Galois Cohomology and Algebraic K-theory of Finite Fields

by

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AN ESSAY

Submitted to the College of Liberal Arts and Sciences,

Wayne State University,

Detroit, Michigan,

in partial fulfillment of the requirements

for the degree of

MASTER OF ARTS

April of 2013

MAJOR: Mathematics

APPROVED BY: ______________________________

Advisor Date

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2^{nd} Reader Date

(if necessary)

Acknowledgements

I would like to extend my utmost graditude to all the people who helped me with this project. First, I would like to thank my family for supporting me in any endeavor I undertake. Second, I would like to thank Jessica McInchak for her patience with me during tough times and for reading drafts along the way. I am in greatest debt to my advisor Dr. Andrew Salch who spent countless hours discussing the material presented here and much more. His enthusiasm for mathematics is an inspiration for me and without him this would not have been possible. I would like to thank my colleagues in the Mathematics department especially Sean Tilson, Michael Catanzaro, and Bogdan Gheorghe for advice on mathematics research and for helping me form relationships with faculty members. Finally, I would like to thank my friends outside of the mathematics department for helping me get through the long days of graduate student life.

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Chapter 1

Introduction

The motivation for this work comes from recent progress in showing a relationship between étale cohomology and algebraic K-theory largely due to Voevodsky and Rost, and in part due to Weibel. This progress begins with the proof of the Milnor conjecture by Voevodsky in 1996 [15]. Milnor conjectured that for k a field with $\mathrm{char}(k) \neq 2$, there are isomorphisms from Milnor K-theory of k modulo 2 to étale cohomology of k with coefficients in $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 0$ [15]. This suggests that Milnor K-theory can be computed using étale cohomology in this special case.

A generalization of this conjecture for primes not equal to 2 is the Bloch-Kato conjecture (or motivic Bloch-Kato conjecture), which has now been proven in a series of 15-20 papers by Voevodsky, Rost, and Weibel [16]. The proof of this conjecture was announced at a conference honoring Grothendieck in 2009 and it is now referred to as the norm-residue theorem. The Bloch-Kato conjecture suggests a relationship between étale cohomology and algebraic K-theory as well and with the proof of the norm-residue theorem this relationship was solidified. Thus, for our purposes, it shows that algebraic K-theory is computable via étale cohomology .

A useful tool we apply is a spectral sequence which has been shown to converge to algebraic K-theory. This story goes back further to 1972 when Lichtenbaum conjectured about a relationship of K-theory to étale cohomology for the purposes of studying zeta functions [16]. Later Quillen and Beilinson expanded and generalized these conjectures. One construction of this spectral sequence came from Friedlander and Suslin in a preprint in 1994. The most useful versions of the spectral sequence for our purposes can be found in Weibel's book on algebraic K-theory, yet to arrive in print [16], and Thomason's paper on algebraic K-theory and étale cohomology [13].

All theorems used directly in this paper will be stated in their original generality and then translated to the special case of finite fields. The goal of this paper is to approach algebraic K-theory from a different direction than direct computation using some recent results. We begin with some results from basic galois theory and group cohomology making specific computations where they will be helpful for later use. Next we define and compute profinite group cohomology and continuous galois cohomology. We follow by identifying continuous galois cohomology with étale cohomology and then étale cohomology with motivic cohomology in this special case. We use this information to compute algebraic K-theory of finite fields which we find to be the following for $i \geq 1$

$$\begin{array}{rcccl} K_0(\mathbb{F}_p;\mathbb{Z}/\ell\mathbb{Z})[\beta^{-1}] & \cong & \mathbb{Z}/\ell\mathbb{Z} & & \\ K_{2i}(\mathbb{F}_p;\mathbb{Z}/\ell\mathbb{Z}) & \cong & \mathbb{Z}/\ell\mathbb{Z} & \text{if} & \ell|p^i-1. \\ K_{2i-1}(\mathbb{F}_p;\mathbb{Z}/\ell\mathbb{Z}) & \cong & \mathbb{Z}/\ell\mathbb{Z} & \text{if} & \ell|p^i-1. \\ K_*(\mathbb{F}_p;\mathbb{Z}/\ell\mathbb{Z}) & \cong & 0 & & \text{otherwise} \end{array}$$

We show that comparison with Quillen's original results confirms our answer. We then suggest other generalizations of this approach.

Chapter 2

Basic Galois Cohomology

Since galois cohomology will be useful for our calculations, we begin with a brief summary of important facts from galois theory and group cohomology and use them to compute cohomology of finite galois groups. Though we consider the finite case here, ultimately we will be interested in the profinite case called continuous galois cohomology.

2.1 Basic Facts

We summarize some facts about galois groups and G-modules.

2.1.1 Basic Galois Theory

Let $\mathbb{F}_{p^n}$ denote the field of order p^n, for p a prime and $n \geq 1$ an integer. Recall that any finite field is isomorphic to $\mathbb{F}_{p^n}$ for some prime and some integer $n \geq 1$.

Proposition 2.1.1. *If $\mathbb{F}_{p^n}$ is a finite extension of a finite field $\mathbb{F}_p$, then $\mathbb{F}_{p^n}$ is finite and is galois over K. The galois group Gal($\mathbb{F}_{p^n}/\mathbb{F}_p$) $= G_n$ is cyclic and $G_n \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. See [3] or any other basic abstract algebra text. □

Proposition 2.1.2. *Let* $G_{mn} = \mathrm{Gal}(\mathbb{F}_{p^{mn}}/\mathbb{F}_p)$*, and* $G_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ *and let* $H = \mathrm{Gal}(\mathbb{F}_{p^{mn}}/\mathbb{F}_{p^n})$*. Then*

$$G_n \cong G_{mn}/H$$

Proof. This follows from the fundamental theorem of galois theory; see [3]. □

These facts will be usefull for future calculations. We note here that Prop. 2.1.1 gives us the galois groups as cyclic groups, so it will suffice to understand cohomology of cyclic groups.

2.1.2 G-Modules

We will be dealing with G-modules where G is the galois group of an extension of $\mathbb{F}_p$.

Definition 2.1.3. Let G be a group. Then a G-module M is an abelian group along with a (left) action of G, by which we mean a map $G \times M \to M$ such that the following properties hold for $a, b \in M$, $s, t \in G$

1. 1.a=a
2. s.(a+b)=s.a+s.b
3. s.(t.a)=(s.t).a

where s.a denotes s acting on a; i.e the map above sends (s,a) to s.a.

2.2 Computing Group Cohomology

Let A be a G-module, then we let A^G denote the elements of A fixed by the action of G. Given a function $f : A \to B$ there is an induced map $f : A^G \to B^G$ making A^G functorial. It is an additive left exact functor. We define cohomology in the following way:

Definition 2.2.1. Given A a G-module, we take the right derived functors R^i of the functor A^G and define the following

$$H^i(G, A) = R^i(A^G).$$

We call this cohomology of G with coefficients in A.

We consider cohomology of cyclic groups here as they are the most pertinent for our goal. Our resources for this material are [1], [4] along with [10] and [11] for the applications.

Proposition 2.2.2. *Let $H^i(\mathbb{Z}/n\mathbb{Z}, A)$ be group cohomology of $\mathbb{Z}/n\mathbb{Z}$ with with coefficients in A. We let A have the trivial action. For A=$\mathbb{Z}/\ell\mathbb{Z}$ we get*

$$H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/\ell\mathbb{Z} & \textit{if} \quad i = 0 \\ \mathbb{Z}/\gcd(n, \ell)\mathbb{Z} & \textit{if} \quad i > 0. \end{cases}$$

Proof. We let $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ be the group ring and we use the fact that

$$H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z}) \cong Ext^i_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z}).$$

We then compute Ext in this case. First, we take a projective resolution of $\mathbb{Z}$ (suffices in most cases to consider a free resolution since a free module is a projective module). We write a general element of the group ring $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ as a(b) where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}/n\mathbb{Z}$.

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \xleftarrow{D} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \xleftarrow{N} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \longleftarrow \ldots$$

by letting ϵ be the the map which sends 1(m) to 1 for all $0 \leq m \leq n-1$ (called the augmentation map). The map D sends 1(0) to 1(1)-1(0). We let N be the map which sends 1(1) to $\sum_{i=0}^{n-1} 1(i)$. The following maps are D again and N and this pattern repeats. We then truncate and apply $\mathrm{Hom}(-, \mathbb{Z}/\ell\mathbb{Z})$. Resulting in the sequence,

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}], \mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{D*} \mathrm{Hom}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}], \mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{N*} \mathrm{Hom}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}], \mathbb{Z}/\ell\mathbb{Z})...$$

and using a simple fact from algebra $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}], \mathbb{Z}/\ell\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z}$ [3], we get the sequence

$$0 \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{D^*} \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{N^*} \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{D^*} \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{N^*} \mathbb{Z}/\ell\mathbb{Z}...$$

We are then left to determine the maps between these modules [D^* and N^*]. The isomorphism from Hom to $\mathbb{Z}/\ell\mathbb{Z}$ takes $f \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}], \mathbb{Z}/\ell\mathbb{Z})$ to f(1(0)), so the map D* is $(f \circ D)(1(0)) = f(1(1) - f(1(0))$. Since f is a linear $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ module map and since $\mathbb{Z}/n\mathbb{Z}$ acts trivially on $\mathbb{Z}/\ell\mathbb{Z}$, we have

$$f(1(1) - 1(0)) = f(1(1)) - f(1(0)) = 1(1)f(1(0)) - f(1(0)) = f(1(0)) - f(1(0)) = 0.$$

Thus, D^* is the 0 map. We then find N^*. Similarly we have $f(1(0))$ goes to $(f \circ N)(1(0))$ and for the same reasons

$$\begin{aligned}(f \circ N)(1(0)) &= f(1(0) + 1(1) + ... + 1(n-1)) \\ &= f(1(0) + 1(1)f(1(0)) + 1(2)f(1(0) + ... + 1(n-1)f(1(0)) \\ &= nf(1(0)).\end{aligned}$$

Thus, the map N^* is multiplication by n.

We are now in a position to compute the functor Ext:

$$\mathrm{Ext}^i_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z}) \cong H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z})$$

$$
\begin{array}{lcllll}
\mathrm{Ext}^0_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z}) & \cong & \ker D^*/\{0\} & = & \mathbb{Z}/\ell\mathbb{Z} & \\
\mathrm{Ext}^1_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z}) & \cong & \ker N^*/\mathrm{im} D^* & = & (\mathbb{Z}/\gcd(n,\ell)\mathbb{Z})/\{0\} & = \quad \mathbb{Z}/\gcd(n,l)\mathbb{Z} \\
\mathrm{Ext}^2_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z}) & \cong & \ker D^*/\mathrm{im} N^* & = & (\mathbb{Z}/\ell\mathbb{Z})/n\mathbb{Z} & = \quad \mathbb{Z}/\gcd(n,l)\mathbb{Z} \\
\mathrm{Ext}^i_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z}) & \cong & \mathbb{Z}/\gcd(n,\ell)\mathbb{Z} & & \text{for} & i \geq 3 \quad .
\end{array}
$$

This gives us a simple expression in terms of n and l of $H^i(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z})$. We have therefore computed the following:

$$
\begin{array}{rcl}
H^0(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z}) & \cong & \mathbb{Z}/\ell\mathbb{Z} \\
H^i(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z}) & \cong & \mathbb{Z}/\gcd(n,\ell) \text{ for i} > 0.
\end{array}
$$

□

It will also be useful to know cohomology of cyclic groups with coefficients A in more generality. If g is a map from A to A, then we denote ${}_gA$ the kernel of g contained in A and gA the image of the map g in A. We state what they are explicitly in the proposition below.

Proposition 2.2.3. *Let G be a cyclic group of order n with generator t and A a G-module then*

$$
H^i(G,A) \cong \begin{cases} A^G/N^*A & \text{if} \quad i \text{ is even} \\ {}_{N^*}A/D^*A & \text{if} \quad i \text{ is odd} \end{cases}
$$

for $i > 0$. *And we have,*

$$
H^0(G,A) \cong A^G
$$

where

$$
\begin{array}{rcl}
A^G & = & \{a \in A \mid t(a) = a \text{ for all } t \in G\} \\
N^*A & = & \{a \in A \mid (1 + t + t^2 + \ldots + t^{n-1})(a') = a \text{ for some } a' \in A\} \\
{}_{N^*}A & = & \{a \in A \mid (1 + t + t^2 + \ldots + t^{n-1})(a) = 0\} \\
D^*A & = & \{a \in A \mid (t-1)(a') = a \text{ for some } a' \in A\}
\end{array}
$$

with $t(a)$ the action of t on a.

Proof. Similar to the previous proof, we use the fact that $H^i(G, A) \cong \mathrm{Ext}^i_{\mathbb{Z}[G]}(Z, A)$. We take a projective resolution of $\mathbb{Z}$.

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \xleftarrow{D} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \xleftarrow{N} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \longleftarrow ...$$

where ϵ, N, and D are the same maps as before. The only difference here is the next step. We truncate and apply $\mathrm{Hom}(-, A)$ to get

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}[G], A) \arrow[r, "D^*"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbb{Z}[G], A) \arrow[r, "N^*"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbb{Z}[G], A) \arrow[r, "D^*"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbb{Z}[G], A) \arrow[d, "\cong"'] \\
0 \arrow[r] & A \arrow[r, "D^*"] & A \arrow[r, "N^*"] & A \arrow[r, "D^*"] & A
\end{tikzcd}$$

We then need to find the maps D^* and N^*. We can compute that the map $D^* : A \to A$ sends a to $a - ta$ and $N^* : A \to A$ sends a to $a + ta + ... + t^{n-1}a$. Thus, we have enough to get a general formula for cohomology. Note that $\ker D^* \cong \{a \in A \mid ta - a = 0\} \cong \{a \in A \mid ta = a\} \cong A^G$

$$\begin{aligned}
H^0(G, A) &\cong A^G \\
H^i(G, A) &\cong {}_{N^*}A/D^*A \quad \text{if} \quad i = 2m - 1 \quad m \geq 1 \\
H^i(G, A) &\cong A^G/N^*A \quad \text{if} \quad i = 2m \qquad\quad m \geq 1
\end{aligned}$$

thus we have shown our claim.

□

2.3 Finite Galois Cohomology

Once we have cohomology of finite cyclic groups, we get the finite galois cohomology we need by Prop. 2.1.1. The difference that may arise is the action that the galois group has on our coefficient module. We will discuss this more in the next section.

Proposition 2.3.1. *Let Gal($\mathbb{F}_{p^n}/\mathbb{F}_p$)=$G_n$ be the galois group of order n and $\mu_\ell(\mathbb{F}_{p^n})$ be the ℓ-th roots of unity in $\mathbb{F}_{p^n}$ and suppose $\ell | p^n - 1$ then we compute galois cohomology as follows.*

$$H^i(G_n;\mu_l(\mathbb{F}_{p^n})) \cong \begin{cases} \mathbb{Z}/\ell\mathbb{Z} & \text{if} \quad i = 0 \\ \mathbb{Z}/\gcd(n,\ell)\mathbb{Z} & \text{if} \quad i > 0. \end{cases}$$

If ℓ does not divide $p^n - 1$ then $\mu_\ell(\mathbb{F}_{p^n}) = 0$.

Proof. The proof follows from Prop. 2.1.1 and Prop. 2.2.2. □

In the above proposition, we use that the order of $\mu_\ell(\mathbb{F}_{p^n})$ is $\gcd(\ell, p^n - 1)$, which is a well known fact. We also get a general formula for finite galois cohomology.

Proposition 2.3.2. *Let G_n be defined as above, A a G_n-module, then*

$$H^i(G_n, A) \cong \begin{cases} A^{G_n}/N^*A & \text{if} \quad i \text{ is even} \\ {}_{N^*}A/D^*A & \text{if} \quad i \text{ is odd} \end{cases}$$

for $i > 0$. And we have,

$$H^0(G_n, A) = A^{G_n}$$

where

$$\begin{array}{rcl} A^{G_n} & = & \{a \in A \mid g(a) = a \text{ for all } g \in G\} \\ N^*A & = & \{a \in A \mid (1 + g + g^2 + ... + g^{n-1})(a) = a\} \\ {}_{N^*}A & = & \{a \in A \mid (1 + g + g^2 + ... + g^{n-1})(a) = 0\} \\ D^*A & = & \{a \in A \mid (1 - g)(a) = a\} \end{array}$$

with $g(a)$ the action of g on a, and g the generator of G_n.

Proof. Combine Prop. 2.1.1 with Prop. 2.2.3. □

We use these computations to introduce cohomology and give the flavor of our future computations, though we ultimately consider profinite galois cohomology, also called continuous galois cohomology.

Chapter 3

Continuous Galois Cohomology

Here we will introduce continuous galois cohomology, which is cohomology of a profinite galois group. We lead up to a specific computation that will be useful in the next section. First, we define profinite groups, then we define cohomology of a profinite galois group. We then move to specific computations.

3.1 Profinite Groups

For this section our main reference is Serre's book on galois cohomology [11].

Definition 3.1.1. A profinite group is a topological group defined as the projective limit of finite groups each with the discrete topology.

We can say more about the topology of a profinite group as well:

Proposition 3.1.2. *A profinite group is compact and totally disconnected and a compact and totally disconnected topological group is a profinite group.*

Proof. The first statement follows from the definition and for the converse see [11] □

The important example for us is the following:

Example 3.1.3. *Let L be a finite galois extension of K, K a field. Then Gal(L/K) is a finite group. We give Gal(L/K) the discrete topology. We then take the projective limit* $\varprojlim Gal(L/K)$ *where the limit is taken over all finite extensions* L/K*. This projective limit is by definition the galois group Gal(*K_{sep}/K*) or in the case when* $K = \mathbb{F}_p$ *we get Gal(*$\overline{\mathbb{F}}_p/\mathbb{F}_p$*) where* $\overline{\mathbb{F}}_p$ *is the algebraic closure of* $\mathbb{F}_p$*, since the separable closure agrees with the algebraic closure in this case.*

3.2 Profinite Group Cohomology

Given a profinite group G, we may consider the category of all G-modules, abelian groups with an action of G. The discrete abelian groups with a continuous action form a full subcategory which is an abelian category. These type of G-modules are called discrete G-modules and we will follow with a formal definition. In words, a discrete G-module, M, is one such that the stabilizer of each element in M is an open subgroup of G.

Definition 3.2.1. A discrete G-module, M, is a G-module with the condition

$$M = \bigcup M^H$$

where the union is taken over all open subgroups H of G. M has the discrete topology and the action of G on M is continuous.

Given M as above we define a complex $C^*(G, M)$ in the following way. Let $C^n(G, M)$ denote the set of continuous maps from an n-fold cartesian product G^n to M. Define a coboundary map $d : C^n \longrightarrow C^{n+1}$ by the formula

$$\begin{aligned} (df)(g_1, .., g_{n+1}) &= g_1 f(g_2, ..., g_{n+1}) \\ &= + \sum_{i=1}^{i=n} (-1)^i f(g_1, ..., g_i g_{i+1}, ..., g_{n+1}) \\ &= +(-1)^{n+1} f(g_1, ..., g_n). \end{aligned}$$

The cohomology groups of this complex are $H_c^i(G, M)$ [11]. Alternatively, these cohomology groups can be defined as the right derived functors of the functor M^G, a fact which will be used later. It will also be useful to understand how limits are incorporated into the definition of profinite group cohomology.

Proposition 3.2.2. *Let G be a profinite group and $\{A_\alpha\}$ be G-modules such that $\varinjlim A_i = A$ then*

$$H^q(G, A) = \varinjlim H^q(G, A_\alpha)$$

Proof. This is a consequence of Serre's Prop. 8 in Sec. 2.2 of [11]. □

Proposition 3.2.3. *Let A be a discrete G-module then*

$$H^q(G, A) = \varinjlim H^q(G/H, A^H) \text{ for all } q \geq 0 \tag{3.2.1}$$

where the limit is taken over all open normal subgroups H of G.

Proof. This is another consequence of Serre's Prop. 8 in Sec. 2.2 of [11]. □

3.3 Computations in Continuous Galois Cohomology

We now apply profinite group cohomology to the case of $G = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Let $\hat{\mathbb{Z}}$ be the completion of $\mathbb{Z}$ for the topology of subgroups of finite index. Then $\hat{\mathbb{Z}} \cong \varprojlim \mathbb{Z}/n\mathbb{Z}$ and is therefore a profinite group. Since, as defined earlier, $G_n \cong \mathbb{Z}/n\mathbb{Z}$ we get that

$$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \varprojlim G_n \cong \hat{\mathbb{Z}}.$$

We also define $H_n = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n}) \cong n\hat{Z}$ and we get from the isomorphism $G_n \cong G/H_n$ (See Prop. 2.1.2), that $G = \varprojlim G/H_n$. We note that every open subgroup of G is of the form H_n for some n. Given A a G-module, G has a single topological generator which is the Frobenius automorphism on A ($F(a) = a^p$ where p is the characteristic of the base field). A a discrete G-module necessitates that for $a \in A$ there is a positive integer n such that $F^n(a) = a$. This gives that $A = \bigcup A^{H_n}$, where the union ranges over all n, as required. We can now define the following

Definition 3.3.1. Let A be a discrete G-module, where $G = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ and $H_n = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$. The cohomology of G with coefficients in A is

$$H^q(G, A) = \varinjlim H^q(G/H_n, A^{H_n}) \tag{3.3.2}$$

for all $q \geq 0$.

In general, we have some useful propositions of Serre for low degree cohomology. First, we note that for G any profinite group with an action on A degree 0 cohomology is equal to the elements in A fixed by the action:

$$H^0(G, A) = A^G.$$

We now consider specifically $G \cong \hat{\mathbb{Z}} \cong Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Let N_n be the map $1 + F + ... + F^{n-1}$ and let

$$A' = \{a \in A | N_n(a) = 0 \text{ for some } n > 0\}.$$

Given that G acts on A by F, we have the following:

Proposition 3.3.2.

$$H^1(G, A) \cong A'/(F-1)A.$$

Proof. See [10]. □

We may also understand the cohomology in degree 2 for certain coefficient modules.

Proposition 3.3.3. *Let A be a torsion group, then*

$$H^2(G, A) = 0.$$

Proof. Suppose A is finite then from Prop. 2.3.2 we have,

$$H^2(G_n, A^{H_n}) = A^G/N_nA^{H_n}$$

where N_n is defined as above. We can show that the homomorphism

$$A^G/N_nA^{H_n} \longrightarrow A^G/N_{nm}A^{H_{nm}}$$

is just multiplication by m. We have that for $a \in A^G/N_nA^{H_n}$, $F(a) = a$ and

$$(1 + F + ... + F^{n-1})(a) = 0$$

so $na = 0$ and for the same reasons $nma = 0$ in $A^G/N_{nm}A^{nmG}$. Thus, a must go to ma making the map multiplication by m. If m is a multiple of the order of A the homomorphism is 0 and since this homomorphism is a part of the directed system of equation 3.3.2, which gives us the definition of $H^2(G, A)$, then the claim is proved for A finite.

If A is torsion then $A = \varinjlim A_\alpha$ for A_α finite. Thus, we have proven our claim by Prop. 3.2.2.

$$H^2(G, A) \cong \varinjlim H^2(G, A_\alpha) \cong 0.$$

□

We now want to consider higher degree cohomology. From the following proposition, we get that cohomology vanishes in degree 3 and higher.

Proposition 3.3.4. *Let A be a discrete G-module; then for each* $q \geq 3$

$$H^q(G, A) \cong 0.$$

Proof. We make use of computations of the cohomology of a cyclic group and determine the homomorphisms from

$$H^q(G_n, A^{H_n}) \longrightarrow H^q(G_{nm}, A^{H_{nm}}).$$

We have the following due to Prop. 2.3.2,

$$H^q(G_n, A^{H_n}) \cong \begin{cases} A^G/N_n A^{H_n} & \text{for} \quad n = 0 \bmod 2 \\ {}_{N_n}A^{H_n}/D^* A^{H_n} & \text{for} \quad n = 1 \bmod 2. \end{cases}$$

When q=2 the map is multiplication by m as shown in the previous proof. Let q=3, then we want to know the map induced in cohomology

$${}_{N_n}A^{H_n}/D^* A^{H_n} \rightarrow_{N_{nm}} A^{H_{nm}}/D^* A^{H_{nm}}.$$

If $a \in_{N_n} A^{H_n}/D^* A^{H_n}$, then $(1 + ... + F^{n-1})(a) = 0$ and $(F - 1)(a) = 0$ so $F(a) - a = 0$, which means $F(a) = a$, so $na = 0$. In ${}_{N_{nm}}A^{H_{nm}}/D^* A^{H_{nm}}$, $(1 + ... + F^{nm-1})(a) = 0$ and $(F - 1)(a) = 0$.

Thus, the map after taking the quotient sends na to nma so before before taking quotients the map must be multiplication by m. Thus after quotienting the map is multiplication by m modulo the given cosets. We see here also that ${}_{N_n}A^{H_n}/D^*A^{H_n}$ is torsion, so when we go far enough out in the limit the map will be multiplication by the order of this group and the rest of the maps will be 0. This means that $H^3(G, A) = 0$ for any G-module A. We proceed with a shifting argument. Given an injective resolution of A,

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow ...$$

we may truncate to get

$$0 \longrightarrow A \xrightarrow{f} I_0 \longrightarrow \mathrm{coker}(f) \longrightarrow 0.$$

We then consider a part of the long exact sequence we get from this exact sequence:

$$... \longrightarrow H_c^3(G, \mathrm{coker}(f)) \longrightarrow H_c^4(G, A) \longrightarrow H_c^4(G, I_0) \longrightarrow ...$$

We know $H_c^3(G, \mathrm{coker}(f)) \cong 0$ since $H_c^3(G, M) \cong 0$ for any discrete G-module M. Also, $H_c^4(G, I_0) \cong 0$ since I_0 is injective. Thus, $H_c^4(G, A) \cong 0$. Since this is true for any discrete G-module A, we can use an inductive argument to show that for all $q \geq 3$ $H_c^q(G, A) \cong 0$. Therefore, we have proven our desired result. □

We now compute the relevant cohomology for algebraic K-theory of finite fields. Let $A = \mu_\ell^{\otimes t}$ where μ_ℓ are the ℓ-th roots of unity in $\overline{\mathbb{F}}_p$ and where we think of A as a G-module with a different action depending on the tensor power t. If $t = 0$ then G acts trivially on A. If $t = 1$ then G acts on A by the Frobenius map $F(a) = a^p$. If $t > 1$ then G acts on A by F^t. We then notice that the ℓ-th roots of unity in $\overline{\mathbb{F}}_p$ are isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ as abelian groups. We use the propositions above to compute the following.

Proposition 3.3.5. *Let $A = \mu_\ell^{\otimes t}$ be a G-module with the action described above. Then*

$$\begin{aligned}
H^0(G,A) &\cong \begin{cases} \{a \in A | F^t(a) = a\} & \textit{if} \quad t > 0 \\ A & \textit{if} \quad t = 0 \end{cases} \\
&\cong \begin{cases} \mathbb{Z}/\gcd(\ell, p^t - 1)\mathbb{Z} & \textit{if} \quad t > 0 \\ \mathbb{Z}/\ell\mathbb{Z} & \textit{if} \quad t = 0 \end{cases} \\
H^1(G,A) &\cong \{a \in A | (F^t - 1)(a) = 0\} \\
&\cong \mathbb{Z}/\gcd(\ell, p^t - 1)\mathbb{Z} \\
H^s(G,A) &\cong 0 \textit{ if } s > 1
\end{aligned}$$

Proof. Let $A = \mu_\ell^{\otimes t}$, first we may compute $H^0(G,A)$ easily by the fact that in general $H^0(G,A) = A^G$, where A^G are the elements fixed by the action of G and since the action is trivial when $t = 0$ we get just $A \cong \mathbb{Z}/\ell\mathbb{Z}$. When $t > 0$ we get the elements in A such that $F^t(a) = a$ which is isomorphic to the elements $a \in \mathbb{Z}/\ell\mathbb{Z}$ such that $(p^t - 1)a = 0$ so it is isomorphic to $\mathbb{Z}/\gcd(\ell, p^t - 1)\mathbb{Z}$. Thus, we have computed degree 0 cohomology.

For degree 1 cohomology, we need to use Prop. 3.3.2. First, we claim that A'=A since in our case A is torsion. In general, let A be a topological G module. Then denote A_f the torsion subgroup of A. If $a \in A_f$ then $na = 0$ for some n. Also, $F^m(a) = a$ for some m since $a \in A$ by definition of the Frobenius map. It follows that

$$\begin{aligned}
(1 + F + ... + F^{mn-1})a &= (1 + F + .. + F^{m-1} + F^m + F^{m+1} ... + F^{mn-1})(a) \\
&= n(1 + F + ... + F^{m-1})(a) \\
&= na' \text{ for some } a' \in A \\
&= 0
\end{aligned}$$

which shows that $a \in A'$. Thus, the torsion elements of A are contained in A'; i.e. if A is torsion then $A \subset A'$. Since $A' \subset A$ by definition we proved the claim. We now interpret the result of Prop. 3.3.2:

$$H^1(G, A) \cong A/(F^t - 1)A.$$

In the proof of Prop. 3.3.2, we are using that G acts on A by the Frobenius map, so if G acts on A by F^t then we get the above definition. Thus,

$$H^1(G, A) \cong (\mathbb{Z}/\ell\mathbb{Z})/((F^t - 1)\mathbb{Z}/\ell\mathbb{Z})$$

We note that for $a \in \mu_\ell$, $F^t(a) - a = 0$, so $a^{p^t - 1} = 0$. Thus, for $a \in \mathbb{Z}/\ell\mathbb{Z}$, $(p^t - 1)a = 0$, and we get the following

$$H^1(G, A) \cong \mathbb{Z}/\gcd(\ell, p^t - 1)\mathbb{Z}$$

proving our claim.

We now need to show $H^s(G, A) = 0$ for $s > 1$. From Prop. 3.3.3 we know that since A is torsion,

$$H^2(G, A) \cong 0.$$

Then we note that for any G-module A we have from Prop. 3.3.4

$$H^s(G, A) \cong 0 \text{ for } s \geq 3.$$

Therefore, we have computed all the cohomology of $G = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ with coeffictients $\mu_\ell^{\otimes t}$ as required. □

Chapter 4

Algebraic K-theory of finite fields

This computation of Algebraic K-theory for finite fields involves continuous galois cohomology, étale cohomology and motivic cohomology, although it suffices to compute continuous galois cohomology and then pass to étale or motivic cohomology where the results will be equivalent in this special case. The ability to identify continuous galois cohomology to étale cohomology or étale cohomology to motivic cohomology is nontrivial and it shall be the first topic of discussion. Once we have a computation of étale or motivic cohomology, we use a spectral sequence where the E_2 page can be identified with étale or motivic cohomology with appropriate bi-grading. The reason for using either motivic or étale category in this setting is that we have two versions of an Atiyah-Hirzebruch type spectral sequence to work with: one starting at motivic cohomology and one starting at étale cohomology, each converging to similar versions of algebraic K-theory. The computation proceeds as follows; (1) we identify continuous galois cohomology with étale cohomology, (2) we identify étale cohomology with motivic cohomology, (3) we describe the two versions of the spectral sequence we have, (4) we compute algebraic K-theory of finite fields, and (5) we compare to results of Quillen.

4.1 Continuous Galois Cohomology to Étale Cohomology

For our purposes, we need enough étale cohomology to show an equivalence with continuous galois cohomology. If we let $G = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, then consider the category of discrete G-modules, **G-mod.** It can be shown that there is an equivalence of categories between **G-mod** and the category of sheaves of abelian groups over the étale site, X_{et}, of X=spec($\mathbb{F}_p$). We denote this category **S(**X_{et}**)**. We begin by defining some terms used already to clarify the necessary propositions.

4.1.1 Definitions: basic algebraic geometry

Our main sources for these sections are Milne's book on étale cohomology [7] and Hartshorne's *Algebraic Geometry* [2]. These definitions are meant to help the inexperienced reader get a brief introduction to the material and it should be skipped by any reader with previous knowledge of algebraic geometry on the level of sheaves and schemes. It should be acknowledged that at the time of writing the author considers himself in the former category.

Definition 4.1.1. A presheaf P is a contravariant functor from $\mathcal{T}op(X)$, the category of open subsets of X with morphisms given by inclusions, to a category $\mathcal{C}$, usually the category of sets, abelian groups or commutative rings.

Definition 4.1.2. A sheaf $\mathcal{F}$ is a presheaf with the additional properties:

1. Given U_i a covering of U and $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$, and
2. if for each i there exists a section s_i of $\mathcal{F}$ such that for U_i, U_j and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a section $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for each i.

Definition 4.1.3. The *stalk* of a sheaf $\mathcal{F}$ gives local information about the sheaf at a point. It is given by the limit taken over all U such that the point $x \in U$

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U)$$

Definition 4.1.4. A ringed space is a topological space X, along with a structure sheaf of rings $\mathcal{O}_X$. A ringed space is called a locally ringed space if each stalk of the structure sheaf is a ring with a unique maximal ideal.

Definition 4.1.5. An affine scheme X is a locally ringed space isomorphic to spec(R) for some commutative ring R.

Definition 4.1.6. A scheme X is a locally ringed space along with a covering of open sets U_i such that restricting to the structure sheaf $\mathcal{O}_X$ is an affine scheme.

Remark. This is a similar notion to a covering of a manifold with coordinate maps.

4.1.2 Definitions needed for étale cohomology

We have defined sheaves and schemes. We now want to interpret this in the étale setting. To do this we need morphisms of schemes.

Definition 4.1.7. A morphism of schemes from X to Y is given by a pair (f, ϕ) where f is a continuous map $f : X \longrightarrow Y$ and $\phi : \mathcal{O}_X \longrightarrow f_*(\mathcal{O}_Y)$ where $f_*(\mathcal{O}_Y)$ is the direct image of the structure sheaf of X.

Definition 4.1.8. We say a morphism of schemes is *étale* if it is flat and unramified (this also implies locally of finite type).

Definition 4.1.9. A morphism of schemes $f : Y \longrightarrow X$ is affine if for any affine subset U of X, then $f^{-1}(U)$ is affine in Y. If, for all U with this property, $\Gamma(f^{-1}(U), \mathcal{O}_Y)$ is a finite $\Gamma(U, \mathcal{O}_X)$-algebra, then f is finite.

For definitions of flat and unramified see [7]. It is enough to know that the Zariski topology is too coarse for our purposes and by using étale morphisms as structure morphisms for a scheme, we may get more interesting cohomology. For example, in the case here where we look at spec($\mathbb{F}_p$) which is just a point in the Zariski topology. When we talk about the étale topology, we mean in the sense of a Grothendieck topology on a category. We define a topology on a subcategory of schemes over a base scheme. We call a category along with a Grothendieck topology a site.

Definition 4.1.10. We define X_{et} to be the small local étale site of X. We fix the base scheme X and consider the full subcategory of schemes over X with étale structure morphisms, which we will denote ét/X. This subcategory is given the étale topology in the following way. We define an étale covering of an object Y on ét/X as a family of étale morphisms $g_i : U_i \to Y \; i \in I$ with the property that $Y = \bigcup g_i(U_i)$, for $U_i \in$ ét/X. The class of all such coverings g_i of all such Y forms the étale topology. The category along with this topology is what we refer to as the small local étale site.

For the following proposition, we will also use the notion of a geometric point of a space X with the étale topology. One difference between the étale topology and the Zariski topology for example, is that usually when we consider sheaves over a point, we mean just the set or abelian group assigned to that point, however in the étale setting, sheaves over a point will only give a single set or abelian group if that point is spec(k_{sep}) for k_{sep} a separably closed field. Otherwise, this will not be the case. For X=spec(k) where k is not the separable closure, we denote the geometric point $\bar{x}$ = spec(k_{sep}). In our example, we are considering schemes over spec($\mathbb{F}_p$) so the geometric point will be $\bar{x}$ = spec($\overline{\mathbb{F}}_p$). By contravariance of the functor spec(-), we get a map $u_x : \bar{x} \longrightarrow X$ induced by the inclusion of $\mathbb{F}_p$ in $\overline{\mathbb{F}}_p$.

4.1.3 Equivalence of categories and cohomology theories.

We return to our goal of showing an equivalence of cohomology theories. It begins by showing an equivalence of categories between **G-mod** and $\mathbf{S}(X_{et})$ as laid out in the introduction to this section.

Proposition 4.1.11. *Let $G = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ and X=spec($\mathbb{F}_p$). There is an equivalence of categories between* **G-mod,** *the category of discrete G-modules and* $\mathbf{S}(X_{et})$**,** *the category of sheaves of abelian groups on the small local étale site.*

Proof. We design functors from **G-mod** to $\mathbf{S}(X_{et})$ and from $\mathbf{S}(X_{et})$ to **G-mod** which give us an equivalence of categories. First, we note that by fixing a separable closure of $\mathbb{F}_p$ (in this case we have only one separable closure $\overline{\mathbb{F}}_p$) we are fixing a geometric point $\overline{x} \to X$ and we can write G as the fundamental group of X with respect to that geometric point $G = \pi_1(X, \overline{x})$, in the sense of [7]. By contravariance, G acts on $\overline{\mathbb{F}}_p$ on the left and spec($\overline{\mathbb{F}}_p$) $= \overline{x}$ on the right. Given a presheaf P on the site X_{et}, let

$$M_P = \varinjlim P(\mathrm{spec}(\mathbb{F}_{p^n}))$$

for every finite separable extension of $\mathbb{F}_p$. We inherit a left action of G on P(spec($\mathbb{F}_{p^n}$))) from the usual action of G on $\mathbb{F}_{p^n}$. Thus, we get an action of G on M_P. It is clear that $M_P = \cup M_P^{H_n}$ where the union is taken over all subgroups H_n of G. This makes M_P a discrete G-module.

On the other hand, given a discrete G-module M, we can construct a presheaf $\mathcal{F}_M$ with the following properties

1. $\mathcal{F}_M(\mathbb{F}_{p^n}) = M^{H_n}$, if $H_n = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$

2. $\mathcal{F}_M(\prod \mathbb{F}_{p^i}) = \coprod \mathcal{F}_M(\mathbb{F}_{p^i})$.

We do this in the following way. Let $\mathcal{F}_M$(U) be identified with G-module homomorphisms from a functor $\mathcal{F}$ evaluated at U to the G-module M. We have

$$\mathcal{F}_M(U) = \mathrm{Hom}_G(\mathcal{F}(U), M)$$

where $\mathcal{F}$ is a functor from the category of X-schemes finite and étale over X,

$$\textbf{FEt} / \mathrm{X} \longrightarrow \textbf{G- sets}$$

defined by $\mathcal{F}(U) = \mathrm{Hom}_X(\overline{x}, U)$. The category ét/ X contains morphisms with an extra property of finiteness as in Definition 4.9, and we call the category of such morphisms **FEt** / X. **G- sets** is the category of finite sets with a continuous left action of G. The Hom set is the set of X-scheme maps from the geometric point $\overline{x}$ into U a scheme over X. We see then that $\mathcal{F}(\mathbb{F}_{p^n}) = \mathrm{Hom}_X(\mathrm{spec}(\overline{F}_p), \mathrm{spec}(F_{p^n})) \cong G/H_n$ and $\mathcal{F}(\prod F_{p^i}) = \prod \mathcal{F}(F_{p^i})$. Thus, defining $\mathcal{F}_M$ gives us the presheaf we wish to find . It takes the work of another lemma to prove that this is a sheaf, which can be found in [7]. We then have shown that we can identify any G-module M with a sheaf in $\mathbf{S}(X_{et})$. To get an equivalence of categories we need a little more work. We can see that given a morphism of discrete G-modules $f : M \to M'$ there is clearly an induced morphism on sheaves $\mathcal{F}_M \longrightarrow \mathcal{F}_{M'}$. Given a morphism on sheaves $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ we let $\mathbb{F}_{p^n} = \overline{\mathbb{F}}^{H_n}$ where $H_n = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$, then

$$\phi(\mathbb{F}_{p^n}) : \mathcal{F}(\mathbb{F}_{p^n}) \longrightarrow \mathcal{F}'(\mathbb{F}_{p^n})$$

commutes with the action of G because ϕ is a functor. We can then define a G-homomorphism by $\varinjlim \phi(\mathbb{F}_{p^n})$ from $M_{\mathcal{F}} \longrightarrow M_{\mathcal{F}'}$. We also can see that $Hom_G(M, M') \longrightarrow Hom(\mathcal{F}, \mathcal{F}')$ is an isomorphism and the map $\mathcal{F} \longrightarrow \mathcal{F}_{M_{\mathcal{F}}}$ is an isomorphism. Thus, we have shown that the categories are equivalent. □

This equivalence of categories leads to an equivalence of cohomology theories of étale cohomology of spec($\mathbb{F}_p$) and galois cohomology of Gal($\overline{\mathbb{F}}_p/\mathbb{F}_p$) with appropriate coefficients. In order to make this claim we must recall the definition of continuous galois cohomology as right derived functors of the functor A^G where A is the coefficient module. We define a similar notion of étale cohomology in terms of right derived functors.

Definition 4.1.12. Let $\Gamma(X, -) : \mathbf{S}(X_{et}) \longrightarrow \mathbf{Ab}$ be the functor defined by $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. This is a left exact functor and we can therefore define étale cohomology in terms of the right derived functors R^i of $\Gamma(X, -)$:

$$H^i_{et}(X, -) \cong R^i\Gamma(X, -).$$

Remark. Note that for definition 4.11 we need $\mathbf{S}(X_{et})$ to have enough injectives. A proof of this fact can be found in Milne [7].

Proposition 4.1.13. *Let $\mathbb{F}_p$ be a finite field, $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ and G=Gal($\overline{\mathbb{F}}_p/\mathbb{F}_p$). Let $\mu_l^{\otimes t}$ be the group of l-th roots of unity in $\overline{\mathbb{F}}_p$ with the G acting by F^t. In étale cohomology we see this as the sheaf of abelian groups over spec($\mathbb{F}_p$), which sends spec($\mathbb{F}_{p^n}$) to the ℓ-th roots of unity in $\mathbb{F}_{p^n}$. Then*

$$H^n_c(G, \mu_l^{\otimes t}) \cong H^n_{et}(\mathbb{F}_p, \mu_l^{\otimes t}). \tag{4.1.1}$$

Note that in étale cohomology we use $\mathbb{F}_p$ as shorthand for spec($\mathbb{F}_p$).

Proof. By Definition 4.12, we have,

$$R^i\Gamma(X, -) = H^i_{et}(X, -)$$

for the étale cohomology of X. From Proposition 4.11 we see that

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X) = M^G$$

so for X=spec($\mathbb{F}_p$), $\mathcal{F} = \mu_\ell^{\otimes t}$, and $M = \mu_\ell^{\otimes t(\bar{\mathbb{F}}_p)}$ we have that

$$\begin{aligned} H^i_{et}(X, \mathcal{F}) &\cong R^i\Gamma(X, \mathcal{F}) \\ &\cong R^i M^G \\ &\cong H^i_c(G, M) \end{aligned}$$

.

Thus, we have our desired result. □

4.2 Norm Residue Theorem

The following account shows a correspondence between étale cohomology and motivic cohomology used in computations of a spectral sequence which converges to algebraic K-theory. The theorem of primary importance here is the Norm Residue Theorem. Let $H^n_{mot}(k, \mathbb{Z}/\ell(i))$ be motivic cohomology defined to be the cohomology of the chain complex $\mathbb{Z}/\ell(i)$ of Nisnevich sheaves. For more information on motivic cohomology and Nisnevich sheaves see [5]. Many of the results in this section are not understood by the author in their full generality. They are used here as a way to pass from galois cohomology where computations are made, to motivic cohomology, which is the input for Weibel's version of the spectral sequence used in future computations.

Theorem 4.2.1. Norm Residue Theorem *(Rost-Voevodsky) Let k be a field containing $1/\ell$. The natural map then induces isomorphisms*

$$H^n_{mot}(k, \mathbb{Z}/\ell(i)) \cong \begin{cases} H^n_{et}(k, \mu_\ell^{\otimes i}) & n \leq i \\ 0 & n > i \end{cases} \tag{4.2.2}$$

If X is a smooth scheme over the base field k there is a the natural map

$$H^n_{mot}(X, \mathbb{Z}/\ell(i)) \to H^n_{et}(X, \mu_\ell^{\otimes i})$$

is an isomorphism for $n \leq i$.

This form of the theorem is from Weibel's book [16]. This gives us a correspondence between motivic cohomology and étale cohomology. The coefficient change makes sense due to a lemma in a paper by Voevodsky [14],

Lemma 4.2.2. *Let k be a field and ℓ prime to the char(k), then there is a quasi isomorphism in the etale topology on the category of smooth schemes over k*

$$\mathbb{Z}/\ell\mathbb{Z}(i)_{et} \cong \mu_\ell^{\otimes i}.$$

Thus, one makes the movement in the following way:

$$H^n_{mot}(k, \mathbb{Z}/\ell(i)) \cong H^n_{et}(k, \mathbb{Z}/\ell(i)_{et}) \cong H^n_{et}(k, \mu_\ell^{\otimes i}).$$

We interpret these results for the case of finite fields in the following corollary.

Corollary 4.2.3. *Let $\mathbb{F}_p$ be a finite field of characteristic p, X=spec($\mathbb{F}_p$), and let ℓ be a prime with $\ell \neq p$ (Note that $1/\ell \in \mathbb{F}_p$). Then the natural map induces an isomorphism*

$$H^n_{mot}(X, \mathbb{Z}/\ell(i)) \cong \begin{cases} H^n_{\acute{e}t}(X, \mu_\ell^{\otimes i}) & n \leq i \\ 0 & n > i \end{cases} \tag{4.2.3}$$

This gives us the relevant isomorphism for our computation. In Weibel's version of the spectral sequence, we need this intermediary step to get to motivic cohomology which is the input in the spectral sequence. We also need the intermediary step to get from étale cohomology to galois cohomology, from the previous section. Ultimately, we will use a version of the spectral sequence which takes étale cohomology as input. The fact that the spectral sequences used in the next section compute algebraic K-theory depends on the validity of the norm residue theorem.

4.3 Algebraic K-theory Spectral Sequence

We have two different versions of an Atiyah-Hirzebruch type spectral sequence. One version comes from Weibel, which he calls the “motivic to algebraic K-theory” spectral sequence. His version computes algebraic K-theory with coefficients outright which is an advantage, but his differentials are not specified explicitly, which is a disadvantage. We get from Weibel [16] the following:

Theorem 4.3.1. *Let A be any coefficient group and X be a smooth scheme over a field k. Then there is a spectral sequence from motivic cohomology to algebraic K-theory with E_2 term*

$$E_2^{s,t} = H^{s-t}_{mot}(X, A(-t) \Rightarrow K_{-s-t}(X; A). \tag{4.3.4}$$

If X=spec(k) and $A = \mathbb{Z}/\ell\mathbb{Z}$, where $1/\ell \in k$ then the E_2 terms are just the étale cohomology groups of k, which lie only in the octant where $s \leq t \leq 0$.

This is enough to see what the E_2 page would be, but without knowing what the differentials are we cannot be sure what will happen on the E_r page for $r > 2$. From looking at another version of this type of spectral sequence due to Thomason [13] we can see that the differentials should be trivial in our specific case.

Theorem 4.3.2. *Let $\ell \neq p$ be prime, and let X be a smooth scheme satisfying some other mild hypotheses containing $1/\ell$ (X=spec(k) for k a field satisfies the necessary hypotheses). Let β be the Bott element. The graded ring $K/\ell_*(x)$ may be localized by inverting this β. We can then compute $K/\ell_*(X)[\beta^{-1}]$ in terms of etale cohomology of X:*

$$E_2^{s,t} = \begin{cases} H^s_{et}(X; \mathbb{Z}/\ell(i)), & t = 2i \\ 0, & t \text{ odd} \end{cases} \Rightarrow K/\ell_{t-s}(X)[\beta^{-1}]. \tag{4.3.5}$$

What we lose here is that we are now computing mod ℓ algebraic K-theory with an inverted Bott element instead of outright algebraic K-theory, but this will be enough for our purposes due to Theorem 4.4.2 and Theorem 4.5.2. What we gain here is that Thomason gives the differentials in the spectral sequence:

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,\ t+r-1}$$

in his paper [13]. We can then set up the spectral sequence in the same way one would set up the Adams spectral sequence with s on the vertical axis and t-s on the horizontal axis. In the next section, we use the information from previous sections to input in our spectral sequence and compute algebraic K-theory of finite fields.

4.4 Spectral Sequence Computation

We begin by interpreting Thomason's spectral sequence in the case of finite fields where étale cohomology can be identified with continuous galois cohomology.

Corollary 4.4.1. *Let $G = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ and let μ_ℓ be the ℓ-th roots of unity in $\overline{\mathbb{F}}_p$. Then the following is a spectral sequence from continuous galois cohomology, which converges to mod ℓ algebraic K-theory with an inverted Bott element*

$$E_2^{s,t} = \begin{cases} H_c^s(G;\mu_\ell^{\otimes i}), & t = 2i \\ 0, & t \text{ odd}, s > i \end{cases} \Rightarrow K/\ell_{t-s}(\mathbb{F}_p)[\beta^{-1}]. \tag{4.4.6}$$

We know that we only have degree 0 and degree 1 continuous galois cohomology with these coefficients due to Prop. 3.3.3 and Prop. 3.3.4, which means that the spectral sequence is only nontrivial with s in degree 0 and 1 and with t in even degrees. We have the following which tells us that in most degrees our answer will agree with algebraic K-theory without the inverted Bott element.

Theorem 4.4.2. *Let $\mathbb{F}_p$ be a finite field and $K/\ell_n(\mathbb{F}_p)[\beta^{-1}]$ be algebraic K-theory with an inverted Bott element; then*

$$K/\ell_n(\mathbb{F}_p)[\beta^{-1}] = K_n(\mathbb{F}_p, \mathbb{Z}/\ell\mathbb{Z})$$

for $n > 0$.

Proof. To show that algebraic K-theory of finite fields with an inverted Bott element is equivalent to algebraic K-theory without the inverted Bott element, we need to see where the Bott element comes from. Due to Weibel, we know the Bott element lives in degree 2 of algebraic K-theory of finite fields and multiplication by the Bott element gives an isomorphism to higher even degrees [16]. Similarly, the Bott element gives an isomorphism from degree 1 algebraic K-theory to higher odd algebraic K-theory. Thus, inverting the Bott element does not change the K-theory groups since there is no Bott torsion in algebraic K-theory above degree 0. □

We now have all the information necessary to make our computation.

Theorem 4.4.3. *Let X=spec($\mathbb{F}_p$), $\ell \neq p$ a prime, then we compute the folllowing,*

$$K/\ell_{t-s}(X)[\beta^{-1}] \cong \left\{ \begin{array}{ccc} \mathbb{Z}/\ell & if & t-s=0 \\ \mathbb{Z}/\gcd(\ell, p^i - 1) & if & t-s = 2i-1 \\ \mathbb{Z}/\gcd(\ell, p^i-1) & if & t-s=2i \end{array} \right\}$$

Proof. Using Thomason's version of the spectral sequence and setting up the axis with t-s on the horizontal axis and s on the vertical axis, we can then input our answers from the previous section into the nontrivial positions. Let us recall these calculations,

$$
\begin{aligned}
H_c^0(G, \mu_\ell^{\otimes i}) &\cong \begin{cases} \mathbb{Z}/\ell\mathbb{Z} & \text{if} \quad i = 0 \\ \mathbb{Z}/\gcd(\ell, p^i - 1)\mathbb{Z} & \text{if} \quad i > 0 \end{cases} \\
H_c^1(G, \mu_\ell^{\otimes i}) &\cong \mathbb{Z}/\gcd(\ell, p^i - 1)\mathbb{Z}
\end{aligned}
$$

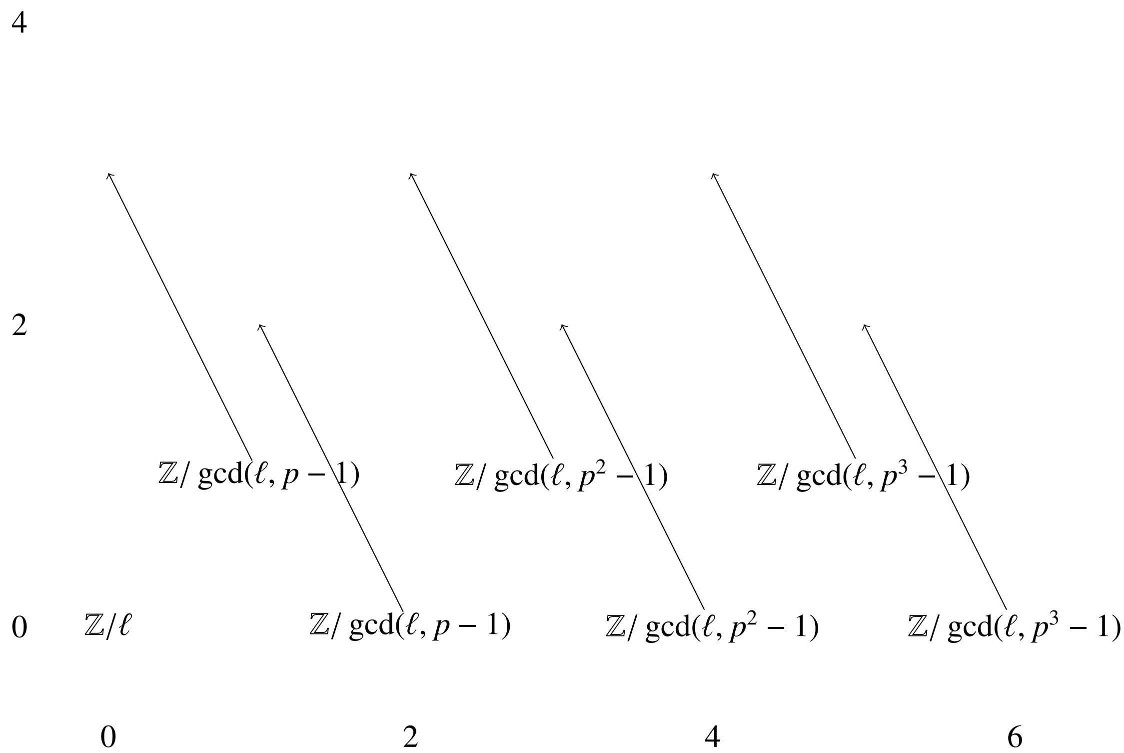
From Thomason we have

$$
E_2^{s,t} \cong \begin{cases} H_c^s(G, \mu_\ell^{\otimes i}) & \text{if} \quad t = 2i \\ 0 & \text{if} \quad t = 2i - 1 \end{cases} \Rightarrow K/\ell_{t-s}(\mathbb{F}_p)[\beta^{-1}]
$$

Thus, the following table will help us input the correct information. Let $i \geq 1$.

s	t	$t-s$	*Group*
0	0	0	$\mathbb{Z}/\ell$
0	$2i-1$	$2i-1$	0
0	$2i$	$2i$	$\mathbb{Z}/\gcd(\ell, p^i - 1)$
1	1	0	0
1	$2i$	$2i-1$	$\mathbb{Z}/\gcd(\ell, p^i - 1)$
1	$2i-1$	$2i$	0

We get the following E_2 page with d_2 differentials given by the arrows. The picture is shown with s on the vertical axis and $t-s$ on the horizontal axis.



We see that there is no room for nontrivial differentials, thus each group will persist to the next page. Therefore, $E_2 \cong E_3$ and since no d_r differentials for $r > 2$ will be nontrivial we get $E_2 \cong E_\infty$. Thus, we get our computation of $K/\ell_{t-s}(\mathbb{F}_p)[\beta^{-1}]$,

$$K/\ell_{t-s}(\mathbb{F}_p)[\beta^{-1}] \cong \left\{ \begin{array}{cll} \mathbb{Z}/\ell & \text{if} & t-s=0 \\ \mathbb{Z}/\gcd(\ell, p^i-1) & \text{if} & t-s=2i-1 \\ \mathbb{Z}/\gcd(\ell, p^i-1) & \text{if} & t-s=2i \end{array} \right\}$$

Since ℓ is prime,

$$\gcd(\ell, p^i-1) = \begin{cases} \ell & \text{if} \quad \ell | p^i - 1 \\ 1 & \text{otherwise} \end{cases}.$$

Thus, wherever ℓ does not divide $p^i - 1$ for some i the mod ℓ algebraic K-theory vanishes.

□

We notice that there are no nontrivial differentials anytime the cohomological dimension $n \leq 1$. In this situation we need only consider cohomological dimension of a profinite group.

Definition 4.4.4. The ℓ-cohomological dimension of a profinite group G is less than or equal to n, written cd(G)≤n if and only if $H^r(G, A) = 0$ for all $r > n$ and all ℓ-torsion G-modules A.

Since we cannot always identify étale cohomology with profinite group cohomology, we need to consider étale cohomological dimension as well. The relevant cohomological dimension here is ℓ-cohomological dimension. We say a sheaf of abelian groups $\mathcal{F}$ is ℓ-torsion if for all quasi-compact U $\mathcal{F}(U)$ is ℓ-torsion. Clearly, the coefficients we are working with have this property.

Definition 4.4.5. The ℓ-cohomological dimension of a site, $(C/X)_{et}$ is the smallest integer n (or ∞) such that $H^i(X_{et}, \mathcal{F}) = 0$ for all $i > n$ and all ℓ-torsion sheaves $\mathcal{F}$.

Using this definition of cohomological dimension we get an easy corollary to Theorem 4.3.2:

Corollary 4.4.6. *Whenever $H^n_{et}(X, \mathbb{Z}_\ell(i))$ has cohomological dimension $n \leq 2$, the differentials will be trivial in all degrees in the spectral sequence of Theorem 4.3.2, and $E_2 \cong E_\infty$.*

4.5 Comparison to results of Quillen

Computing higher algebraic K-theory by definition requires heavier machinery, therefore we prefer to use the results cited in this paper for the computation. However, the only way to compute higher algebraic K-theory groups originally was by definition. The work of computing and defining higher K-theory comes from Quillen [9], [8]. Though the work in these papers is not the focus of this thesis, it is interesting to see the results that Quillen gets in his computation. One difference in our computation is that we are computing mod ℓ K-theory. It is for this reason that we see $\mathbb{Z}/\ell\mathbb{Z}$ in our result depending on a relation to the characteristic of the finite field we consider. In Quillen's computation in [8], he arrives at

Theorem 4.5.1. *(Quillen) If k is a finite field with p elements then*

$$\begin{array}{rcl} K_0(k) & \cong & \mathbb{Z} \\ K_{2i}(k) & \cong & 0 \\ K_{2i-1}(k) & \cong & \mathbb{Z}/(p^i - 1)\mathbb{Z}. \end{array}$$

for $i \geq 1$.

We use two theorems to show the relationship between this calculation and our calculation. One stated earlier, Theorem 4.4.2, gives us algebraic K-theory without the inverted Bott element above degree 0. The second shows how to get from algebraic K-theory with coefficients to algebraic K-theory without coefficients.

Theorem 4.5.2. *(Universal Coefficient Theorem for algebraic K-theory)*
The following is a short exact sequence

$$0 \to K_n(k) \otimes \mathbb{Z}/\ell \to K_n(k; \mathbb{Z}/\ell\mathbb{Z}) \to \ell K_{n-1}(k) \to 0$$

where $\ell K_{n-1}(k)$ *is the* ℓ *torsion elements of* $K_{n-1}(k)$ *for all* $n \in \mathbb{N}$ *and* ℓ *and any ring k.*

Our reference for this theorem is Weibel's book on algebraic K-theory [16]. We use this theorem to make the following computation which relates the results of Quillen to the results found in this paper:

Proposition 4.5.3. *Let $\ell \neq p$ be a prime. Then it follows from Quillen's computation of $K_*(\mathbb{F}_p)$ that*

$$K_*(\mathbb{F}_p;\mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell & \text{if, } \ell | p^i - 1 \text{ and } * = 2i \\ \mathbb{Z}/\ell & \text{if , } \ell | p^i - 1 \text{ and } * = 2i-1 \\ 0 & \text{if } \ell \text{ does not divide } p^i - 1 \end{cases},$$

for $* > 0$.

Proof. We show this using the Universal Coefficient theorem which gives us

$$0 \to K_n(\mathbb{F}_p) \otimes \mathbb{Z}/\ell \to K_n(\mathbb{F}_p;\mathbb{Z}/\ell) \to \ell K_{n-1}(\mathbb{F}_p) \to 0.$$

For even n=2i+1, $i \geq 0$ we get the following from Quillen's computation,

$$0 \to \mathbb{Z}/(p^i - 1)\mathbb{Z} \otimes Z/\ell \to K_n(\mathbb{F}_p;\mathbb{Z}/\ell) \to 0 \to 0.$$

Therefore, by exactness the map is an isomorphism,

$$0 \to \mathbb{Z}/\gcd(p^i - 1, \ell)\mathbb{Z} \cong K_n(\mathbb{F}_p;\mathbb{Z}/\ell) \to 0.$$

Note that $\mathbb{Z}/(p^i - 1) \otimes \mathbb{Z}/\ell \cong \mathbb{Z}/\gcd(p^i - 1, \ell)$. Since we assume ℓ is prime and not equal to p, we get

$$\mathbb{Z}/\gcd(p^i - 1, \ell) \cong \begin{cases} \mathbb{Z}/\ell & \text{if} \quad \ell | p^i - 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

We also have for n=2i, $i \geq 1$,

$$0 \to 0 \to K_n(\mathbb{F}_p;\mathbb{Z}/\ell) \to \ell\mathbb{Z}/(p^i - 1) \to 0$$

which gives us,

$$0 \to K_n(\mathbb{F}_p; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell \to 0$$

if $\ell | p^i - 1$ and $K_n(\mathbb{F}_p; \mathbb{Z}/\ell) \cong 0$ otherwise. This follows because the ℓ-torsion elements in $\mathbb{Z}/(p^i - 1)\mathbb{Z}$ will be a copy of $\mathbb{Z}/\gcd(\ell, p^i - 1)\mathbb{Z}$, which as we said before $\mathbb{Z}/\ell\mathbb{Z}$ if $\ell | p^i - 1$ and 0 otherwise.

In sum,

$$K_n(\mathbb{F}_p; \mathbb{Z}/\ell) \cong K/\ell_n(\mathbb{F}_p) \cong \mathbb{Z}/\ell\mathbb{Z}$$

for $\ell | p^i - 1$ where $n = 2i$ or $n = 2i - 1$. □

This shows us that in computing the mod ℓ algebraic K-theory via continuous galois cohomology our claims are validated by previous work of Quillen. The result of this paper therefore is not new, but the approach generalizes to other possible computations as described in the next section.

Chapter 5

Future Research

The computation we present here is a natural first computation to do using the methods outlined in this paper. Further computations could be made using knowledge of étale cohomology. In this case, étale cohomology was easily computed via continuous galois cohomology, but it would be interesting to consider other cases where we cannot take this approach. Milne computes étale cohomology of curves over an algebraically closed field with coefficients in μ_ℓ. The only augmentation to this result needed for the spectral sequence is to consider the higher tensor power coefficients. Milne's computation in his lecture notes [6] is the following

Theorem 5.0.4. *If X is a complete connected nonsingular curve over k with k algebraically closed and ℓ a prime different from char(k),*

$$\begin{array}{rcc} H^0(X,\mu_\ell) & \cong & \mu_\ell(k) \\ H^1(X,\mu_\ell) & \cong & (\mathbb{Z}/\ell\mathbb{Z})^{2g} \\ H^2(X,\mu_\ell) & \cong & (\mathbb{Z}/\ell\mathbb{Z} \\ H^s(X,\mu_\ell) & \cong & 0 \end{array}$$

for $s > 2$, and g the genus of the curve X.

The next step would be to compute algebraic K-theory for these types of curves.

Another direction would be unveil the underpinnings of the spectral sequence itself in order to understand cases where the differentials are nontrivial, or perhaps to find operations preserved by the differentials. It would also be interesting to understand Quillen's original methods and constructions better.

The methods used in this work were preferred because they introduced cohomology computations and spectral sequence computations, which are important to work in algebraic topology and algebraic geometry. It served as an introduction to different cohomology theories and different categories in which a mathematician may work and showed how one may move between them.

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