) of the domain of \( f(x, y) \) where \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \) or one or both of \( f_x, f_y \) does not exist is called a critical point.

**Definition 16:** \((a, b)\) is a **local maximum** of a function \( f(x, y) \) if \( f(a, b) \geq f(x, y) \) for all \((x, y)\) in an open disk centered at \((a, b)\).

\((a, b)\) is a **local minimum** of a function \( f(x, y) \) if \( f(a, b) \leq f(x, y) \) for all \((x, y)\) in an open disk centered at \((a, b)\).

**Definition 17:** A differentiable function \( f(x, y) \) has a saddle point at a critical point \((a, b)\) if in every open disk there are points \((x, y)\) such that \( f(x, y) \geq f(a, b) \) and \( f(x, y) \leq f(a, b) \).
**Definition 18:** The Hessian of \( f(x, y) \) is

\[
\begin{vmatrix}
  f_{xx} & f_{xy} \\
  f_{xy} & f_{yy}
\end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2
\]

**Main Theorems**

**Theorem 19:** If \( f(x, y) \) and its partial derivatives \( f_x, f_y, f_{xy} \) are defined throughout an open region containing a point \((a, b)\) and are all continuous then mixed partials commute i.e. \( f_{xy}(a, b) = f_{yx}(a, b) \).

**Theorem 20:** A differentiable function is continuous.

**Theorem 21:** If the partial derivatives \( f_x, f_y \) exist and are continuous on an open region \( R \), then \( f \) is differentiable on \( R \) (i.e. at every point of \( R \)).

**Theorem 22:** If \( w = f(x, y) \) is differentiable and \( x = x(t), y = y(t) \) are differentiable functions of \( t \), the composite \( w = f(x(t), y(t)) \) is also differentiable and

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]

**Theorem 23:** Suppose that \( F(x, y) \) is differentiable and that the the equation \( F(x, y) = 0 \) defines \( y \) as a differentiable function of \( x \). Then at any point where \( F_y \neq 0 \),

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}
\]

In 3 variables, \( F(x, y, z) \), if \( F_z \neq 0 \), then \( \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \) and \( \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \).

**Theorem 24:** If \( f(x, y) \) has a local min/max at a point \((a, b)\) and \( f_x(a, b), f_y(a, b) \) exist, then \( f_x(a, b) = f_y(a, b) = 0 \).

**Theorem 25:** Suppose \( f(x, y) \) and its first and second partial derivatives are continuous near \((a, b)\) and \( f_x(a, b) = f_y(a, b) = 0 \). Then

1. \( f \) has a local max at \((a, b)\) if \( f_{xx}(a, b) < 0 \) and \( f_{xx}f_{yy} - (f_{xy})^2 = 0 \)
2. \( f \) has a local min at \((a, b)\) if \( f_{xx}(a, b) > 0 \) and \( f_{xx}f_{yy} - (f_{xy})^2 = 0 \)
3. \( f \) has a saddle point at \((a, b)\) if \( f_{xx}f_{yy} - (f_{xy})^2 < 0 \)
4. The test is inconclusive if \( f_{xx}f_{yy} - (f_{xy})^2 < 0 \)