

$K(n)$ -LOCAL HOMOTOPY GROUPS VIA LIE ALGEBRA COHOMOLOGY

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1. INTRODUCTION

In May's thesis [1], he gives a construction of a spectral sequence from Lie algebra cohomology to Hopf algebra cohomology, which has been profitable for classical computations of homotopy groups of spheres as well as chromatic methods. Evidence for this kind of correspondence between Lie algebras and Hopf algebras was given in Milnor and Moore's paper "On the structure of Hopf algebras." In chromatic stable homotopy theory, we would like to be able to compute $Ext_{BP_*BP}^*(BP_*, M)$ for some BP_*BP -comodule M where M is recognizable as $BP_*(X)$ for some spectrum X , because this is the input for the Adams-Novikov spectral sequence. We compute this for $BP_*(X)$ where X is the $E(2)$ -local Smith-Toda complex $V(1)$ for primes $p > 3$, or equivalently $L_{K(2)}V(1)$. We begin by computing Lie-Algebra cohomology for certain solvable Lie algebras using Hochschild-Serre spectral sequences. Then we use the Lie-May spectral sequence to compute the cohomology of the associated graded of $S(2)$ using Ravenel's filtration. The May spectral sequence allows us to compute $H^*(S(2))$ which, after adjoining $v_2^{\pm 1}$, can be identified via a chain of change of rings isomorphisms with our desired input for the Adams-Novikov spectral sequence. We also show the beginning steps of this computation at primes $p = 2, 3$. Many of these computations are known and can be found in Ravenel's "Complex Cobordism and the Homotopy Groups of Spheres," but we give an expanded version of them here and hope to extend these results in future work.

2. MILNOR-MOORE THEORY

A priori, the input of the chromatic spectral sequence is Hopf algebra cohomology, but due to the seminal work of Milnor and Moore this can often be identified with Lie algebra cohomology. This enables us to use May's spectral sequences to compute our desired Hopf Algebra cohomology. Here we summarize the necessary results of Milnor and Moore.

Definition 2.1. Let A be a graded algebra over k , with product $[,] : A \otimes A \rightarrow A$ satisfying,

$$[x, y] = xy - (-1)^{|x||y|}yx$$

for $|x| = p$, and $|y| = q$. Then we say that $(A, [,])$ is the associated Lie algebra of A . A graded k -module L with a product $[,]$ satisfying the commutative diagram,

$$\begin{array}{ccc} L \otimes L & \longrightarrow & L \\ \downarrow & & \downarrow \\ A \otimes A & \longrightarrow & A, \end{array}$$

with respect to some associated Lie algebra of A over L is a Lie algebra.

A Lie algebra is not an algebra because it is not unital and the product is not associative. However, we do have a universal construction of an algebra from a Lie algebra.

Definition 2.2. Given a Lie algebra L , the *universal enveloping algebra* of L , $U(L)$ is an algebra equipped with a map $L \rightarrow U(L)$ satisfying a universal property; i.e. if there is another algebra B with a Lie algebra map $L \rightarrow B$, then there exists an algebra map

$U(L) \rightarrow B$ such that the diagram of Lie algebras

$$\begin{array}{ccc} L & \longrightarrow & (U(L), [,]) \\ & \searrow & \downarrow \\ & & (B, [,]) \end{array}$$

commutes.

We can construct $U(L)$ by first forming the tensor algebra $T(L) = \bigoplus_n (L)^{\otimes n}$ where, by convention, $L^{\otimes 0} = k$. $T(L)$ has a product coming from the isomorphisms $L^{\otimes n} \otimes L^{\otimes m} \rightarrow L^{\otimes n+m}$. We then form $U(L)$ by $T(L)/I$ where $I = (xy - (-1)^{pq}yx - [x, y])$. $U(L)$ is a Hopf algebra so we can view U as a functor from Lie algebras to Hopf algebras. If k has characteristic zero, then U has an adjoint P where $P(A)$ is the Lie algebra of primitives in A .

Theorem 2.3 (Milnor-Moore [2]). If $\text{char}(k)=0$, then there is an equivalence of categories

$$U : \text{Lie}/k \rightleftarrows p.g.Hopf/k : P$$

where Lie/k is the category of Lie algebras over k and $p.g.Hopf/k$ is the category of primitively generated Hopf algebras over k .

Now we will mainly be concerned with the case when k is characteristic $p > 2$ for some prime p . We will assume our graded algebras are concentrated in even degrees since this is the case in the examples we care about and it makes some statements cleaner.

Definition 2.4. Let A be a commutative ring which is a graded algebra over a finite field of characteristic p . Let $\xi : A_n \rightarrow A_{pn}$ be the map $\xi(x) = x^p$ where n is even. The associated Lie algebra $(A, [,], \xi)$ with this map, called a restriction map, is the associated restricted Lie algebra, $(A, [,], \xi)$.

Definition 2.5. A restricted Lie algebra L is a Lie algebra along with a family of functions $\xi_L : L_n \rightarrow L_{pn}$ such that for some algebra A there exists a monomorphism of Lie algebras $L \rightarrow A$ allowing the diagrams of restricted Lie algebras

$$\begin{array}{ccc} L_n & \xrightarrow{\xi_L} & L_{pn} \\ f_n \downarrow & & \downarrow f_{pn} \\ A_n & \xrightarrow{\xi} & A_{pn} \end{array}$$

to commute.

Definition 2.6. Given a restricted Lie algebra L , the *universal enveloping algebra* of L is an algebra $V(L)$ with structure map $L \rightarrow V(L)$ satisfying a universal property. $V(L) = U(L)/J$ where $J = (x^p - \xi(x))$ for $x \in L_n$. We can see that V is a functor from restricted Lie algebras over k to augmented algebras over k .

If A is a Hopf algebra over k where $\text{char}(k)=p$ then $P(A)$ is a restricted Lie algebra and P is an adjoint to V .

Theorem 2.7 (Milnor-Moore [2]). If $\text{char}(k)=p$, then there is an equivalence of categories

$$V : \text{resLie}/k \rightleftarrows p.g.Hopf/k : P$$

where resLie/k is the category of restricted Lie algebras over k and $p.g.Hopf/k$ is the category of primitively generated Hopf algebras over k .

3. HOPF ALGEBRAS AND LIE ALGEBRAS FROM CHROMATIC STABLE HOMOTOPY THEORY

Let us recall some important objects from chromatic stable homotopy theory. BP is the spectrum constructed as a retract of $MU_{(p)}$ with coefficients $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$.

The spectrum $K(n)$ known as Morava K-theory can be constructed from BP and it has coefficients $\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}]$. The ring $\pi_*(K(n))$ has a BP_* modules structure by sending v_i to 0 if $i \neq n$. We let $\Sigma(n)$ be the Hopf algebra $K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$. Then, there is a change of rings isomorphism by work of Ravenel and Miller.

Theorem 3.1 (Miller-Ravenel [3]). Let M be a BP_*BP -comodule annihilated by $I_n = (p, v_1, \dots, v_{n-1})$, and let $\bar{M} = M \otimes_{BP_*} K(n)_*$. Then there is a natural isomorphism

$$\text{Ext}_{BP_*BP}^*(BP_*, v_n^{-1}M) = \text{Ext}_{\Sigma(n)}^*(K(n)_*, \bar{M}),$$

where $v_n^{-1}M := v_n^{-1}BP_* \otimes_{BP_*} M$.

Now we let \mathbb{F}_p be a $K(n)_*$ module by sending v_n to 1 and define $S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p$. There is an equivalence of categories between $\Sigma(n)$ -modules and $S(n)$ -modules which gives us a correspondence in Ext.

Proposition 3.2. Let M be a $\Sigma(n)$ -module, then there is an isomorphism

$$\text{Ext}_{\Sigma(n)}^*(K(n)_*, M) \otimes_{K(n)_*} \mathbb{F}_p \cong \text{Ext}_{S(n)}^*(\mathbb{F}_p, \bar{M})$$

where $\bar{M} = M \otimes_{K(n)_*} \mathbb{F}_p$ (see [3]).

We will now restrict to the special case when $\bar{M} = \mathbb{F}_p$ to do computations. Following Ravenel we will denote $\text{Ext}_{S(n)}^*(\mathbb{F}_p, \mathbb{F}_p)$ by $H^*(S(n))$. There are spectral sequences which we can use to compute this Hopf algebra cohomology.

Theorem 3.3 (Lie-May spectral sequence). There is a spectral sequence

$$E_2 = H^{s,t}(E_0S(n)) \Rightarrow H^s(S(n))$$

with differential

$$d_r^{s,t} : E_r^{s,t} \longrightarrow E_r^{s+1, t-r},$$

where $E_0(S(n))$ is the associated graded of Ravenel's filtration given explicitly in the next paragraph.

We have another spectral sequence, which we call the Lie-May spectral sequence, that computes the input. First, let us define certain integers, which will be necessary for the filtration of $S(n)$ mentioned above.

Definition 3.4 (Ravenel [3]). Let $d_{n,i}$ be the integers defined by

$$d_{n,i} = \begin{cases} 0 & \text{if } i \leq 0 \\ \max(i, pd_{n,i-n}) & \text{for } i > 0. \end{cases}$$

Then we get a unique increasing filtration of $S(n)$ with $\text{deg}_i^{p^j} = d_{n,i}$ for $0 \leq j < n$.

Theorem 3.5 (May spectral sequence). There is a spectral sequence

$$E_2 = H^*(L(n)) \otimes_{\mathbb{F}_p} P(b_{i,j}) \Rightarrow H^*(E_0(S(n))),$$

where $b_{i,j} \in H^{2, pd_{n,i}}(E_0S(n))$ with internal degree $2p^{j+1}(p^i - 1)$ and $P(b_{i,j})$ is the polynomial algebra on these generators.

$L(n)$ is defined to be $P(E_0(S(n))^*)$ though, for the input, we think of it as a Lie algebra without restriction on the basis $x_{i,j}$ with bracket

$$[x_{i,j}, x_{k,\ell}] = \begin{cases} \delta_{i+j}^\ell x_{i+k,j} - \delta_{k+1}^j x_{i+k,\ell} & \text{for } i+k \leq m, \\ 0 & \text{otherwise} \end{cases}$$

where m is the integer floor of $pn/(p-1)$ and $\delta_i^s = 1$ iff $s = t \pmod{n}$ and zero otherwise.

Now, if we let $L(n, k) = L(n)/I_k$ where $I_k = (x_{i,j} | i > k)$, then the Lie-May spectral sequence can be refined further [3].

Theorem 3.6. The E_2 term of the spectral sequence of Theorem 3.5 can be reduced to

$$H^*(L(n, m)) \otimes P(b_{i,j} : i \leq m - n)$$

where m is the integer floor of $pn/(p - 1)$.

Proof. See Ravenel [3]. □

We have now reduced the computation to understanding $H^*(L(n, m))$, modulo the polynomial tensor factor. Now, we make use of the fact that $L(n, m)$ is a solvable Lie algebra and thus lends itself to a sequence of change of rings spectral sequences derived from the abelian quotients of its finite filtration.

Theorem 3.7. There are Hochschild-Serre spectral sequences

$$E_2 = E(h_{k,j}) \otimes H^*(L(n, k - 1)) \Rightarrow H^*(L(n, k))$$

that collapse at the E_3 page.

4. COMPUTATIONS AT THE HEIGHT 2

We will now restrict to computations at the height $n = 2$. We need to compute $H^*(L(2, j))$ for $j < 2p/(p - 1)$ starting with $j = 1$ and building up as far as we need. We know that $L(2, 1)$ is an abelian Lie algebra since $L(2, 1)$ has $\{x_{10}, x_{11}\}$ as a basis with trivial Lie bracket.

Proposition 4.1. If L is an abelian Lie algebra on two generators x_{10} and x_{11} , then

$$H^*(L) = E(h_{10}, h_{11})$$

and more generally if L is an abelian Lie algebra on n generators, then

$$H^*(L) = E(h_i | 1 \leq i \leq n).$$

Proof. We want to compute $\text{Ext}_{U(L)}^*(k, k)$ where $U(L) \cong k[x_{10}, x_{11}]$. We know that

$$\text{Ext}_{k[x_{10}, x_{11}]}^*(k, k) \cong \text{Ext}_{k[x_{10}]}^*(k, k) \otimes \text{Ext}_{k[x_{11}]}^*(k, k).$$

It suffices to compute $\text{Ext}_{k[x]}^*(k, k)$. An elementary calculation produces $\text{Ext}_{k[x]}^*(k, k) \cong E(h)$, thus $H^*(L) = E(h_{10}, h_{11})$. We have actually proved something stronger; if L is an abelian Lie algebra on n generators $\{x_i\}_{i=1}^n$, then $H^*(L) = E(h_i | 1 \leq i \leq n)$. □

4.1. Hochschild-Serre spectral sequence. Now we can compute the Hochschild-Serre spectral sequences to get $H^*(L(2, m))$. Since m depends on p it will be better to let p be a specific prime or set of primes at the outset. For example, the integer floor of $2p/(p - 1)$ is 2 if $p > 3$.

Proposition 4.2. Let $p > 3$. Then the desired cohomology $H^*(L(2, 2))$ for the input of the Lie-May spectral sequence is

$$H^*(L(2, 2)) = E(\zeta_2) \otimes \mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\}.$$

Proof. We use the Hochschild-Serre spectral sequence with E_2 page $E(h_{2,j} | j = 1, 2) \otimes H^*(L(2, 1))$ converging to $H^*(L(2, 2))$.

1	h_{20}	h_{21}	$h_{20}h_{21}$
h_{10}	$h_{10}h_{20}$	$h_{10}h_{21}$	$h_{10}h_{20}h_{21}$
h_{11}	$h_{11}h_{20}$	$h_{11}h_{21}$	$h_{11}h_{20}h_{21}$
$h_{10}h_{11}$	$h_{10}h_{11}h_{20}$	$h_{10}h_{11}h_{21}$	$h_{10}h_{11}h_{20}h_{21}$

The differentials in this spectral sequence come from the formula

$$d(h_{i,j}) = \sum_{k=1}^{i-1} h_{k,j} h_{i-k,j+k}.$$

We see that the generators that survive are the following:

$$\mathbb{F}_p \left\{ \begin{array}{l} 1, \quad h_{10}, \quad h_{11}, \quad h_{20} + h_{21}, \quad h_{10}h_{20}, \quad h_{11}h_{20}, \\ h_{10}h_{21}, \quad h_{11}h_{21}, \quad h_{10}h_{20}h_{21}, \quad h_{11}h_{20}h_{21}, \quad h_{10}h_{11}h_{20} + h_{10}h_{11}h_{21}, \quad h_{10}h_{11}h_{20}h_{21} \end{array} \right\}.$$

After a change of basis where $\zeta_2 = h_{20} + h_{21}$ and $\eta_2 = h_{20} - h_{21}$, we get our desired result. \square

Since at height 2 and primes $p > 3$ we have $m = n = 2$, there are no polynomial generators in $P(b_{i,j})$. This computation is therefore the full input for the Lie-May spectral sequence. At other primes we need to go further in the Hochschild-Serre spectral sequences. For example the integer floor of $2p/(p - 1)$ is 3 when $p = 3$ so we need to go out one more stage.

Proposition 4.3. The input for the Lie-May spectral sequence at the prime 3 is

$$H^*(L(2, 3)) \otimes_{\mathbb{F}_p} P(b_{10}, b_{11})$$

where $H^*(L(2, 3))$ is an exterior algebra $E(\xi_2)$ tensored with

$$\mathbb{F}_3 \left\{ \begin{array}{l} 1, \quad h_{10}, \quad h_{11}, \quad h_{10}h_{30}, \quad h_{11}h_{31}, \quad h_{10}h_{31} - h_{11}h_{30}, \\ h_{10}\eta_2h_{30}, \quad h_{11}\eta_2h_{31}, \quad h_{10}\eta_2h_{31} - h_{11}\eta_2h_{30}, \quad h_{10}\eta_2h_{30}h_{31}, \quad h_{11}\eta_2h_{30}h_{31}, \quad h_{10}h_{11}\eta_2h_{30}h_{31} \end{array} \right\}.$$

Proof. We will make use of our work from the Prop. 4.2 since that computation is the same as our first step at the prime 3. Since we have an exterior algebra as a tensor factor, the exterior part splits off in the spectral sequence. This produces the following Hochschild-Serre spectral sequence for the remaining tensor factor:

1	h_{30}	h_{31}	$h_{30}h_{31}$
h_{10}	$h_{10}h_{30}$	$h_{10}h_{31}$	$h_{10}h_{30}h_{31}$
h_{11}	$h_{11}h_{30}$	$h_{11}h_{31}$	$h_{11}h_{30}h_{31}$
$h_{10}\eta_2$	$h_{10}\eta_2h_{30}$	$h_{10}\eta_2h_{31}$	$h_{10}\eta_2h_{30}h_{31}$
$h_{11}\eta_2$	$h_{11}\eta_2h_{30}$	$h_{11}\eta_2h_{31}$	$h_{11}\eta_2h_{30}h_{31}$
$h_{10}h_{11}\eta_2$	$h_{10}h_{11}\eta_2h_{30}$	$h_{10}h_{11}\eta_2h_{31}$	$h_{10}h_{11}\eta_2h_{30}h_{31}$

The result follows from observing what survives the spectral sequence. \square

At the prime 2 we need to work harder at the outset because the integer floor of $2p/(p-1)$ is 4 for $p = 2$.

Proposition 4.4. The input for the Lie-May spectral sequence at the prime 2 is $H^*(L(2, 4))$ tensored $P(b_{10}, b_{11}, b_{20}, b_{21})$ where $H^*(L(2, 4))$ is given in Figure 2

Proof. For this computation we need to start from the first Hochschild-Serre spectral sequence since the change of basis is not valid in characteristic 2. We will still have that the following classes survive the first spectral sequence

$$\mathbb{F}_2\{1, h_{10}, h_{11}, h_{20}+h_{21}, h_{10}h_{20}, h_{10}h_{21}, h_{11}h_{20}, h_{11}h_{21}, h_{10}h_{20}h_{21}, h_{11}h_{20}h_{21}, h_{10}h_{11}(h_{20}+h_{21}), h_{10}h_{11}h_{20}h_{21}\}.$$

After a change of basis letting $\zeta_2 = h_{20} + h_{21}$ we get

$$\mathbb{F}\{1, h_{10}, h_{11}, \zeta_2, h_{10}h_{20}, h_{11}h_{20}, h_{10}\zeta_2, h_{11}\zeta_2, h_{10}h_{20}\zeta_2, h_{11}h_{20}\zeta_2, h_{10}h_{11}\zeta_2, h_{10}h_{11}h_{20}\zeta_2\}.$$

We get spectral sequence in Figure 1. We observe that the abutment is the following,

$$\mathbb{F}_2 \left\{ \begin{array}{cccccc} 1, & h_{10}, & h_{11}, & \zeta_2, & h_{10}h_{20}, & h_{11}h_{20}, \\ h_{10}h_{30}, & h_{11}h_{30}, & h_{10}h_{31}, & h_{11}h_{31}, & \zeta_2h_{30}, & \zeta_2h_{31}, \\ \zeta_2h_{30}h_{31}, & h_{10}\zeta_2h_{31}, & h_{11}\zeta_2h_{31}, & h_{10}\zeta_2h_{31}, & h_{10}h_{20}h_{30}, & h_{11}h_{20}h_{31}, \\ h_{10}h_{20}h_{30} + h_{10}h_{20}h_{31}, & h_{10}h_{20}\zeta_2, & h_{11}h_{20}\zeta_2, & h_{10}h_{11}h_{20}, & h_{10}h_{20}\zeta_2h_{30}, & h_{11}h_{20}\zeta_2h_{31}, \\ h_{10}h_{11}\zeta_2h_{30}h_{31}, & h_{10}h_{11}h_{20}h_{30}, & h_{10}h_{11}h_{20}h_{31}, & h_{10}h_{11}h_{20}, & h_{11}h_{30}h_{31}, & h_{10}h_{11}h_{20}h_{30}, \\ h_{10}h_{11}h_{20}h_{31}, & h_{10}h_{20}\zeta_2h_{31}, & h_{11}h_{20}\zeta_2h_{31}, & h_{10}\zeta_2h_{30}h_{31}, & h_{10}h_{20}\zeta_2h_{30}h_{31}, & h_{11}h_{20}\zeta_2h_{30}h_{31} \end{array} \right\}$$

We have computed $H^*(L(2, 3))$ at the prime 2, but we need to do one more Hochschild-Serre spectral sequence in order to get our desired input. We will omit the spectral sequence for $H^*(L(2, 4))$ from this document because the size is too large for the page. What remains in the abutment can be seen in Figure 2. In Figure 2, we let $\zeta_4 = h_{40} + h_{41}$. At the prime 2 there will be a nontrivial tensor factor $P(b_{i,j})$ for $i = 1, 2, j = 0, 1$ in the input of Lie-May spectral sequence. This follows by definition of the spectral sequence at the prime 2. Ultimately, this will increase the complexity of the Lie-May spectral sequence significantly. \square

1	h_{30}	h_{31}	$h_{30}h_{31}$
h_{10}	$h_{10}h_{30}$	$h_{10}h_{31}$	$h_{10}h_{30}h_{31}$
h_{11}	$h_{11}h_{30}$	$h_{11}h_{31}$	$h_{11}h_{30}h_{31}$
ζ_2	ζ_2h_{30}	ζ_2h_{31}	$\zeta_2h_{30}h_{31}$
$h_{10}\zeta_2$	$h_{10}\zeta_2h_{30}$	$h_{10}\zeta_2h_{31}$	$h_{10}\zeta_2h_{30}h_{31}$
$h_{11}\zeta_2$	$h_{11}\zeta_2h_{30}$	$h_{11}\zeta_2h_{31}$	$h_{11}\zeta_2h_{30}h_{31}$
$h_{10}h_{20}$	$h_{10}h_{20}h_{30}$	$h_{10}h_{20}h_{31}$	$h_{10}h_{20}h_{30}h_{31}$
$h_{11}h_{20}$	$h_{11}h_{20}h_{30}$	$h_{11}h_{20}h_{31}$	$h_{11}h_{20}h_{30}h_{31}$
$h_{10}h_{20}\zeta_2$	$h_{10}h_{20}\zeta_2h_{30}$	$h_{10}h_{20}\zeta_2h_{31}$	$h_{10}h_{20}\zeta_2h_{30}h_{31}$
$h_{11}h_{20}\zeta_2$	$h_{11}h_{20}\zeta_2h_{30}$	$h_{11}h_{20}\zeta_2h_{31}$	$h_{11}h_{20}\zeta_2h_{30}h_{31}$
$h_{10}h_{11}h_{20}$	$h_{10}h_{11}h_{20}h_{30}$	$h_{10}h_{11}h_{20}h_{31}$	$h_{10}h_{11}h_{20}h_{30}h_{31}$
$h_{10}h_{11}h_{20}\zeta_2$	$h_{10}h_{11}h_{20}\zeta_2$	$h_{10}h_{11}h_{20}\zeta_2h_{30}$	$h_{10}h_{11}h_{20}\zeta_2h_{30}h_{31}$

$$(2) \quad \mathbb{F}_2 \left\{ \begin{array}{ccccc} 1, & h_{10}, & h_{11}, & \zeta_2, & \zeta_4, \\ h_{10}h_{20}, & h_{11}h_{20}, & h_{10}h_{30}, & h_{11}h_{31}, & h_{10}h_{31}, \\ \zeta_2h_{30}, & \zeta_2h_{31}, & h_{10}h_{40}, & h_{10}h_{41}, & h_{11}h_{41}, \\ \zeta_2h_{40}, & \zeta_2h_{41}, & h_{10}h_{11}h_{20}, & h_{10}h_{20}\zeta_2, & h_{11}h_{20}\zeta_2, \\ h_{10}h_{20}h_{30}, & h_{10}\zeta_2h_{30}, & h_{10}\zeta_2h_{31}, & h_{10}h_{20}h_{31} + h_{11}h_{20}h_{30}, & h_{11}h_{20}h_{31}, \\ h_{11}\zeta_2h_{31}, & h_{10}h_{30}h_{31}, & h_{11}h_{30}h_{31}, & \zeta_2h_{30}h_{31}, & h_{10}h_{20}\zeta_4, \\ h_{11}h_{20}\zeta_4, & h_{10}h_{30}h_{40}, & h_{10}h_{30}h_{41}, & h_{10}h_{31}h_{40}, & h_{10}h_{31}h_{41}, \\ h_{11}h_{30}h_{40}, & h_{11}h_{31}h_{40}, & h_{11}h_{31}h_{41}, & \zeta_2h_{30}\zeta_4, & \zeta_2h_{31}\zeta_4, \\ h_{10}h_{40}h_{41}, & h_{11}h_{40}h_{41}, & h_{10}h_{20}\zeta_2h_{30}, & h_{11}h_{20}\zeta_2h_{30}, & h_{10}h_{20}\zeta_2h_{31}, \\ h_{11}h_{20}\zeta_2h_{31}, & h_{10}h_{11}h_{20}h_{40}, & h_{10}h_{11}h_{20}h_{41}, & h_{10}\zeta_2h_{31}h_{40}, & h_{10}\zeta_2h_{31}h_{41}, \\ h_{10}\zeta_2h_{30}h_{41}, & h_{10}\zeta_2h_{30}h_{40}, & h_{10}h_{20}h_{30}h_{41}, & h_{10}h_{20}h_{30}h_{40}, & h_{10}h_{20}\zeta_2h_{40}, \\ h_{10}h_{20}\zeta_2h_{41}, & h_{11}h_{20}\zeta_2h_{40}, & h_{11}h_{20}\zeta_2h_{41}, & h_{11}h_{20}h_{31}h_{40}, & h_{11}h_{20}h_{31}h_{41}, \\ h_{11}\zeta_2h_{31}h_{40}, & h_{11}\zeta_2h_{31}h_{41}, & h_{10}h_{11}h_{20}h_{31}, & h_{10}h_{11}h_{20}h_{30}, & \zeta_2h_{31}h_{40}h_{41}, \\ \zeta_2h_{30}h_{40}h_{41}, & h_{10}h_{20}h_{30}h_{40}, & h_{10}h_{20}h_{30}h_{41}, & h_{10}\zeta_2h_{30}h_{40}, & h_{10}\zeta_2h_{30}h_{41}, \\ h_{10}h_{30}h_{31}h_{40}, & h_{10}h_{30}h_{31}h_{41}, & h_{11}h_{30}h_{31}h_{40}, & h_{11}h_{30}h_{31}h_{41}, & h_{11}\zeta_2h_{31}h_{41}, \\ h_{11}\zeta_2h_{31}h_{40}, & h_{11}h_{20}h_{31}h_{41}, & h_{11}h_{20}h_{31}h_{40}, & h_{10}h_{20}h_{31}h_{40} + \\ & & & h_{11}h_{20}h_{30}h_{40}, & h_{11}h_{20}h_{30}h_{41}, \\ & & & h_{11}h_{30}h_{40}h_{41}, & h_{10}h_{31}h_{40}h_{41}, \\ & & & h_{10}h_{11}h_{20}h_{30}h_{40}, & h_{10}h_{11}h_{20}h_{31}h_{41}, \\ & & & h_{11}h_{20}\zeta_2h_{30}h_{40}, & h_{11}h_{20}\zeta_2h_{30}h_{41}, \\ & & & h_{11}\zeta_2h_{30}h_{31}h_{41}, & h_{10}h_{11}h_{20}h_{40}h_{41}, \\ & & & h_{10}\zeta_2h_{30}h_{40}h_{41}, & h_{10}\zeta_2h_{31}h_{40}h_{41}, \\ & & & h_{10}h_{30}h_{31}h_{40}h_{41}, & h_{11}h_{30}h_{31}h_{40}h_{41}, \\ \zeta_2h_{30}h_{31}h_{40}h_{41}, & \zeta_2h_{30}h_{31}h_{41}, & h_{10}h_{30}h_{40}h_{41}, & h_{11}h_{30}h_{40}h_{41}, & h_{10}h_{31}h_{40}h_{41}, \\ h_{11}h_{31}h_{40}h_{41}, & h_{10}h_{20}\zeta_2h_{30}h_{31}, & h_{11}h_{20}\zeta_2h_{30}h_{31}, & h_{10}h_{11}h_{20}h_{30}h_{40}, & h_{10}h_{11}h_{20}h_{31}h_{41}, \\ h_{10}h_{20}\zeta_2h_{30}h_{40}, & h_{10}h_{20}\zeta_2h_{30}h_{41}, & h_{10}h_{20}\zeta_2h_{31}h_{41}, & h_{11}h_{20}\zeta_2h_{30}h_{40}, & h_{11}h_{20}\zeta_2h_{30}h_{41}, \\ h_{11}h_{20}\zeta_2h_{31}h_{40}, & h_{11}h_{20}\zeta_2h_{31}h_{41}, & h_{10}\zeta_2h_{30}h_{31}h_{41}, & h_{11}\zeta_2h_{30}h_{31}h_{41}, & h_{10}h_{11}h_{20}h_{40}h_{41}, \\ h_{10}h_{20}\zeta_2h_{40}h_{41}, & h_{11}h_{20}\zeta_2h_{40}h_{41}, & h_{10}h_{20}h_{30}h_{40}h_{41}, & h_{10}\zeta_2h_{30}h_{40}h_{41}, & h_{10}\zeta_2h_{31}h_{40}h_{41}, \\ h_{10}h_{20}h_{31}h_{40}h_{41} + & h_{11}\zeta_2h_{31}h_{40}h_{41}, & h_{11}\zeta_2h_{31}h_{40}h_{41}, & h_{10}h_{30}h_{31}h_{40}h_{41}, & h_{11}h_{30}h_{31}h_{40}h_{41}, \\ h_{11}h_{20}h_{30}h_{40}h_{41}, & & & & & \\ \zeta_2h_{30}h_{31}h_{40}h_{41}, & h_{10}h_{11}h_{20}h_{30}h_{31}h_{40}, & h_{10}h_{11}h_{20}h_{30}h_{31}h_{41}, & h_{10}h_{20}\zeta_2h_{30}h_{31}h_{40}, & h_{10}h_{20}\zeta_2h_{30}h_{31}h_{41}, \\ h_{11}h_{20}\zeta_2h_{30}h_{31}h_{40}, & h_{11}h_{20}\zeta_2h_{30}h_{31}h_{41}, & h_{10}h_{11}h_{20}h_{30}h_{40}h_{41}, & h_{10}h_{11}h_{20}h_{31}h_{40}h_{41}, & h_{10}h_{20}\zeta_2h_{31}h_{40}h_{41} + \\ & & & & h_{11}h_{20}\zeta_2h_{30}h_{40}h_{41}, \\ & & & & h_{10}h_{11}h_{20}h_{30}h_{31}h_{40}h_{41}, \\ & & & & h_{10}h_{20}\zeta_2h_{30}h_{40}h_{41}, & h_{11}h_{20}\zeta_2h_{30}h_{31}h_{40}h_{41}, \\ & & & & h_{10}h_{11}h_{20}\zeta_2h_{30}h_{31}h_{40}h_{41}, & h_{10}h_{11}h_{20}\zeta_2h_{30}h_{31}h_{40}h_{41}, \\ & & & & h_{10}h_{11}h_{20}\zeta_2h_{30}h_{31}h_{40}h_{41}, & h_{10}h_{11}h_{20}\zeta_2h_{30}h_{31}h_{40}h_{41}, \end{array} \right.$$

4.2. Lie-May spectral sequence. We will now use the results from the previous section to compute $H^*(E_0(S(2)))$ using the Lie-May spectral sequence at certain primes.

Proposition 4.5. At the primes $p > 3$,

$$H^*(L(2, 2)) \cong H^*(E_0(S(2))).$$

Proof. The nontrivial differentials in the Lie-May spectral sequence hit the polynomial generators and at primes $p > 3$ there are no polynomial generators to hit. Thus, the spectral sequence collapses at the E_2 page. \square

We will now consider the case when $p = 3$. This computation remains unfinished, but we present the beginning as a way of showing where the differentials come from. Recall that the Lie-May spectral sequence has input $H^*(L(n, m)) \otimes P(b_{i,j})$ and converges to $H^*(E_0(S(n)))$ and the computation for $p = 3$ can be found in Proposition 4.3. Heuristically, we want to know how far $d(h_{i,j})$, the differential in the DGA with cohomology $H^*(L(n, m))$, is from $\Delta(t_{i,j})$ in the Hopf algebra $E_0(S(n))$. Our formula for $\Delta(t_{i,j})$ from [3] is

$$\Delta(t_{m,j}) = \sum_{k=0}^m t_{k,j} \otimes t_{m-k,k+j} + \bar{b}_{m-n,j+n-1},$$

where $t_{0,s} = 1$ by convention. For example, we should get a differential in the Lie-May spectral sequence $d(h_{30}) = b_{11}$ since $d(h_{30}) = h_{10}h_{21} + h_{20}h_{10}$ in the DGA computing $H^*(L(2, 3))$ and $\Delta(t_{30}) = t_{10} \otimes t_{21} + t_{20}t_{10} + b_{11}$ where h_{ij} is represented by t_{ij} . Now the issue is that h_{30} does not exist as an individual element in $H^*(L(2, 3))$. What we can do is

to represent b_{11} as a cocycle in the cobar complex. The explicit formula from [3] is

$$b_{1,j} = -\frac{1}{p} \sum_{i=1}^{p^{j+1}-1} \binom{p^{j+1}}{i} t_1^i \otimes t_1^{p^{j+1}-i}.$$

For example, at the prime 3,

$$b_{10} = -t_1 \otimes t_1^2 - t_1^2 \otimes t_1$$

and $b_{11} = -3t_1 \otimes t_1^8 - 12t_1^2 \otimes t_1^7 - 28t_1^3 \otimes t_1^6 - 3t_1^8 \otimes t_1$ so mod 3

$$b_{11} = -t_1^3 \otimes t_1^6 - t_1^6 \otimes t_1^3.$$

This gives us a representative for b_{ij} in the cobar complex for $E_0(S(n))$. Now we need to compute differentials on the elements with an h_{30} or h_{31} factor since these will hit the b_{11} and b_{10} elements. For example, $d(h_{10}h_{30})$ can be computed by computing the differential in the cobar complex on the representative of $h_{10}h_{30}$

$$\begin{aligned} d(t_{10} \otimes t_{30}) &= 1 \otimes t_{10} \otimes t_{30} - \Delta(t_{10}) \otimes t_{30} + t_{10} \otimes \Delta(t_{30}) - t_{10} \otimes t_{30} \otimes 1 \\ &= t_{10} \otimes t_{10} \otimes t_{21} + t_{10} \otimes t_{20} \otimes t_{10} - t_{10} \otimes t_{11}^2 \otimes t_{11} - t_{10} \otimes t_{11} \otimes t_{11}^2. \end{aligned}$$

Since $d(h_{10}h_{30}) = h_{10}h_{10}h_{21} + h_{10}h_{20}h_{10}$ and this is represented by $t_{10} \otimes t_{10} \otimes t_{21} + t_{10} \otimes t_{20} \otimes t_{10}$, the differential in the Lie-May spectral sequence is $d(h_{10}h_{30}) = h_{10}b_{11}$. A similar computation gives that $d(h_{11}h_{31}) = h_{11}b_{10}$ and $d(h_{10}h_{31} - h_{11}h_{30}) = h_{10}b_{10} - h_{11}b_{11}$.

We would now like to compute $d(h_{10}\eta_2h_{30})$ because by Poincare duality we could produce the rest of the differentials from knowing this one. We represent $h_{10}\eta_2h_{30}$ by $t_{10} \otimes t_{20} \otimes t_{30} - t_{10} \otimes t_{21} \otimes t_{30}$ and compute the following,

$$\begin{aligned} d(t_{10} \otimes t_{20} \otimes t_{30} - t_{10} \otimes t_{21} \otimes t_{30}) &= t_{10} \otimes t_{10} \otimes t_{11} \otimes t_{30} - t_{10} \otimes t_{20} \otimes t_{20} \otimes t_{10} - t_{10} \otimes t_{20} \otimes t_{10} \otimes t_{21} \\ &\quad + t_{10} \otimes t_{20} \otimes t_{11}^2 \otimes t_{11} + t_{10} \otimes t_{20} \otimes t_{11} \otimes t_{11}^2 - t_{10} \otimes t_{11} \otimes t_{10} \otimes t_{30} \\ &\quad + t_{10} \otimes t_{21} \otimes t_{20} \otimes t_{10} + t_{10} \otimes t_{21} \otimes t_{10} \otimes t_{21} - t_{10} \otimes t_{21} \otimes t_{11}^2 \otimes t_{11} \\ &\quad - t_{10} \otimes t_{21} \otimes t_{11} \otimes t_{11}^2. \end{aligned}$$

Now the goal would be to add coboundaries to our class to reduce the output of this differential as much as possible and whatever remains would be in the image of our differential. We leave this part of the computation unfinished until we have developed a method for computing these differentials using a computer algorithm.

4.3. May spectral sequence. We will now compute the May spectral sequence from Theorem 3.5 at primes $p > 3$ and height $n = 2$. First, it will be useful to record each class and its cohomological, Ravenel-May, and topological degrees. Recall from Proposition 4.2 that

$$H^*(L(2, 2)) = E(\zeta_2) \otimes \mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\}.$$

We consider topological degree mod $2p^2 - 2$ since we will be adjoining $v_2^{\pm 1}$ which has degree $2p^2 - 2$.

(3)

class	s (coh. deg.)	t (Rav.-May deg.)	$u \bmod 2p^2 - 2$ (top. deg.)	$u - s$
1	0	0	0	0
h_{10}	1	1	$2p - 2$	$2p - 3$
h_{11}	1	1	$-2p + 2$	$-2p + 1$
$h_{10}\eta_2$	2	3	$2p - 2$	$2p - 4$
$h_{11}\eta_2$	2	3	$-2p + 2$	$-2p$
$h_{10}h_{11}\eta_2$	3	4	0	-3
ζ_2	1	2	0	-1
$\zeta_2 h_{10}$	2	3	$2p - 2$	$2p - 4$
$\zeta_2 h_{11}$	2	3	$-2p + 2$	$-2p$
$\zeta_2 h_{10}\eta_2$	3	5	$2p - 2$	$2p - 5$
$\zeta_2 h_{11}\eta_2$	3	5	$-2p + 2$	$-2p - 1$
$\zeta_2 h_{10}h_{11}\eta_2$	4	6	0	-4

We will refer to this table for the following Proposition as well as the computation of the Adams-Novikov spectral sequence in the subsequent section.

Proposition 4.6. At primes $p > 3$ and height $n = 2$ the abutment of the May spectral sequence is

$$H^*(S(2)) \cong E(\zeta_2) \otimes \mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\}.$$

Proof. We simply note that the differentials in this spectral sequence

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+t,t-r}$$

must be trivial for degree reasons insuring that the spectral sequence collapses at the E_2 page. \square

4.4. Adams-Novikov spectral sequence and $K(n)$ -local homotopy groups of a Smith-Toda complex. We have computed $H^*(S(2))$ and now we would like to recall why algebraic topologists care about this. In Section 3 Proposition 3.2, we discussed the isomorphism

$$H^*(S(2))[v_2^{\pm 1}] = \text{Ext}_{S(2)}^*(\mathbb{F}_p, \mathbb{F}_p)[v_2^{\pm 1}] \cong \text{Ext}_{\Sigma(2)}^*(K(2)_*, K(2)_*)$$

where $|v_2| = 2p^2 - 2$. Then the change of rings isomorphism of Miller and Ravenel in Theorem 3.1 gives

$$\text{Ext}_{\Sigma(2)}^*(K(2)_*, K(2)_*) \cong \text{Ext}_{BP_*, BP}^*(BP_*, v_2^{-1}BP_*/I_2).$$

It remains to show that the right hand side is the input for an Adams-Novikov spectral sequence for some spectrum X . Since we are at height 2 and primes $p > 3$, the Smith-Toda complex

$$V(1) = \text{cof}(\Sigma^{2p-2}S/p \longrightarrow S/p)$$

is known to exist; i.e. the multiplication by v_1 map on S/p can be realized geometrically. We have a map $v_2 : \Sigma^{2p^2-2}V(1) \longrightarrow V(1)$, which is known to be periodic so we can realize a spectrum as the telescope

$$v_2^{-1}V(1) = \underset{\longrightarrow}{\text{hocolim}}(V(1) \longrightarrow \Sigma^{-(2p^2-2)}V(1) \longrightarrow \dots).$$

The telescope conjecture says that this telescope homotopy equivalent to $L_{K(2)}V(1)$. This conjecture is thought to be false in general, but perhaps it is true in this particular case. To be safe we will refer to the spectrum $L_{K(2)}V(1)$ since it is known that there is an isomorphism

$$BP_*(L_{K(2)}V(1)) \cong v_2^{-1}BP_*/I_2.$$

Theorem 4.7. At primes $p > 3$, let $V(1)$ be a Smith-Toda complex as defined above, then

$$\pi_*(L_{K(2)}V(1)) \cong E(\zeta_2) \otimes \mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\} \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_2^{\pm 1}]$$

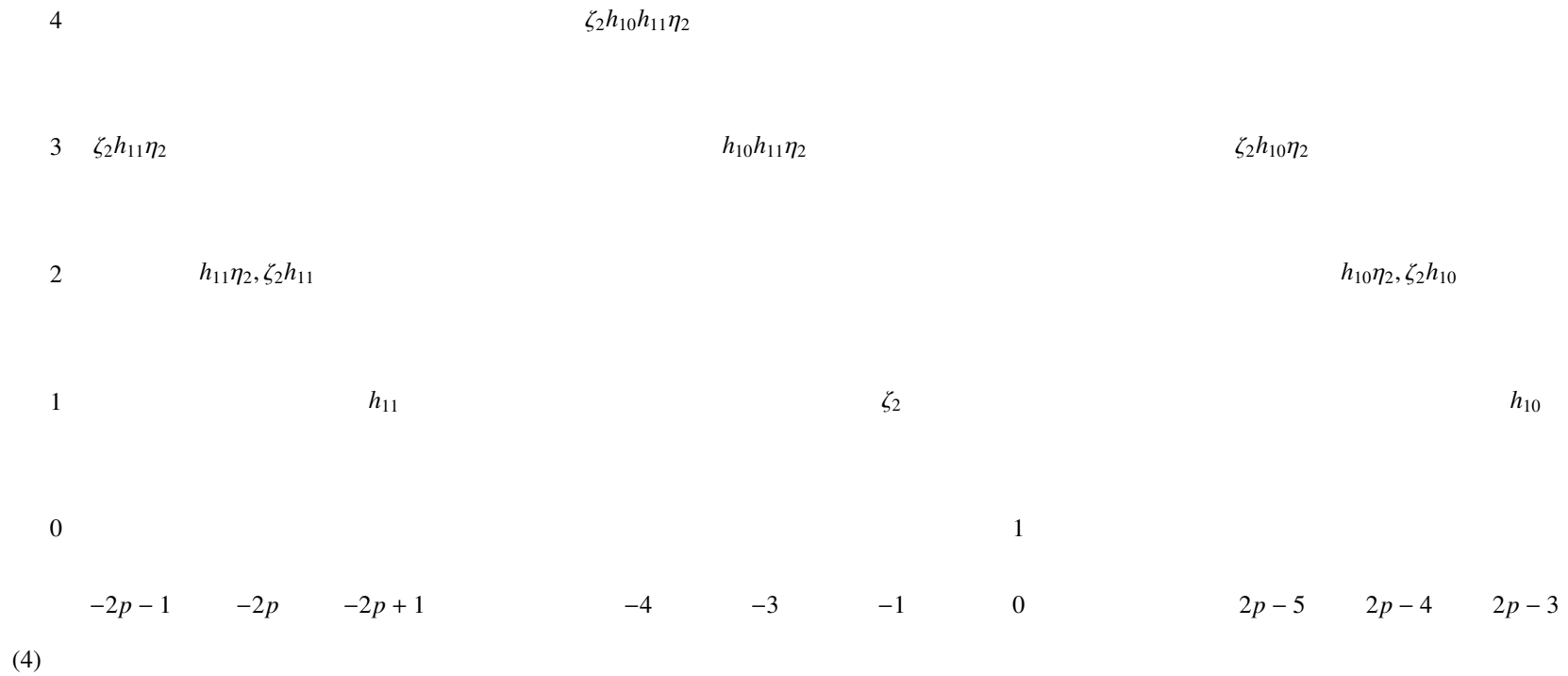
where degrees are given in Table 3.

Proof. By the remarks above, we need only compute the Adams-Novikov spectral sequence with E_2 page $\text{Ext}_{BP, BP}^{s, u}(BP_*, v_2^{-1}BP_*/I_2)$ given by

$$E(\zeta_2) \otimes_{\mathbb{F}_p} \{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\} \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_2^{\pm 1}].$$

We will see that for degree reasons the spectral sequence collapses. The differentials in this spectral sequence follow the Adams convention they move one place to the left and up r on the E_r page. We are beginning on the E_2 page so the shortest differential goes to the left one and up two places. Observe that in the spectral sequence of Figure 4 no elements lie in the column to the right of an element and two more more rows above the element. Thus, the spectral sequence collapses and we have proven our claim. \square

On the following page in Figure 4 we depict the Adams-Novikov spectral sequence used to compute the homotopy groups of $L_{K(2)}V(1)$. The spectral sequence is drawn with the Adams convention of $t - s$ on the horizontal axis and s on the vertical axis. The spectral sequence repeats every $2p^2 - 2$ on the horizontal axis because the input is $H^*(S(2))$ adjoin a periodic element $v_2^{\pm 1}$ in degree $2p^2 - 2$. The picture reveals a collapse of the spectral sequence as acknowledged in the proof above.



5. FUTURE RESEARCH

We suggest two projects to extend these results.

First, we observe that these computations used trivial coefficients. To understand computations of this nature with nontrivial coefficients we would need to incorporate a group action. Nontrivial coefficients would allow us to bypass the Bockstein spectral sequences needed to compute the input for the chromatic spectral sequence for the sphere spectrum. We could therefore compute the input for the chromatic spectral sequence right away leading to computations of homotopy groups of more complicated spectra than Smith-Toda complexes.

Second, it became evident quite quickly that for certain primes and heights it would be necessary to write computer programs to compute differentials in the Lie-May spectral sequence. Algorithms for computing such differentials could be useful for other types of differential computations as well and would be interesting in their own right. To push the known computations to higher heights and certain primes we would need this technology. These two projects are not mutually exclusive and in fact we propose to use both of them to produce new results.

REFERENCES

- [1] J. Peter May. The cohomology of restricted Lie algebras and of Hopf algebras. *Bull. Amer. Math. Soc.*, 71:372–377, 1965.
- [2] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.
- [3] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.