4.6 (Part A) Exponential and Logarithmic Equations

In this section you will learn to:

- solve exponential equations using like bases
- solve exponential equations using logarithms
- solve logarithmic equations using the definition of a logarithm
- solve logarithmic equations using 1-to-1 properties of logarithms
- apply logarithmic and exponential equations to real-world problems
- convert $y = ab^x$ to an exponential equation using base e

Definition of a Logarithm	$y = \log_b x$ is equivalent to $b^y = x$
Inverse Properties	$\log_b b^x = x$ $\quad$ $b^{\log_b x} = x$
Log Properties Involving One	$\log_b b = 1$ $\quad$ $\log_b 1 = 0$
Product Rule	$\log_b (MN) = \log_b M + \log_b N$
Quotient Rule	$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$
Power Rule	$\log_b M^p = p \log_b M$
One-to-One Properties	If $b^M = b^N$ then $M = N$.
	If $\log_b M = \log_b N$ then $M = N$.
	If $M = N$ then $\log_b M = \log_b N$.

Example 1: Solve each equation by expressing each side as a power of the **same base**.

(a) $5^{x+1} = 25^{x-3}$

(b) $9^{2x} = \frac{1}{\sqrt[5]{3}}$

(c) $e^2 e^x = \frac{e^6}{e^x}$

<u>Steps for solving EXPONENTIAL EQUATIONS</u>:
(Examples 2 – 6)

1. Isolate the exponential "factor".

2. Take the common/natural log of both sides.

3. Simplify (Recall: $\ln b^x = x \ln b$; $\ln e^x = x$)

4. Solve for the variable.

5. Check your answer.

Example 2: Solve $5e^{2x} = 60$

Example 3: Solve $3^x = 30$ using (a) common logarithms, (b) natural logarithms, and (c) the definition of a logarithm.

Example 4: Solve $10^x + 3 = 835$

Example 5: Solve $5^{x-2} = 50$

Example 6: Solve $2^{x+2} = 3^{x-1}$

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Example 7: Use **FACTORING** to solve each of the following equations. (Hint: Use substitution or short-cut method learned in Section 1.6.)

(a) $e^{2x} - 2e^x - 3 = 0$

(b) $3^{2x} - 4 \cdot 3^x - 12 = 0$

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<u>Steps for solving LOGARITHMIC EQUATIONS</u>: (Examples 8 – 11)

Example 8: Solve $\log_4(x+3)=2$

1. Write as a single logarithm. ($\log_b M = c$)

2. Change to exponential form. ($b^c = M$)

3. Solve for the variable.

4. Check your answer.

Example 9: Solve $\log_2 x + \log_2(x+7) = 3$

Example 10: Solve $3\ln 2x = 12$

Example 11: Solve $\log_2(x+2) - \log_2(x-5) = 1$

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<u>Steps for solving equations using 1-to-1 properties:</u> (Examples 12 – 14)

Example 12: $\log(x+7) - \log 3 = \log(7x+1)$

1. Write the equation in $\log_b M = \log_b N$ form.

2. Use 1-to-1 property. (Write without logarithms.)

3. Solve for the variable.

4. Check your answer.

Example 13: $2\log x - \log 7 = \log 112$

Example 14: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

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Periodic Interest Formula	Continuous Interest Formula
$A = P\left(1+\frac{r}{n}\right)^{nt}$	$A = Pe^{rt}$

Example 15: How long will it take $25,000 to grow to $500,000 at 9% interest compounded continuously?

Example 16: How long will it take $25,000 to grow to $500,000 at 9% interest compounded quarterly?

Example 17: What interest rate is needed for $25,000 to double after 8 years if compounded continuously? (Round rate to nearest hundredth of a percent.)

4.6 (Part B) Exponential Growth and Decay

Decay Model	$A = Pe^{rt}$ $f(t) = A_0 e^{kt}$ $A = A_0 e^{kt}$ $A_0 =$ $A =$ $k =$ $t =$	Growth Model

Example 18: In 2001 the world population was approximately 6.2 billion. If the annual growth rate averaged about 1.3% per year, write an exponential equation that models this situation. Use your model to estimate the population for this year.

Example 19: An account has a continuous interest rate of *k*.

(a) How long will it take your money to double if compounded continuously?

(b) How long will it take it to triple?

(c) At 3% interest, how long will it take an investment to double? Triple?

(d) What interest rate is needed for an investment to double after 5 years?

Steps for finding growth/decay model (when growth or decay rate is not given:

1. Use $A = A_0 e^{kt}$ to find k. (Estimate k to 6 decimal places.)

2. Substitute k into growth/decay model: $A = A_0 e^{kt}$

Example 20: The **minimum wage** in 1970 was $1.60. In 2000 it was $5.15.

(a) Find a growth model for this situation.

(b) Estimate the minimum wage for this year.

(c) Estimate the minimum wage 10 years from now.

(d) Based on this model, when will the minimum wage be $10.00?

Example 21: Carbon-14 testing is used to determine the age of fossils, artifacts and paintings. Carbon-14 has a half-life of 5715 years.

(a) Find an exponential decay model for Carbon-14.

(b) A painting was discovered containing 96% of its original Carbon-14. Estimate the age of the painting.

(c) An art collector plans to purchase a painting by Leonardo DaVinci for a considerable amount of money. (DaVinci lived from 1452-1519). Could the painting in part (b) possibly be one of DaVinci's works?

Example 22 (optional): Arsenic-74 has a half-life of 17.5 days.

(a) Find an exponential decay model for this situation.

(b) How long will it take for 5% of the original amount of arsenic-74 to remain in a blood system? (Round to nearest day.)

(c) What is the decay rate (per day) to the nearest tenth of a percent.

Expressing an Exponential Model in Base *e*: $y = ab^x$ is equivalent to $y = ae^{(\ln b)x}$ or $y = ae^{x \ln b}$

Example 23: Rewrite each equation in terms of base *e*.

(a) $y = 68(2)^x$ (b) $y = 5000(.5)^x$ (c) $y = 2.7(0.25)^x$ (d) $y = 25(3.5)^x$

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4.6 (Part C) Linear vs Exponential Functions

Example 24: Which of the following tables could represent linear functions? Which could represent exponential functions? If linear or exponential, determine the equation for the function.

(a)

x	0	1	2	3	4	5
$f(x)$	36	30	24	18	12	6

(b)

x	0	1	2	3	4	5
$g(x)$	32	16	8	4	2	1

(c)

t	2	4	6	8	10
$Q(t)$	3	9	15	21	27

(d)

q	0	1	2	3	4	5
$h(q)$	4	1	0	1	4	9

(e)

x	-2	-1	0	1	2
y	32	48	72	108	162

Example 25: Spartanville had a population of 28,000 in 2007. Give a formula for its population, P, in terms of t years after 2007 if P decreases by

(a) 2.75% per year.

(b) 750 people per year.

Example 26: The following functions give the populations of several towns with time t in years.

A. $P(t) = 12,000(1.05)^t$

B. $P(t) = 6000(1.035)^t$

C. $P(t) = 15,000(.94)^t$

D. $P(t) = 1200 + 300t$

E. $P(t) = 900(1.15)^t$

F. $P(t) = 5000 - 25t$

(a) Which town had a constant population decrease? Find the **average rate of change** per year.

(b) Which town has the fastest annual growth rate? What is the annual growth rate?

(c) Which towns were decreasing in size?

(d) What is the annual rate of decrease (decay) for Town C? At this rate how many people will be left after 150 years?

(e) Which town had the smallest initial population?

4.6 Homework Problems

Solve each equation by expressing each side as a power of the same base.

1. $3^{x+3} = 9^{x-2}$
2. $5^{2x} = \dfrac{1}{\sqrt[3]{5}}$
3. $e^x e = \dfrac{e^7}{e^x}$
4. $\left(\dfrac{1}{3}\right)^{-3x-2} = 9^{x+1}$

Solve the exponential equations. Round answers to two decimal places.

5. $5^x = 27$
6. $2e^{3x} = 30$
7. $10^x + 36 = 150$
8. $5^{x-3} = 137$
9. $e^{4x-5} - 7 = 11{,}243$
10. $5^{2x+3} = 3^{x-1}$
11. $e^{x+1} = 10^{2x}$
12. $40e^{0.6x} - 3 = 237$

Solve by factoring. Write the answer using exact values.

13. $e^{2x} - 4e^x + 3 = 0$
14. $e^{2x} - 3e^x + 2 = 0$
15. $e^{2x} - 2e^x = 3$
16. $e^{4x} + 5e^{2x} = 24$

Solve each logarithmic equation.

17. $\log_4(3x+2) = 3$
18. $5\ln(2x) = 20$
19. $\log_5 x + \log_5(4x-1) = 1$
20. $\log_3(x-5) + \log_3(x+3) = 2$
21. $\log(x+4) - \log x = \log 4$
22. $\log(5x+1) = \log(2x+3) + \log 2$
23. $\log(x+4) - \log 2 = \log(5x+1)$
24. $\ln(x-4) + \ln(x+1) = \ln(x-8)$
25. How long will it take \$2000 to grow to \$10,000 at 5% if compounded (a) continuously and (b) compounded quarterly.
26. What interest rate is needed for \$1000 to double after 10 years if compounded continuously?
27. The formula $A = 22.9e^{0.0183t}$ models the population of Texas, A, in millions, t years after 2005. (a) What was the population in Texas in 2005? (b) Estimate the population in 2012? (c) When will the population reach 27 million?
28. You have \$2300 to invest. What interest rate is needed for the investment to grow to \$3000 in two years if the investment is compounded quarterly?
29. What interest is needed for an investment to triple in four years if it is compounded (a) semiannually and (b) continuously?
30. How much money would you need to invest now in order to have \$15,000 saved in two years if the principal is invested at an interest rate of 6.2% compounded (a) quarterly and (b) continuously?
31. A certain type of radioactive iodine has a half-life of 8 days.
    (a) Find an exponential decay model, $A = A_0 e^{kt}$, for this type of iodine. Round the *k* value in your formula to six decimal places.

    (b) Use your model from part (a) to determine how long it will take for a sample of this type of radioactive iodine to decay to 10% of its original amount. Round your final answer to the nearest whole day.

32. The population in Tanzania in 1987 was about 24.3 million, with an annual growth rate of 3.5%. If the population is assumed to change continuously at this rate.

(a) estimate the population in 2008.

(b) in how many years after 1987 will the population be 30,000,000?

33. The half-life of aspirin in your bloodstream is 12 hours. How long will it take for the aspirin to decay to 70% of the original dosage?

34. The growth model $A = 107.4e^{0.012t}$ describes Mexico's population, A, in millions t years after 2006.

(a) What is Mexico's growth rate?

(b) How long will it take Mexico to double its population?

4.6 Homework Answers: **1.** 7 **2.** $-\frac{1}{6}$ **3.** 3 **4.** 0 **5.** {2.05} **6.** {.90} **7.** {2.06} **8.** {6.06}

9. {3.58} **10.** {-2.80} **11.** {.28} **12.** {2.99} **13.** {0, ln 3} **14.** {0, ln 2} **15.** {ln 3}

16. $\left\{\frac{\ln 3}{2}\right\}$ **17.** $\left\{\frac{62}{3}\right\}$ **18.** $\left\{\frac{e^4}{2}\right\}$ **19.** $\left\{\frac{5}{4}\right\}$ **20.** {6} **21.** $\left\{\frac{4}{3}\right\}$ **22.** {5} **23.** $\left\{\frac{2}{9}\right\}$ **24.** ϕ

25. (a) 32.19 years; (b) 32.39 years **26.** 6.93% **27.** (a) 22.9 million; (b) 26.03 million; (c) 2014

28. 13.5% **29.** (a) 29.44%; (b) 27.47% **30.** (a) \$13,263.31; (b) \$13,250.70 **31.** (a) $A = A_0e^{-0.086643t}$

(b) 27 days **32.** (a) 50.68 million; (b) 6.02 years **33.** 6.2 hours **34.** (a) 1.2%; (b) about 58 years