Logistics

- All course materials available on class page (www.math.msu.edu)
- Syllabus too
- Course textbook is S.A. Broverman’s ”Mathematics of Investment and Credit” , 5th edition or later
- Some questions on these slides, and on in class exam preparation slides, are taken from the third edition of Broverman’s book. Please note that Actex owns the copyright for that material. No portion of the ACTEX textbook material may be reproduced in any part or by any means without the permission of the publisher. We are very thankful to the publisher for allowing posting of these notes on our class website.
- Supplementary book is Finan, available online
The value of money is a function of the time that passes while it is stuffed under a mattress, deposited in a bank account, or invested in an asset. Just what that function is depends on many circumstances, and we will spend our time investigating many real-world examples.
Calculus, and the study of Geometric Series, provides...
Main Tools

- Calculus, and the study of Geometric Series, provides
- the machinery necessary for solving for the final value of an investment
- the optimal time to switch accounts
- the interest rate needed to plan fixed income investments
- among other wonderful things
Main Tools

- Calculus (1-d only!)
Main Tools

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- Can you take a derivative?
- Can you find the maximum of a function?
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- Can you find the maximum of a function?
- Sequences and Series
- Do you know what a geometric series is? What happens as $n \to \infty$?
Main Tools

- Calculus (1-d only!)
- Can you take a derivative?
- Can you find the maximum of a function?
- Sequences and Series
- Do you know what a geometric series is? What happens as $n \to \infty$?

$$\sum_{k=0}^{n} \lambda^k = \frac{1 - \lambda^{n+1}}{1 - \lambda} \quad (1)$$
Begin by investing 1000 at 9% per annum. Compound *annually* for 3 years. End up with...
Sometimes, interest rate is quoted per annum, but *compounded monthly*. For previous example, assume again 9% but compounded monthly. Then

\[
FV = 1000 \cdot \left[ \left( 1 + \frac{0.09}{12} \right)^{12} \right]^3
\]

\[
= 1000 \cdot (1.0938)^3
\]  \hspace{1cm} (2)
Sometimes, interest rate is quoted per annum, but *compounded monthly*. For previous example, assume again 9% but compounded monthly. Then

\[
FV = 1000 \cdot \left[ \left( 1 + \frac{0.09}{12} \right)^{12} \right]^3
\]

\[
= 1000 \cdot (1.0938)^3
\]

In other words, the equivalent or *effective annual rate* is 9.38%
It may be that over a longer period of time, the rate of return on an initial investment $A(0)$ may fluctuate. At time $n$, the value of the investment is now $A(n)$. 
Guess: \( A(n) = A(0) \cdot (1 + r)^n \)

Reality: \( A(n) = A(0) \cdot (1 + r_1) \cdot \ldots \cdot (1 + r_n) \)

In a bank account or bond, the interest rate, \( r \), is guaranteed to be small but positive. For a general investment, there is no guarantee that an investment will not shrink in value over a period of time.
Arithmetic vs Geometric Means

One other possible guess for an average rate of return is \( r = \frac{r_1 + \ldots + r_n}{n} \). The question of ordering, ie which average rate is bigger, reduces to the question:

\[
\text{Is } \left(1 + \frac{r_1 + \ldots + r_n}{n}\right)^n > (1 + r_1) \cdot \ldots \cdot (1 + r_n)\,? \tag{3}
\]

To answer this, define for \( k \in \{1, 2, \ldots, n\} \),

\[
x_k = 1 + r_k. \tag{4}
\]

Then, our question is reframed as:

\[
\text{Is } \left(\frac{x_1 + \ldots + x_n}{n}\right)^n > (x_1) \cdot \ldots \cdot (x_n)\,? \tag{5}
\]
Arithmetic vs Geometric Means

The answer is yes, and there are many proofs. One of them is by induction. Another is to apply Jensen’s Inequality:

For

- a concave function $f$, for example if $f''(x) \leq 0$ for all $x$ in our domain
- real numbers $a_k$ such that $\sum_{k=1}^{n} a_k \neq 0$,
- real numbers $x_k$,

it follows that

$$f \left( \frac{\sum_{k=1}^{n} a_k x_k}{\sum_{k=1}^{n} a_k} \right) \geq \frac{\sum_{k=1}^{n} a_k f(x_k)}{\sum_{k=1}^{n} a_k}. \quad (6)$$
By **Jensen’s Inequality**, if we define $f(x) = \ln(x)$ and each $a_k = \frac{1}{n}$, then we retain

$$\ln \left( \frac{x_1 + \ldots + x_n}{n} \right) \geq \frac{1}{n} \sum_{k=1}^{n} \ln(x_k) = \ln \left( \sqrt[n]{x_1 \cdot \ldots \cdot x_n} \right)$$  \hspace{1cm} (7)

and so by taking exponentials of both sides we are done.

The inequality can be shown to be strict if the $\{x_k\}_{k=1}^{n}$ are not all the same. In other words, if even one interest rate is different between periods, the the arithmetic average rate is strictly greater than the geometric rate.
We can calculate the average interest rate $i_{av}$ via

\[
(1 + i_{av})^n = (1 + r_1) \cdot (1 + r_2) \cdot \ldots \cdot (1 + r_n)
\]  

(8)

**Table:** Average vs. Annual Rate of Return

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual Rate</td>
<td>6.9%</td>
<td>6.4%</td>
<td>9.4%</td>
<td>-3.0%</td>
<td>7.8%</td>
</tr>
<tr>
<td>Average Rate</td>
<td>6.9%</td>
<td>6.65%</td>
<td>7.56%</td>
<td>4.82%</td>
<td>5.41%</td>
</tr>
</tbody>
</table>
A very nice example is the following: On Jan 1, 2000, Smith deposits 1000 into an account with 5% annual interest. The interest is paid on every Dec 31. Smith withdraws 200 on Jan 1, 2002, deposits 100 on Jan 1 2003 and again withdraws from the account on Jan 1 2005, this time 250. What is the balance in the account just after the interest is paid on Dec 31, 2006?

**Hint:** Think of a deposit as a loan to the bank, a withdrawal as a loan to Smith, and at the end of the term (7 years,) both bank and Smith settle up accounts.
Example 1.3

We can replicate the cash flows of this example by stating that Smith has invested both positive and negative amounts of money over the period of time:

<table>
<thead>
<tr>
<th>Year</th>
<th>Period</th>
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<tr>
<td>1000</td>
<td>7 years</td>
</tr>
<tr>
<td>−200</td>
<td>5 years</td>
</tr>
<tr>
<td>100</td>
<td>4 years</td>
</tr>
<tr>
<td>−250</td>
<td>2 years</td>
</tr>
</tbody>
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Hence, $A(7) = 1000 \cdot (1.05)^7 + (-200) \cdot (1.05)^5 + 100 \cdot (1.05)^4 + (-250) \cdot (1.05)^2 = 997.77$
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Hence,

\[
A(7) = 1000 \cdot (1.05)^7 + (-200) \cdot (1.05)^5 \\
+ 100 \cdot (1.05)^4 + (-250) \cdot (1.05)^2 = 997.77
\]
Another way of computing $A(7)$ is to do so recursively:

\[
A(0) = 1000 \\
A(1) = A(0) \cdot 1.05 = 1000 \cdot (1.05) \\
A(2) = A(1) \cdot (1.05) - 200 \\
\hspace{1cm} = 1000 \cdot (1.05)^2 - 200 \\
A(3) = A(2) \cdot (1.05) + 100 \\
\hspace{1cm} = 1000 \cdot (1.05)^3 - 200 \cdot (1.05) + 100 \\
A(4) = A(3) \cdot (1.05) \\
A(5) = A(4) \cdot (1.05) - 250 \\
A(6) = A(5) \cdot (1.05) \\
A(7) = A(6) \cdot (1.05) \\
\hspace{1cm} = 1000 \cdot (1.05)^7 + (-200) \cdot (1.05)^5 \\
\hspace{1cm} + 100 \cdot (1.05)^4 + (-250) \cdot (1.05)^2 = 997.77
\]
Another way interest can be designed to accrue is linearly Symbolically, if $A(0)$ is the initial value of our investment, then the final value is

$$A_{\text{linear}}(t) = A(0) \cdot (1 + i \cdot t)$$  \hspace{1cm} (11)

Notice that at any time $k \leq t = n$,

$$A_{\text{compound}}(n) = A(k) \cdot (1 + i)^{n-k}$$

$$= A(0) \cdot (1 + i)^k \cdot (1 + i)^{n-k}$$

$$A_{\text{linear}}(n) = A(0) \cdot (1 + i \cdot n)$$

$$\neq A(k) \cdot (1 + i \cdot (n - k))$$

$$= A(0) \cdot (1 + i \cdot k) \cdot (1 + i \cdot (n - k))$$  \hspace{1cm} (12)
Assume you can select

- from two interest rates \( j, i \) where \( 0 < i < j < 1 \) and
- time \( t \in [0, 1] \)

where you can decide to switch from a bank paying \( j \) for a time of length \( t \) to one paying \( i \) for the remaining time \( 1 - t \).

At what time \( t \) would this be optimal to do, if at all?
In general, if we switch at, then it must be that

\[ 1 + j < (1 + jt)(1 + (i(1 - t))) \]

\[ \Rightarrow 1 + j < 1 + jt + i(1 - t) + ijt(1 - t) \]

\[ \Rightarrow (j - i)(1 - t) < ijt(1 - t) \]  \hspace{1cm} (13) \]

\[ \Rightarrow \frac{j - i}{ij} < t. \]

It follows that if \( 0 < \frac{j-i}{ij} < 1 \), we are able to justify such a switch!

The inequality linking \( i \) to \( j \) is now

\[ i < j < \frac{i}{1-i}. \]  \hspace{1cm} (14) \]
The **optimal** time to switch is when $f(t) := (1 + jt)(1 + i(1 - t))$ is maximized:

$$0 = f'(t) = \frac{d}{dt} \left[(1 + jt)(1 + i(1 - t))\right]$$

$$= (j - i) + ij(1 - 2t).$$

$$\Rightarrow t_{optimal} = \frac{1}{2} + \frac{1}{2} \cdot \frac{j - i}{ij}. \quad (15)$$

Of course, $t_{optimal} < 1$ as $i < j < \frac{i}{1 - i}$ and so the optimal value of the bank account is

$$f(t_{optimal}) = \left[1 + j \cdot \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{j - i}{ij}\right)\right] \left[1 + i \cdot \left(\frac{1}{2} - \frac{1}{2} \cdot \frac{j - i}{ij}\right)\right]. \quad (16)$$
Accumulated Amount Function

$A(t)$ is also known as the A.A.F., and

\[
A(t_2) = A(t_1)(1 + i_{t_2})
\]

\[
i_{t_2} := \frac{A(t_2) - A(t_1)}{A(t_1)} \tag{17}
\]

$i_{t_2}$ is known as the \textit{effective rate} for our investment from $t_1$ to $t_2$. 
A dollar today is certainly more valuable than a dollar tomorrow, and even more valuable than a dollar next year. To reflect this idea, we say that the present value of a unit of currency one year from now is $\nu$, where

$$\nu \equiv \frac{1}{1 + i}. \quad (18)$$

$\nu$ is also known as the *discount factor*. This factor works as the inverse of the interest gained on an investment of a unit of currency for one year. For example, the present value of 25,000 at a rate of 5% per annum, 25 years from now, is $\frac{25000}{(1.05)^{25}}$. 

In calculating the present value of an investment, we are working backwards from a fixed outcome to calculate it’s value to us *today*.

But, there may be more than one payment expected in the future, such as when planning for retirement. They may also be balanced by further investment made in the future.
Loan Shark pays you 1000 every week, starting today. There are 4 payments in total, at 8% weekly interest. Starting the week after the last 1000, you are to repay the loan in 3 consecutive weekly installments of 1100 at 8% weekly interest, plus a fourth payment $X$ at 8% weekly interest. What is $X$?
Example 1.5

<table>
<thead>
<tr>
<th>Week</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment in</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
</tr>
</tbody>
</table>

Table: Example 1.5: Payments In

<table>
<thead>
<tr>
<th>Week</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment out</td>
<td>1100</td>
<td>1100</td>
<td>1100</td>
<td>X</td>
</tr>
</tbody>
</table>

Table: Example 1.5: Payments Out
Example 1.5

\[ PV_0(in) = PV_0(out) \]
Example 1.5

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\[ PV_0(in) = 1000 + \frac{1000}{1.08} + \frac{1000}{1.08^2} + \frac{1000}{1.08^3} \]

\[ PV_0(out) = \frac{1100}{1.08^4} + \frac{1100}{1.08^5} + \frac{1100}{1.08^6} + \frac{X}{1.08^7} \]

\[ X = 2273.79 \]
When a credit card has a 24% *nominal* annual interest rate, you do not only pay 24% on the balance. Rather, that 24% is broken down into 12 monthly interest charges on the average balance over a 30 day billing cycle. So, one pays an *effective annual rate of*

\[
\left(1 + \frac{0.24}{12}\right)^{12} - 1 = 0.2682. \tag{20}
\]
Example 1.9

Which is better:

- (A) 15.25% compounded semi-annually, or
- (B) 15% compounded monthly?
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- (B) 15% compounded monthly?

\[
i_A(\text{eff}) = \left(1 + \frac{0.1525}{2}\right)^2 - 1 = 0.1583
\]

\[
i_B(\text{eff}) = \left(1 + \frac{0.15}{12}\right)^{12} - 1 = 0.1608
\]
Actuaries reserve $i$ for effective annual rate and $i^{(m)}$ for the nominal rate compounded $m$ times annually. The link between the two is

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$$
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$$< i^{(m)} \quad \forall m$$
Example 1.10

Assume \( i = 0.12 \). Then

\[
\begin{array}{c|c}
 m & m \cdot \left( (1 + i)^{\frac{1}{m}} - 1 \right) \\
1 & 0.12 \\
2 & 0.1166 \\
6 & 0.1144 \\
52 & 0.1135 \\
365 & 0.113346 \\
\infty & 0.113329 \\
\end{array}
\]
Assume that we are promised 12% annual interest, and we wish to calculate the daily interest gained on 10,000,000:

\[
\text{interest}_{\text{overnight}} = 10,000,000 \cdot 0.12 \cdot \frac{1}{365} = 3287.67
\]

\[
\text{interest}_{\text{continuous}} = 10,000,000 \cdot \left( e^{\frac{1}{365}} - 1 \right) = 3288.21,
\]

a difference of 0.54 on a principle of 10 million
Sometimes, *interest paid up front*: Receive $A(0)$ up front, pay back $A(1) > A(0)$.
Consider the example

$$A(0) = 900$$
$$A(1) = 1000$$

$$d = \frac{A(1) - A(0)}{A(1)}$$
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Consider the example

\[ \begin{align*}
A(0) &= 900 \\
A(1) &= 1000 \\
d &= \frac{A(1) - A(0)}{A(1)} \\
\frac{1}{1 + i} &= \nu = 1 - d
\end{align*} \]  

Equation (24)

Essentially, the discount is the forward price of a dollar (or any other unit of currency) in the interest market with no carrying cost or dividends (coupons) paid out.
Example 1.11

A T-bill represents the forward value today of 100 delivered at time $T$. The price is calculated via simple discount:

$$P(0) = P(T) \cdot (1 - d \cdot t) = 100 \cdot \left(1 - d \cdot \frac{T}{365}\right)$$
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\[
100 = P(T) = P(0) \cdot (1 + i \cdot t) = P(0) \cdot \left(1 + i \cdot \frac{T}{365}\right)
\]

(25)
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Table: Example 1.11

<table>
<thead>
<tr>
<th>Term</th>
<th>Discount (%)</th>
<th>Investment (%)</th>
<th>Price per 100</th>
</tr>
</thead>
<tbody>
<tr>
<td>12–day</td>
<td>0.965</td>
<td>0.974</td>
<td>99.680</td>
</tr>
<tr>
<td>28–day</td>
<td>0.9940</td>
<td>0.952</td>
<td>99.927</td>
</tr>
<tr>
<td>91–day</td>
<td>1.130</td>
<td>1.150</td>
<td>99.174</td>
</tr>
<tr>
<td>128–day</td>
<td>1.400</td>
<td>1.430</td>
<td>99.292</td>
</tr>
</tbody>
</table>
$d^{(m)}$ is the quoted annual discount rate that is applied $m$ times over the year, with the effective discount rate as $\frac{d^{(m)}}{m}$ for the period of $\frac{1}{m}$ years. Also, $d$ is the effective discount rate:

$$1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m$$

$$d^{(m)} = m \cdot \left(1 - (1 - d)^{\frac{1}{m}}\right)$$

$$d^{(\infty)} = \lim_{m \to \infty} d^{(m)} = \ln \left(\frac{1}{1 - d}\right) = i^{(\infty)}$$
Force of Interest

From time $t_1$ to $t_2$, an investment grows by a rate of

$$i_{t_1 \rightarrow t_2} := \frac{A(t_2) - A(t_1)}{A(t_1)}$$

(27)

Now, fix a time $t_1 = t$ and correspondingly, let $t_2 = t + \frac{1}{m}$. It follows that the nominal rate that gives the same annual growth from $t$ to $t + \frac{1}{m}$ is

$$i^{(m)} = m \cdot \frac{A(t + \frac{1}{m}) - A(t)}{A(t)} = \frac{1}{A(t)} \cdot \frac{A(t + \frac{1}{m}) - A(t)}{\frac{1}{m}}$$
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$$\lim_{m \to \infty} i^{(m)} = \frac{1}{A(t)} \cdot \lim_{m \to \infty} A(t + \frac{1}{m}) - A(t)$$
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$$\lim_{m \to \infty} i^{(m)} = \frac{1}{A(t)} \cdot \lim_{m \to \infty} \frac{A(t + \frac{1}{m}) - A(t)}{\frac{1}{m}}$$

$$= \frac{1}{A(t)} \frac{dA}{dt}$$
From time $t_1$ to $t_2$, an investment grows by a rate of

$$i_{t_1 \to t_2} := \frac{A(t_2) - A(t_1)}{A(t_1)}$$  \hspace{1cm} (27)

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$$i^{(m)} = m \cdot \frac{A\left(t + \frac{1}{m}\right) - A(t)}{A(t)} = \frac{1}{A(t)} \cdot \frac{A\left(t + \frac{1}{m}\right) - A(t)}{\frac{1}{m}}$$

$$\lim_{m \to \infty} i^{(m)} = \frac{1}{A(t)} \cdot \lim_{m \to \infty} \frac{A\left(t + \frac{1}{m}\right) - A(t)}{\frac{1}{m}} = \frac{1}{A(t)} \frac{dA}{dt} =: \delta(t)$$  \hspace{1cm} (28)
Since \( \delta(t) = \frac{1}{A(t)} \frac{dA}{dt} = \frac{d}{dt} [\ln (A(t))], \) it follows that

\[
\int_0^t \delta(s) ds = \int_0^t \frac{d}{ds} [\ln (A(s))] ds = \ln (A(t)) - \ln (A(0)) \tag{29}
\]

and so

\[
A(t) = A(0)e^{\int_0^t \delta(s) ds} \tag{30}
\]
Some Examples...

Calculate $\delta(t)$ for

- $A(t) = e^{-t^2}$
- $A(t) = t \cdot e^{-t}$
- $A(t) = (t - 1)^2$
- $A(t) = A(0) \cdot (1 + i \cdot t)$
- $A(t) = A(0) \cdot (1 + i)^t$
Example 1.14

Assume $\delta(t) = 0.08 + 0.005t$ and your asset’s initial value is $A(0) = 1000$. Then

$$A(t) = A(0)e^{\int_0^t (0.08+0.005s)\,ds} = 1000e^{0.08t+0.0025t^2}$$

$$A(5) = 1000e^{0.08\cdot 5+0.0025\cdot 5^2} = 1588.04 \quad (31)$$

But, if we were given $A(2) = 1000$, then

$$A(7) = A(2)e^{\int_2^7 \delta(s)\,ds} = 1000e^{0.08\cdot 5+0.0025\cdot (7^2-2^2)} = 1669.46 \quad (32)$$
This limiting term, known as the **force of interest** represents the theoretical upper limit for interest gained on this asset. However, in the derivation above, there is no randomness involved in obtaining the ordinary differential equation

$$\frac{dA}{dt} = \delta(t)A(t)$$ (33)

To handle this lack of stochasticity, one common asset model (Geometric Brownian Motion) is defined by the stochastic differential equation

$$dA(t) = \delta(t)A(t)dt + \sigma(A(t))dW(t),$$ (34)

where $W(.)$ is the Wiener process.
An *annuity* is a series of periodic payments.
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\[
X_n := 1 + x + \ldots + x^n
\]
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\[ X_n := 1 + x + \ldots + x^n \]
\[ (1 - x) \cdot X_n = 1 - x^{n+1} \]
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\[
X_n := 1 + x + \ldots + x^n
\]

\[
(1 - x) \cdot X_n = 1 - x^{n+1}
\]

\[
\Rightarrow X_n = \frac{1 - x^{n+1}}{1 - x}
\]  \hspace{1cm} (35)
Example 2.1

The federal government sends Smith a family allowance of 30 every month for Smith’s child. Smith deposits the payments in bank account on the last day of each month. The account earns interest at the annual rate of 9% compounded monthly and payable on the last day of each month, on the minimum monthly balance. If the first payment is deposited on May 31, 1998, what is the account balance on December 31, 2009, including the payment just made?
Example 2.1

There are 140 total payments
Example 2.1

- There are 140 total payments
- \( i_{\text{monthly}} = \frac{i}{12} = 0.0075 \)
Example 2.1

- There are 140 total payments
- \( i_{\text{monthly}} = \frac{i}{12} = 0.0075 \)

\[
\text{TotalValue} = 30 + 30 \cdot (1.0075)^1 + \ldots + 30 \cdot (1.0075)^{139} \\
= 30 \cdot \sum_{k=0}^{139} 1.0075^k = 30 \cdot \frac{1 - 1.0075^{140}}{1 - 1.0075} \\
= 7385.91
\]
Number of payments in a series of payments is called the *term* of the annuity.

Time between the successive payments is called the payment period, or *frequency*.

A series of payments whose value is found at the time of the final payment is known as *accumulated annuity immediate*.
Level Payment Annuities

- Number of payments in a series of payments is called the *term* of the annuity.
- Time between the successive payments is called the payment period, or *frequency*.
- A series of payments whose value is found at the time of the final payment is known as *accumulated annuity immediate*.

\[
s_{\overline{n|}} := \sum_{k=0}^{n-1} (1 + i)^k = \frac{(1 + i)^n - 1}{i} \tag{37}
\]
Level Payment Annuities

- Number of payments in a series of payments is called the term of the annuity
- Time between the successive payments is called the payment period, or frequency
- A series of payments whose value is found at the time of the final payment is known as accumulated annuity immediate

\[
s_{\overline{n|}} := \sum_{k=0}^{n-1} (1 + i)^k = \frac{(1 + i)^n - 1}{i}
\]  

(37)

Equivalently, \((1 + i)^n = 1 + i \cdot s_{\overline{n|}}\).
What level amount must be deposited on May 1 and Nov 1 each year from 1998 to 2005, inclusive, to accumulate to 7000 on November 1, 2005 if the nominal annual interest rate, compounded semi-annually, is 9%?
16 total deposits

\( i^{(2)} = 0.09 \)

\( X \) denotes the level amount deposited per period
Interest Compounded Monthly

- 16 total deposits
- \( i^{(2)} = 0.09 \)
- \( X \) denotes the level amount deposited per period

\[
7000 = X \cdot (1 + (1.045)^1 + \ldots + (1.045)^{15})
\]
\[
= X \cdot \overline{s_{16|0.045}}
\]
\[
\Rightarrow X = 308.11
\]
Suppose that in Example 2.1, Smith’s child is born in April 1998 and the first payment is received in May (and deposited at the end of May.) The payments continue and the deposits are made at the end of the month until, and including the month of, the child’s 16\textsuperscript{th} birthday. The payments stop after the 16\textsuperscript{th} birthday, but the balance continues to accumulate with interest until the end of the month of the child’s 21\textsuperscript{st} birthday. What is the balance $X$ in the account at that time?
Example 2.3

Suppose that in Example 2.1, Smith’s child is born in April 1998 and the first payment is received in May (and deposited at the end of May.) The payments continue and the deposits are made at the end of the month until, and including the month of, the child’s 16\textsuperscript{th} birthday. The payments stop after the 16\textsuperscript{th} birthday, but the balance continues to accumulate with interest until the end of the month of the child’s 21\textsuperscript{st} birthday. What is the balance $X$ in the account at that time?

$$X = 1.0075^{60} \cdot 30 \cdot s_{\overline{192}}^{0.0075} = 20,028.68$$ (39)
Some Arithmetic

After accumulating cash via a series of payments at rate $i$ until time $n$, we allow this amount to grow until time $n + k$:

$$Value(n + k) = s_{n|i} \cdot (1 + i)^k = \left(\frac{(1 + i)^n - 1}{i}\right) \cdot (1 + i)^k$$

$$= \frac{(1 + i)^{n+k} - (1 + i)^k}{i}$$

$$= \frac{(1 + i)^{n+k} - 1}{i} - \frac{(1 + i)^k - 1}{i}$$

$$= s_{n+k|i} - s_{k|i}$$
Some Arithmetic

After accumulating cash via a series of payments at rate $i$ until time $n$, we allow this amount to grow until time $n + k$:

$$\text{Value}(n + k) = s_{n|i} \cdot (1 + i)^k = \frac{(1 + i)^n - 1}{i} \cdot (1 + i)^k$$

$$= \frac{(1 + i)^{n+k} - (1 + i)^k}{i}$$

$$= \frac{(1 + i)^{n+k} - 1}{i} - \frac{(1 + i)^k - 1}{i}$$

$$= s_{n+k|i} - s_{k|i}$$

$$s_{n+k|i} = s_{k|i} + s_{n|i} \cdot (1 + i)^k$$
Suppose that in Example 2.1, the nominal annual interest rate earned on the account changes to 7.5%, still compounded monthly, as of January 2004. What is the accumulated value of the account on December 31 2009?
Example 2.4

Suppose that in Example 2.1, the nominal annual interest rate earned on the account changes to 7.5%, still compounded monthly, as of January 2004. What is the accumulated value of the account on December 31 2009?

\[ Value = 30 \cdot (s_{\overline{68}|0.0075} \cdot 1.00625^{72} + s_{\overline{72}|0.00625}) = 6865.22 \quad \text{(41)} \]
Suppose 10 monthly payments of 50 each are followed by 14 monthly payments of 75 each. If interest is at an effective monthly rate of 1%, what is the accumulated value of the series at the time of the final payment?
Suppose 10 monthly payments of 50 each are followed by 14 monthly payments of 75 each. If interest is at an effective monthly rate of 1%, what is the accumulated value of the series at the time of the final payment?

\[ Value = 50 \cdot s_{24|0.01} + 25 \cdot s_{14|0.01} \]  

(42)
Make a lump sum payment $X$ now to receive periodic payments $C$, starting one period from today. If the market bears a constant interest of $i$, the present value of this Annuity Immediate is calculated as

$$X = \frac{C}{1 + i} + \frac{C}{(1 + i)^2} + \ldots + \frac{C}{(1 + i)^n}$$

$$= C \cdot \sum_{k=1}^{n} \nu^k = C \cdot \nu \cdot \sum_{k=0}^{n-1} \nu^k$$

$$= C \cdot \nu \cdot \frac{1 - \nu^n}{1 - \nu}$$

$$= C \cdot \frac{1 - \nu^n}{i} = C \cdot \frac{1 - (1 + i)^{-n}}{i}$$

$$\equiv C \cdot a_{\overline{n}|i}$$
Brown has bought a new car and requires a loan of 12000 to pay for it. The car dealer offers Brown two alternatives on the loan:

- **A**: Monthly payments for 3 years, starting one month after purchase, with an annual interest rate of 12% compounded monthly
- **B**: Monthly payments for 4 years, also starting one month after purchase, with an annual interest rate of 15% compounded monthly.

Find Brown’s monthly payment and the total amount paid over the course of the repayment period under each of the two options.
Example 2.7

\[ 12000 = P_A \cdot a_{36|0.01} \Rightarrow P_A = 398.57 \]
\[ 12000 = P_B \cdot a_{48|0.0125} \Rightarrow P_B = 333.97 \]

TotalValue(A) = 36 \cdot P_A = 14348.52

TotalValue(B) = 48 \cdot P_B = 16030.56

(44)
Example 2.8

Suppose that in Example 2.7, Brown can repay the loan, still with 36 payments under option $A$ or 48 payments under option $B$, with the first payment made 9 months after the car is purchased in either case. Assuming interest accrues from the time of the car purchase, find the payments required under options $A$ and $B$. This is known as a *deferred annuity*.
Example 2.8

\[ 12000 = P'_A \cdot (1.01^{-9} + 1.01^{-10} + \ldots + 1.01^{-44}) \]
\[ = P'_A \cdot \nu_A^8 \cdot a_{36|0.01} \]
\[ \Rightarrow P'_A = 431.60 \]

\[ 12000 = P'_B \cdot (1.0125^{-9} + 1.0125^{-10} + \ldots + 1.0125^{-56}) \]
\[ = P'_B \cdot \nu_B^8 \cdot a_{48|0.0125} \]
\[ \Rightarrow P'_B = 368.86 \]
Some Facts About Annuities

There is a duality between $a_{\bar{n}}|i$ and $s_{\bar{n}}|i$:

$$s_{\bar{n}}|i = (1 + i)^n a_{\bar{n}}|i$$

$$a_{\bar{n}}|i = \nu^n s_{\bar{n}}|i$$

(46)

As $n \to \infty$, we have

$$a_{\infty}|i = \lim_{n \to \infty} a_{\bar{n}}|i$$

$$= \lim_{n \to \infty} \frac{1 - \nu^n}{i} = \frac{1}{i}$$

(47)
A perpetuity immediate pays $X$ per year. Brian receives the first $n$ payments, Colleen receives the next $n$ payments, and Jeff receives the remaining payments. Brian’s share of the present value of the original perpetuity is 40%, and Jeff’s share is $K$. Calculate $K$. 
Example 2.9

\[ PV(Brian) = X \cdot a_{\overline{n}|i} = X \cdot \frac{1 - \nu^n}{i} \]

\[ = 0.4 \cdot X \cdot a_{\overline{\infty}|i} = 0.4 \frac{X}{i} \]

\[ \Rightarrow 1 - \nu^n = 0.4 \]

\[ PV(Colleen) = \nu^n \cdot X \cdot a_{\overline{n}|i} = 0.4 \cdot 0.6 \frac{X}{i} = 0.24 \frac{X}{i} \]
Example 2.9

\[ PV(Brian) = X \cdot a_{n\mid i} = X \cdot \frac{1 - \nu^n}{i} \]

\[ = 0.4 \cdot X \cdot a_{\infty \mid i} = 0.4 \frac{X}{i} \]

\[ \Rightarrow 1 - \nu^n = 0.4 \]

\[ PV(\text{Colleen}) = \nu^n \cdot X \cdot a_{n\mid i} = 0.4 \cdot 0.6 \frac{X}{i} = 0.24 \frac{X}{i} \quad (48) \]

\[ \frac{X}{i} = PV = PV(Brian) + PV(Jeff) + PV(\text{Colleen}) \]

\[ = 0.4 \frac{X}{i} + K + 0.24 \frac{X}{i} \]

\[ \Rightarrow K = 0.36 \frac{X}{i} \]
Recall...

\[ s_{\overline{m}|i} = 1 + (1 + i) + (1 + i)^2 + \ldots + (1 + i)^{n-1} = \frac{(1 + i)^n - 1}{i} \quad (49) \]

\[ a_{\overline{m}|i} = \nu + \nu^2 = \ldots + \nu^n = \frac{1 - \nu^n}{i} \]

Of course, \( s_{\overline{m}|i} \) is the accumulation of all the payments at the final time, whereas \( a_{\overline{m}|i} \) is the present value of all the payments, one period before the first payment. Now, define

\[ \ddot{s}_{\overline{m}|i} := (1 + i) + (1 + i)^2 + \ldots + (1 + i)^n = (1 + i)s_{\overline{m}|i} \]

\[ = \text{annuity due} \]

\[ = \text{accumulated value one-period after final payment} \quad (50) \]

\[ \ddot{a}_{\overline{m}|i} := (1 + i)a_{\overline{m}|i} \]

\[ = \text{present value at time of first payment} \]
What happens when compounding interest period and annuity payment period don’t coincide? For example: Consider 4 deposits made per year, over 16 years, of 1000 each. What is the balance after the last payment if interest is quoted at 9% nominal annual interest rate, compounded monthly?

In this case, the equivalent rate \( j \) for the \( 3 \)-month periods satisfies

\[
1 + j = \left(1 + \frac{0.09}{12}\right)^3.
\]

It follows that

\[
\text{Value} = 1000 \times s_{64}^{-j} = 141,076 (51)
\]
What happens when compounding interest period and annuity payment period don’t coincide? For example: Consider 4 deposits made per year, over 16 years, of 1000 each. What is the balance after the last payment if interest is quoted at 9% nominal annual interest rate, compounded monthly?

In this case, the equivalent rate $j$ for the 3-month periods satisfies $1 + j = \left(1 + \frac{0.09}{12}\right)^3$. It follows that

$$\text{Value} = 1000s_{64}^{-j} = 141,076$$  \hspace{1cm} (51)
$m^{thly}$ payable annuities: Some Generalizations.

Imagine that for each of $n$ periods in the term of an annuity, 1 is paid out over $m$ payments of $\frac{1}{m}$ each. Then the present value is computed using $j = \frac{i^{(m)}}{m}$, where

$$a_{m|i}^{(m)} = \nu \cdot \frac{1}{m} s_{m|j} + \nu^2 \cdot \frac{1}{m} s_{m|j} + \ldots + \nu^n \cdot \frac{1}{m} s_{m|j}$$

$$= (\nu + \nu^2 + \ldots + \nu^n) \cdot \frac{1}{m} \cdot \frac{\left(1 + \frac{i^{(m)}}{m}\right)^m - 1}{i^{(m)}}$$

$$= a_{n|i} \cdot \frac{\left(1 + \frac{i^{(m)}}{m}\right)^m - 1}{i^{(m)}}$$

$$= a_{n|i} \cdot \frac{i}{i^{(m)}}$$
Imagine that for each of $n$ periods in the term of an annuity, 1 is paid out over $m$ payments of $\frac{1}{m}$ each. Then the present value is computed using

$$j = \frac{i^{(m)}}{m},$$

where

$$a_{\frac{m}{i}}^{(m)} = \nu \cdot \frac{1}{m} s_{\overline{m}|j} + \nu^2 \cdot \frac{1}{m} s_{\overline{m}|j} + \ldots + \nu^n \cdot \frac{1}{m} s_{\overline{m}|j}$$

$$= (\nu + \nu^2 + \ldots + \nu^n) \cdot \frac{1}{m} \cdot \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$$

$$= a_{\overline{n}|i} \cdot \frac{\left(1 + \frac{i^{(m)}}{m}\right)^m}{i^{(m)}} - 1 = a_{\overline{n}|i} \cdot \frac{i}{i^{(m)}}$$

$$s_{\overline{m}|i}^{(m)} = s_{\overline{m}|i} \cdot \frac{i}{i^{(m)}}$$
As $m \to \infty$, we have $i^{(m)} \to \ln (1 + i)$, and so it follows that

$$a_{n|i}^{(m)} \to a_{\infty|i} \cdot \frac{i}{\ln (1 + i)}$$

$$s_{n|i}^{(m)} \to s_{\infty|i} \cdot \frac{i}{\ln (1 + i)}$$

Let’s analyze this using calculus!
Take $t_2 - t_1 = \frac{1}{m}$, and assume that 1 is paid out at a continuous rate. Then, from $t$ to $t + dt$, the amount paid out is $dt$. At time $n$, the value of $dt$ accumulated from $t$ to $n$ is $dt \cdot (1 + i)^{n-t}$.
Take \( t_2 - t_1 = \frac{1}{m} \), and assume that 1 is paid out at a continuous rate. Then, from \( t \) to \( t + dt \), the amount paid out is \( dt \). At time \( n \), the value of \( dt \) accumulated from \( t \) to \( n \) is \( dt \cdot (1 + i)^{n-t} \). Add up all of these bits and we obtain

\[
\bar{s}_{\frac{m}{i}} = \int_0^n (1 + i)^{n-t} dt = \int_0^n e^{\ln(1+i)(n-t)} dt
\]

\[
= e^{\ln(1+i)n} \cdot \left[ \frac{e^{-\ln(1+i)t}}{-\ln(1+i)} \right]_0^n
\]

\[
= \frac{(1 + i)^n - 1}{\ln(1 + i)} = s_{\frac{n}{i}} \cdot \frac{i}{\ln(1 + i)} = \lim_{m \to \infty} s_{\frac{m}{i}}^{(m)}
\]
Consider depositing 12 per day in 2004 and 2005 and 15 per day in 2006. The earned interest in 2004-05 is \( i_1 = 9\% \) effective per year, and \( i_2 = 12\% \) effective per year in 2006. Compute the total accumulated amount at the end of 2006 if computed via \((a)\) daily deposits and \((b)\) continuous deposits. Assume no leap years!
For daily deposits, we compute the equivalent daily interest rates $j_1$ for 2004-05 and $j_2$ for 2006 by

$$j_1 = (1.09)^{\frac{1}{365}} - 1 = 0.00023631$$

$$j_2 = (1.12)^{\frac{1}{365}} - 1 = 0.00031054$$

So,

$$AccValue = 12 \cdot s_{730|j_1} \cdot (1 + i_2) + 15 \cdot s_{365|j_2}$$

$$= 12 \cdot s_{730|0.00023631} \cdot (1.12) + 15 \cdot s_{365|0.00031054}$$

$$= 16502.59$$
An Example: Daily vs Continuous deposits

For continuous deposits, we compute that $12 \cdot 365 = 4380$ is invested per year for the period 2004-05 and $15 \cdot 365 = 5475$ for 2006. So,

\[
AccValue = 4380 \cdot \bar{s}_{2\|0.09} \cdot (1.12) + 5475 \cdot \bar{s}_{1\|0.12} \\
= 4380 \cdot \frac{1.09^2 - 1}{\ln (1.09)} \cdot (1.12) + 5475 \cdot \frac{1.12 - 1}{\ln (1.12)} \\
= 16504.75
\]

(57)
Define $a(t_1, t_2)$ as the accumulated value at $t_2$ of an amount 1 invested at $t_1$. 
Define $a(t_1, t_2)$ as the accumulated value at $t_2$ of an amount 1 invested at $t_1$. Hence,

$$\int_{t_1}^{t_2} a(t, t_2) \, dt \quad (58)$$

is the accumulated value at $t_2$ of a continuous annuity paying 1 per unit time over the interval $(t_1, t_2)$, and

$$\int_{t_1}^{t_2} \frac{1}{a(t, t_2)} \, dt \quad (59)$$

is the present value at $t_1$ of a continuous annuity paying 1 per unit time over the interval $(t_1, t_2)$,
If \( \delta_s \) is the \textit{force of interest} at time \( s \), then

\[
a(t_1, t_2) = e^{\int_{t_1}^{t_2} \delta_s ds}
\]

\[
\bar{a}_{n|\delta_s} = \int_0^n e^{-\int_0^t \delta_s ds} dt
\]

\[
\bar{s}_{n|\delta_s} = \int_0^n e^{\int_t^n \delta_s ds} dt
\]
Assuming \( n \) level payments of \( J \) each at interest rate \( i \), the accumulated value is
Unknown Number of Payments in an Annuity

Assuming $n$ level payments of $J$ each at interest rate $i$, the accumulated value is

$$M = J \left[ (1 + i)^{n-1} + \ldots + (1 + i) + 1 \right]$$

$$= J \cdot \frac{(1 + i)^n - 1}{i} = J \cdot s_{\bar{n}|i}$$
Assuming $n$ level payments of $J$ each at interest rate $i$, the accumulated value is

$$M = J \left[ (1 + i)^{n-1} + \ldots + (1 + i) + 1 \right]$$

$$= J \cdot \frac{(1 + i)^n - 1}{i} = J \cdot s_{\bar{n}|i}$$

$$\Rightarrow (1 + i)^n = 1 + i \cdot \frac{M}{J}$$

$$\Rightarrow n = \frac{\ln \left( 1 + i \cdot \frac{M}{J} \right)}{\ln 1 + i}$$

(61)
Smith wishes to accumulate 1000 by means of semi-annual deposits earning interest at nominal rate $i^{(2)} = 0.08$, with interest credited semi-annually. Regular deposits of 50 are made. Find the number of periods and regular deposits needed, along with any additional \textit{fractional} deposits if such a fractional deposit is made at the time of the last regular deposit \textit{or} the very next period (i.e. last regular deposit can be 0.)
Here, the parameters are $M = 1000$, $J = 50$, $i = 0.04$. It follows that

$$n = \frac{\ln (1 + i \cdot \frac{M}{J})}{\ln 1 + i} = 14.9866$$

(62)
Here, the parameters are $M = 1000$, $J = 50$, $i = 0.04$. It follows that

$$n = \frac{\ln (1 + i \cdot \frac{M}{J})}{\ln 1 + i} = 14.9866$$

(62)

Rounding down to $n = 14$, we obtain $50s_{14|0.04} = 914.60$. 

Example 2.13

Here, the parameters are $M = 1000$, $J = 50$, $i = 0.04$. It follows that

$$n = \frac{\ln \left(1 + i \cdot \frac{M}{J}\right)}{\ln (1 + i)} = 14.9866$$

(62)

Rounding down to $n = 14$, we obtain $50s^{\frac{14}{14}0.04} = 914.60$. However, 50 is not enough to carry over to get a final amount of 1000, so let the 914.60 accrue for 1 period to $914.60 \cdot (1.04) = 951.18$, and then add the fractional payment of 48.82. Hence 15 periods are needed.
Smith pays 100 per month to a fund earning $i^{(12)} = 0.09$, with interest credited on the last day of each month. At the time of each deposit, 10 is deducted from the deposit for expenses and administration fees. The first deposit is made on the last day of Jan 2000. In which month does the accumulated value become greater than the total gross contribution to that point?
Example 2.13

We wish to solve for the time when the Value of the account exceeds the total amount invested, i.e. the value $n$ that satisfies

$$90 \cdot \left[1 + 1.0075 + \ldots + 1.0075^{n-1}\right] \geq 100n$$

Essentially, we are solving a fixed point equation for $f(x) = 0$, where

$$f(x) = e^{kx} - 1 - 0.00833x$$

In this case, plotting the above function of $x$ gives an intercept of approximately 28.6. Hence, $n = 29$. 
We wish to solve for the time when the Value of the account exceeds the total amount invested, i.e. the value $n$ that satisfies

$$90 \cdot [1 + 1.0075 + \ldots + 1.0075^{n-1}] \geq 100n$$

$$s_{\bar{n}|i} \geq \frac{100}{90} n$$
We wish to solve for the time when the Value of the account exceeds the total amount invested, i.e. the value $n$ that satisfies

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$$s_{\bar{n}|i} \geq \frac{100}{90}n$$

$$1.0075^n \geq 1 + 0.00833n$$
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\[
90 \cdot \left[ 1 + 1.0075 + \ldots + 1.0075^{n-1} \right] \geq 100n
\]

\[
s_{ni} \geq \frac{100}{90}n
\]

\[
1.0075^n \geq 1 + 0.00833n
\]

Essentially, we are solving a fixed point equation for \( f(x) = 0 \), where

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\[
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\]

\[
s_{n|i} \geq \frac{100}{90} n
\]

\[
1.0075^n \geq 1 + 0.00833n
\]

Essentially, we are solving a fixed point equation for \( f(x) = 0 \), where

\[
f(x) = e^{kx} - 1 - 0.00833x
\]

\[
k = \ln (1.0075)
\]
Example 2.13

We wish to solve for the time when the Value of the account exceeds the total amount invested, i.e. the value \( n \) that satisfies

\[
90 \cdot \left[ 1 + 1.0075 + \ldots + 1.0075^{n-1} \right] \geq 100n
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s_{n|i} \geq \frac{100}{90} n
\]

Essentially, we are solving a fixed point equation for \( f(x) = 0 \), where

\[
f(x) = e^{kx} - 1 - 0.00833x
\]

\[
k = \ln (1.0075)
\]

In this case, plotting the above function of \( x \) gives an intercept of approximately 28.6. Hence, \( n = 29 \).
We now have a philosophy for pricing assets that bring a future stream of payments. Simply put, the value *today* of the asset is simply the Net Present Value, opr $NPV$, of that stream of payments, when adjusted for inflation or (compound) growth, if applicable.
Dividend Discount Model for Stock Shares

If a table, or number line of future payments looks like

<table>
<thead>
<tr>
<th>Week</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>0</td>
<td>$K$</td>
<td>$K \cdot (1 + r)$</td>
<td>$K \cdot (1 + r)^{n-1}$</td>
<td></td>
</tr>
</tbody>
</table>

then the NPV, valued with rate $i$, is

$$NPV = \frac{K}{1 + i} + \frac{K \cdot (1 + r)}{(1 + i)^2} + \ldots + \frac{K \cdot (1 + r)^{n-1}}{(1 + i)^n}$$

$$= \frac{K}{1 + i} \cdot \sum_{j=0}^{n-1} \left(\frac{1 + r}{1 + i}\right)^j = \frac{K}{1 + i} \cdot \frac{1 - \left(\frac{1+r}{1+i}\right)^n}{1 - \frac{1+r}{1+i}}$$

(65)
As $n \to \infty$, we approach a *perpetuity*, and

\[
NPV \to \frac{K}{1+i} \cdot \frac{1}{1 - \frac{1+r}{1+i}} = \frac{K}{i - r}
\]  

(66)

In words, we have a perpetuity that is *adjusted for the growth rate* $r$. We use this formula to price a stock or any other asset as a perpetuity with possible compound growth at rate $r$. 
Example 2.13

Stock $X$ pays dividend 50 per year with growth of 5% after the first year. John purchases $X$ at the theoretical price corresponding to an effective yield of 10%. After receiving the the $10^{th}$ dividend, John sells the stock for price $P$. If annual yield for John was 8%, what is the fair price for $P$?
Stock X pays dividend 50 per year with growth of 5% after the first year. John purchases X at the theoretical price corresponding to an effective yield of 10%. After receiving the the 10th dividend, John sells the stock for price $P$. If annual yield for John was 8%, what is the fair price for $P$?

The equation of value is

$$\frac{50}{0.10 - 0.05} = 1000 = \frac{50}{1.08} \cdot \sum_{j=0}^{9} \left(\frac{1.05}{1.08}\right)^j + \frac{P}{1.08^{10}}$$
Example 2.13

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$$\Rightarrow P = 1000(1.08)^{10} - 50(1.08)^9 \cdot \sum_{j=0}^{9} \left( \frac{1.05}{1.08} \right)^j$$

$$= 1000(1.08)^{10} - 50(1.08)^9 \left( \frac{1 - (1.05/1.08)^{10}}{1 - (1.05/1.08)} \right)$$

$$= 1275.54.$$
Increasing Annuities

Consider now a payment schedule where the payment, at time $k$, was $K_m = m$. 

\[ X = \sum_{m=1}^{n} K_m \nu_m = \sum_{m=1}^{n} m \nu_m = \sum_{m=1}^{n} m (1 + i)^{m-1} \]

Then
\[ (1 + i) \cdot X = \sum_{m=1}^{n} m \nu_m - 1 = \sum_{m=1}^{n} m (1 + i)^{m-1} i \cdot X = 1 + \nu + \nu^2 + \ldots + \nu^{n-1} - n \nu^n \]

\[ = 1 - \nu - \nu^2 - \ldots - \nu^{n-1} \]

\[ X = \ddot{a}_n \]

(69)
Increasing Annuities

Consider now a payment schedule where the payment, at time $k$, was $K_m = m$. Now, the net present value is

$$X = \sum_{m=1}^{n} K_m \nu^m = \sum_{m=1}^{n} m \nu^m = \sum_{m=1}^{n} \frac{m}{(1 + i)^m}$$

(68)

Then
Consider now a payment schedule where the payment, at time $k$, was $K_m = m$.

Now, the net present value is

$$X = \sum_{m=1}^{n} K_m \nu^m = \sum_{m=1}^{n} m \nu^m = \sum_{m=1}^{n} \frac{m}{(1 + i)^m}$$

Then

$$(1 + i) \cdot X = \sum_{m=1}^{n} m \nu^{m-1} = \sum_{m=1}^{n} \frac{m}{(1 + i)^{m-1}}$$

$$i \cdot X = 1 + \nu + \nu^2 + \ldots + \nu^{n-1} - n\nu^n$$

$$= \frac{1 - \nu^n}{1 - \nu} - n\nu^n$$

$$X = \ddot{a}_{n|i} - n\nu^n = (Ia)_{n|i}$$
An increasing perpetuity is defined as

\[
\lim_{n \to \infty} (la)_{n|i} = \lim_{n \to \infty} \frac{1-n^n - n\nu^n}{1-\nu} i
\]

\[
= \frac{1 + i}{i^2}
\]

(70)

Notice that the level payment perpetuity of \( K_l \equiv 1 \) has present value of only \( \frac{1}{i} \). Since \( i \approx 0 \), we have \( \frac{1}{i^2} \gg \frac{1}{i} \).
An increasing perpetuity is defined as

\[
\lim_{n \to \infty} (la)_{ni} = \lim_{n \to \infty} \frac{\frac{1-\nu^n}{1-\nu} - n\nu^n}{i} = \frac{1 + i}{i^2}
\]  

(70)

Notice that the level payment perpetuity of \( K_l \equiv 1 \) has present value of only \( \frac{1}{i} \). Since \( i \approx 0 \), we have \( \frac{1}{i^2} >> \frac{1}{i} \).

HW: apply same analysis for decreasing annuities, with \( K_m = n - m + 1 \), where \( m \in \{1, 2, .., n\} \)
Consider perpetuities $X$ and $Y$ with payment

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>..n..</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment in</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>..n..</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>..2k−1</th>
<th>2k..</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment in</td>
<td>q</td>
<td>q</td>
<td>2q</td>
<td>2q</td>
<td>..kq</td>
<td>kq..</td>
</tr>
</tbody>
</table>

at the end of each year. It is known that $NPV(X) = NPV(Y)$ when valued at annual interest rate $i = 0.10$. Solve for $q$. 
Example 2.22

If we take $\nu_{eq} := \nu^2 = \left(\frac{1}{1.1}\right)^2 = \frac{1}{1.21}$, and so $i_{eq} = 0.21$. It follows that

$$NPV(X) = \frac{1 + i}{i^2} = \frac{1.1}{0.01} = 110$$

$$NPV(Y) = q\nu + q\nu^2 + 2q\nu^3 + 2q\nu^4 + 3q\nu^5 + 3q\nu^6 + \ldots$$

$$= q \cdot (\nu + 2\nu^3 + 3\nu^5 + \ldots) + q \cdot (\nu^2 + 2\nu^4 + 3\nu^6 + \ldots)$$

$$= \left(q + \frac{q}{\nu}\right) \cdot \left(\nu^2 + 2\nu^4 + 3\nu^6 + \ldots\right)$$

$$= \left(q + \frac{q}{\nu}\right) \cdot \left(\nu_{eq} + 2\nu_{eq}^2 + 3\nu_{eq}^3 + \ldots\right)$$

$$= 2.1 \cdot q \cdot \frac{1 + i_{eq}}{i_{eq}^2} = 2.1 \cdot q \cdot \frac{1.21}{0.21^2} = 57.62q$$
Example 2.22

If we take \( \nu_{eq} := \nu^2 = \left(\frac{1}{1.1}\right)^2 = \frac{1}{1.21} \), and so \( i_{eq} = 0.21 \). It follows that

\[
NPV(X) = \frac{1 + i}{i^2} = \frac{1.1}{0.01} = 110
\]

\[
NPV(Y) = q\nu + q\nu^2 + 2q\nu^3 + 2q\nu^4 + 3q\nu^5 + 3q\nu^6 + \ldots
\]

\[
= q \cdot (\nu + 2\nu^3 + 3\nu^5 + \ldots) + q \cdot (\nu^2 + 2\nu^4 + 3\nu^6 + \ldots)
\]

\[
= \left( q + \frac{q}{\nu} \right) \cdot (\nu^2 + 2\nu^4 + 3\nu^6 + \ldots)
\]

\[
= \left( q + \frac{q}{\nu} \right) \cdot (\nu_{eq} + 2\nu_{eq}^2 + 3\nu_{eq}^3 + \ldots)
\]

\[
= 2.1 \cdot q \cdot \frac{1 + i_{eq}}{i_{eq}^2} = 2.1 \cdot q \cdot \frac{1.21}{0.21^2} = 57.62q
\]

\[
q = 1.91
\]
What if the payments $K_m$ were just discrete samples at time $t_m$ of a continuous payment scheme? For example, the payment at time $t = t_1$ was $K_i = h(t_i)\, dt$. 

Then all payments must be summed to obtain the NPV and AccValue:

$$NPV = \int_{n_0}^{n_1} (1 + i)^t h(t) \, dt$$

$$AccVal = \int_{n_0}^{n_1} (1 + i)^{n - t} h(t) \, dt$$

(72)

A special case is when $h(t) = t$. Then

$$\bar{I}_{\bar{a}} = \int_{n_0}^{n_1} (1 + i)^t t \, dt$$

$$\bar{I}_{\bar{s}} = \int_{n_0}^{n_1} (1 + i)^{n - t} t \, dt$$

(73)
What if the payments $K_m$ were just discrete samples at time $t_m$ of a continuous payment scheme? For example, the payment at time $t = t_1$ was $K_i = h(t_i)dt$.

Then all payments must be summed to obtain the NPV and AccValue:

$$NPV = \int_0^n \frac{1}{(1 + i)^t} h(t) dt$$

$$AccVal = \int_0^n (1 + i)^{n-t} h(t) dt$$

A special case is when $h(t) = t$. Then

$$(\bar{\bar{a}})_{\bar{m}_i} = \int_0^n \frac{1}{(1 + i)^t} t dt$$

$$(\bar{\bar{s}})_{\bar{m}_i} = \int_0^n (1 + i)^{n-t} t dt$$
Integration by parts allows us to obtain

\[
(\bar{I}\bar{a})_{\bar{m}i} = \int_0^n \frac{1}{(1 + i)^t} t \, dt = \frac{\bar{a}_{\bar{m}i} - n\nu^n}{\delta}
\]

\[
(\bar{I}\bar{s})_{\bar{m}i} = \int_0^n (1 + i)^{n-t} t \, dt = \frac{\bar{s}_{\bar{m}i} - n}{\delta}
\]

where \(\delta := \ln(1 + i)\)

(74)

where \(\delta\) is the constant force of interest.
Integration by parts allows us to obtain

\[
(\bar{I}_a)_{\bar{m}\bar{i}} = \int_0^n \frac{1}{1+i} t dt = \frac{\bar{a}_{\bar{m}\bar{i}} - n\nu^n}{\delta}
\]

\[
(\bar{I}_s)_{\bar{m}\bar{i}} = \int_0^n (1+i)^{n-t} t dt = \frac{\bar{s}_{\bar{m}\bar{i}} - n}{\delta}
\]

where \(\delta := \ln(1+i)\)

If the force of interest varies, then

\[
(\bar{I}_a)_{\bar{m}\delta_\star} = \int_0^n \exp(-\int_0^t \delta_s ds) t dt
\]

\[
(\bar{I}_s)_{\bar{m}\delta_\star} = \int_0^n \exp(\int_t^n \delta_s ds) t dt
\]
Returning to the discrete setting, consider the following important theorem:

**Theorem**

Assume that an annuity $X$ pays amount $K_j > 0$ at time $t_j$ for all times $0 < t_1 < t_2 \ldots < t_n$. Suppose that you are given $L > 0$. Then there exists a unique $i > -1$ such that $NPV(X)(i) := \sum_{l=1}^{n} \frac{K_l}{(1+i)^{t_l}} = L$ under rate $i$.

Without this theorem, we would not be able to uniquely price returns on assets.
Proof.

It follows that since \( \{ (K_l, t_l) \}_{l=1}^n \subset \mathbb{R}_+^2 \), we have \( NPV(X)'(i) < 0 \). This also follows from financial reasoning as a higher rate \( i \) would demand a lower up front payment \( L \). Since

\[
\lim_{i \to \infty} NPV(X)(i) = 0 \\
\lim_{i \to -1} NPV(X)(i) = \infty
\]

(76)

it follows from the IVT that there exists a unique \( i \) (due to strict decreasing nature of \( NPV(X)(i) \)) such that \( NPV(X)(i) = L \)
Assume a loan amount $L$ for a term of $n$ years, and a total value $M$ of all payments after the $n$ years. Then it must be that for level payments $K$ the yield rate $i$ satisfies

$$L = \frac{M}{(1 + i)^n} = \frac{K}{1 + i} + \frac{K}{(1 + i)^2} + \cdots + \frac{K}{(1 + i)^n} = K \cdot a_{\overline{m}|i}$$

(77)

On a loan, the Internal Rate of Return, or IRR, is the rate of interest for which the loan amount upront is equal in value to the NPV of all loan payments. Also known as the loan rate. Sometimes, the entity that loans $L$ can reinvest these loan repayments at a higher rate.
Ex. 2.25

Smith owns a $10000$ savings bond that pays $i_{Bond}^{(12)} = 0.06$. Upon receipt of an interest payment, he immediately deposits it into an account earning interest, payable monthly, at a rate of $i_X^{(12)} = 0.12$. Find the accumulated value of this account just after the $12^{th}$, $24^{th}$, and $36^{th}$ deposit. In each case, find the average annual yield $i_{avg}^{(12)}$ based on his initial investment of $10000$. Assume that the savings bond may be cashed in any time for $10000$. 
The interest paid each month is at a rate of \( \frac{i^{(12)}}{12} = \frac{0.06}{12} = 0.5\% \). On an investment of 10000, this means payments of 50 each month. The value of his reinvestment of these payments at rate 0.01 is

<table>
<thead>
<tr>
<th>Month</th>
<th>12</th>
<th>24</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acc Val</td>
<td>(50 \cdot s_{12|0.01} )</td>
<td>(50 \cdot s_{24|0.01} )</td>
<td>(50 \cdot s_{36|0.01} )</td>
</tr>
<tr>
<td></td>
<td>634.13</td>
<td>1348.67</td>
<td>2153.84</td>
</tr>
</tbody>
</table>
After 12 months, the equivalent rate $j$ needed to accumulate 
$10000 + 50 \cdot s_{12|0.01} = 10634.13$ satisfies

$$10000 \left( 1 + \frac{i_{avg}^{(12)}}{12} \right)^{12} = 10634.13$$

$\Rightarrow \quad i_{avg}^{(12)} = 0.0616$

Do this for the other cases.
In general, we have

\[
10000 \left( 1 + \frac{i_{avg}(n)}{12} \right)^n = 10000 \left[ 1 + 0.005 s_{0.01} \right]
\]

\[
= 10000 \left[ 1 + \frac{(1.01)^n - 1}{0.01} \right]
\]

\[
\Rightarrow i_{avg}(n) = 12 \left[ \left( \frac{1}{2} + \frac{1}{2} (1.01)^n \right)^{\frac{1}{n}} - 1 \right]
\]
Sinking Fund Method of Valuation

It often happens that interest payments returned to an investor at rate $i$ can only be reinvested at "market" rate $j$. If $j \neq i$, then the yield rate to value the investment is also $\neq i$ and we consider another way to determine the value. Instead of using the present value method, we now consider the value at the end of the term $n$ of the investment.

When an investment is made, it can be considered a loan (principal) to the person receiving the money up front. By the end of the term, the entire principal has been repaid and the investor has enjoyed (consumed) periodic interest payments for making the initial investment.

Consider, for example, the fair purchase price $P$ of an annuity with level payments $K$ over $n$ periods.
Sinking Fund Method of Valuation

- Receive periodic payments $K$ in return for initial investment $P$.
- The interest payment is $P \cdot i$, received every unit of time over interval of length $n$.
- These interest payments are consumed, and the excess $K - P \cdot i$ every period is reinvested at a rate $j$ into a *sinking fund*.
- The fair value $P$ is now determined via the recursive equation that says investor recovers $P$ at end of term, and this is equivalent to the amount accumulated by reinvesting the excess $K - P \cdot i$ per period:

$$P = (K - P \cdot i) s_{\bar{n}|j}$$
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\[
P = (K - P \cdot i) s_{\bar{n}|j}
\]

\[
P = \frac{K s_{\bar{n}|j}}{1 + i \cdot s_{\bar{n}|j}}
\]
Sinking Fund Method of Valuation

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$$P = (K - P \cdot i) s_{\bar{n}|j}$$

$$P = \frac{Ks_{\bar{n}|j}}{1 + i \cdot s_{\bar{n}|j}} = \frac{Ks_{\bar{n}|j}}{1 + \frac{i}{j} \cdot ((1 + j)^n - 1)}$$
Receive periodic payments $K$ in return for initial investment $P$.

The interest payment is $P \cdot i$, received every unit of time over interval of length $n$.

These interest payments are consumed, and the excess $K - P \cdot i$ every period is reinvested at a rate $j$ into a sinking fund.

The fair value $P$ is now determined via the recursive equation that says investor recovers $P$ at end of term, and this is equivalent to the amount accumulated by reinvesting the excess $K - P \cdot i$ per period:

$$P = (K - P \cdot i) s_{n|j}$$

$$P = \frac{K s_{n|j}}{1 + i \cdot s_{n|j}} = \frac{K s_{n|j}}{1 + \frac{i}{j} \cdot ((1 + j)^n - 1)}$$

$$\lim_{j \to i} P = \frac{K s_{n|i}}{1 + (1 + i)^n - 1} = Ka_{n|i}$$

(80)
When repaying a loan \( L \) with periodic payments \( K_m \) at time \( m \), we have the equation of value

\[
L = \sum_{m=1}^{n} \frac{K_m}{(1 + i)^m} \tag{81}
\]

In this method, we track the *outstanding balance* on a loan, denoted by \( OB(i) \) or sometimes \( OB_i \).
Amortization

\[ OB_0 = L \]
Amortization

\[ OB_0 = L \]
\[ OB_1 = L \cdot (1 + i) - K_1 \]
Amortization

\[
\begin{align*}
OB_0 & = L \\
OB_1 & = L \cdot (1 + i) - K_1 \\
OB_2 & = OB_1 \cdot (1 + i) - K_2
\end{align*}
\]
Amortization

\[ OB_0 = L \]
\[ OB_1 = L \cdot (1 + i) - K_1 \]
\[ OB_2 = OB_1 \cdot (1 + i) - K_2 \]
\[ OB_{t+1} = OB_t \cdot (1 + i) - K_{t+1} = OB_t - PR_{t+1} \]
\[ PR_{t+1} = K_{t+1} - i \cdot OB_t \]
\[ I_{t+1} = i \cdot OB_t \]
Amortization

\[ OB_0 = L \]
\[ OB_1 = L \cdot (1 + i) - K_1 \]
\[ OB_2 = OB_1 \cdot (1 + i) - K_2 \]
\[ OB_{t+1} = OB_t \cdot (1 + i) - K_{t+1} = OB_t - PR_{t+1} \]  \( (82) \)
\[ PR_{t+1} = K_{t+1} - i \cdot OB_t \]
\[ l_{t+1} = i \cdot OB_t \]
\[ OB_n = 0 = OB_{n-1} \cdot (1 + i) - K_n \]

Also note that the total interest and cash paid out are, respectively,
\[ \sum_{t=1}^{n} I_t \text{ and } \sum_{t=1}^{n} K_t - \sum_{t=1}^{n} l_t. \]
Retrospective and Prospective Forms

At time $t$, if we look backwards to balance out how the original loan amount has grown and subtract from that our total payments, then

$$OB_t = L \cdot (1 + i)^t - \sum_{m=1}^{t} K_m \cdot (1 + i)^{t-m}$$

(83)

If we take the NPV viewpoint, then the outstanding balance today is simply the present value of all future payments until the loan is paid off:

$$OB_t = \sum_{m=t+1}^{n} \frac{K_m}{(1 + i)^{m-t}}$$

(84)
A loan of 3000 at an effective quarterly rate of $j = 0.02$ is amortized by means of 12 quarterly payments, beginning one quarter after the loan is made. Each payment consists of a principal repayment of 250 plus interest due on the previous quarters outstanding balance. Construct the amortization schedule.
In this case, we have level principal repayments, and so $PR_t = 250$ for all $0 \leq t \leq 12$. Since $I_{t+1} = 0.02 \cdot OB_t$, it follows that

$$OB_{t+1} = OB_t - 250$$

$$\Rightarrow OB_t = 3000 - 250t$$

$$I_t = 0.02 \cdot (3000 - 250 \cdot (t - 1)) = 65 - 5t$$

$$K_t = I_t + PR_t = 250 + 65 - 5t = 315 - 5t$$

(85)
In this case, which is common, the payments are held constant, say at $K$, but the proportion of interest paid versus principal paid varies with time.

Examples such as mortgages, car payments, etc..

Can use retrospective method to value outstanding balance at time $t$. 
Here, the outstanding balance is of course $OB_t = K \cdot a_{n-t|i}$, and the total cash paid out by time $t$ is $\sum_{m=1}^{t} K_t = K \cdot t$. It follows that, with $\delta = \ln (1 + i)$

$$I_t = i \cdot OB_{t-1} = i \cdot K \cdot a_{n-t+1|i} = K \cdot (1 - (1 + i)^{t-n-1})$$

$$PR_t = K_t - I_t = \frac{K}{(1 + i)^{n+1-t}}$$

$$OB_t = \frac{K}{i} \cdot (1 - e^{-\delta \cdot (n-t)})$$

(86)
Even though the outstanding balance is paid off only at the end of the term of the loan, we can still interpolate and say that the outstanding balance decreases as

\[ OB_t = L - \frac{L}{s_{\bar{n}|j}} \cdot s_{t|j} \]  

where the balance decreases by the amount paid into the account earning at a rate \( j \).
Note that we can also calculate the net interest payment each month via

\[ K_t = L \cdot i + \frac{L}{s_{nj}} \]

\[ PR_t = OB_{t-1} - OB_t = L \cdot \left( \frac{s_{t|j} - s_{t-1|j}}{s_{nj}} \right) \]  \hspace{1cm} (88)

\[ I_t = K_t - PR_t = L \cdot \left( i - j \cdot \frac{s_{t-1|j}}{s_{nj}} \right) \]
Ronaldo borrows $L$ for 10 years at an annual effective interest rate of $100 \cdot i\%$. At the end of each year, he pays the interest on the loan and deposits the level amount necessary to repay the principal to a sinking fund earning an annual effective interest rate of $100 \cdot j\%$. The total payments made by Ronaldo over the 10-year period is $X$. Calculate $X$. 


Answer:

- Annual interest payment $= L \cdot i$
- Annual sinking fund deposit is $\frac{L}{s_{10|j}}$
- Total annual payment $= L \cdot i + \frac{L}{s_{10|j}}$
- Total of all payments $= 10 \cdot \left( L \cdot i + \frac{L}{s_{10|j}} \right)$
Consider the following scenario: An investor makes a loan at interest rate \( i \), with only interest payments for the term of the loan, and then a lump sum \( L \) at the end of term. The investor then sells the loan to a speculator, who values the expected cash flows at a rate \( j \). Then the value the speculator puts on the investment is

\[
A = \frac{L}{(1 + j)^n} + L \cdot i \cdot \frac{a_{n|j}}{(1 + j)^n} = \frac{L}{(1 + j)^n} + \frac{i}{j} \cdot \left( L - \frac{L}{(1 + j)^n} \right) \tag{89}
\]
Makeham’s Formula

This idea generalizes to the case where the principal $L$ is paid over a series of lump sum payments $L = \sum_{m=1}^{n} L_m$ and periodic repayments $K = \sum_{m=1}^{n} K_m$. 
This idea generalizes to the case where the principal $L$ is paid over a series of lump sum payments $L = \sum_{m=1}^{n} L_m$ and periodic repayments $K = \sum_{m=1}^{n} K_m$.

Each of these lump sum payments can be viewed as corresponding to an individual loan:

$$A_s = \frac{L_s}{(1 + j)^{t_s}} + L_s \cdot i \cdot a_{ts|j} = \frac{L_s}{(1 + j)^{t_s}} + \frac{i}{j} \left( L_s - \frac{L_s}{(1 + j)^{t_s}} \right) \quad (90)$$
Makeham’s Formula

We can find the total value now by summing up the individual loan values

\[
A = \sum_{s=1}^{n} A_s = \sum_{s=1}^{n} \frac{L_s}{(1 + j)^{t_s}} + \frac{i}{j} \cdot \left( L_s - \frac{L_s}{(1 + j)^{t_s}} \right) \\
= \sum_{s=1}^{n} \frac{L_s}{(1 + j)^{t_s}} + \frac{i}{j} \cdot \left( \sum_{s=1}^{n} L_s - \sum_{s=1}^{n} \frac{L_s}{(1 + j)^{t_s}} \right) \\
= K + \frac{i}{j} \cdot (L - K)
\]
A loan \( L \) is being repaid with \( n \) annual level payments of \( K \) each. With the \( m^{th} \) payment, \( m < n \), the borrower pays an extra amount \( A \), and then agrees to repay the remaining balance over \( l \) years with a revised annual payment. The effective rate of interest is \( 100 \cdot i\% \). Calculate the amount of the revised annual payment.
With $n - m$ yearly payments left right before the extra repayment, the remaining balance is $K \cdot a_{(n-m)|i}$. Right after the extra payment $A$, the remaining balance is $K \cdot a_{(n-m)|i} - A$. Hence, the new payment over the new 10–year schedule is

$$K_{\text{new}} = \frac{K \cdot a_{(n-m)|i} - A}{a_{\parallel|i}}$$

(92)
A Bond is a financial contract where an up front payment is exchanged for coupon payments (periodic interest paid to the debt-holder) plus a face amount the maturity date (end of term). The Bond is valued using a yield rate used to value future cash flows. The notation used with bond valuation is

- $j =$ the Yield rate per period
- $F =$ the Face amount
- $r =$ the Coupon rate
- $C =$ the redemption amount, usually equal to the face value
- $N =$ the number of coupon payments until maturity
- $P =$ the price of the bond, where
The bond can now be valued as

\[
P = \frac{F}{(1+j)^n} + F \cdot r \cdot \left( \frac{1}{1+j} + \ldots + \frac{1}{(1+j)^n} \right)
\]

\[
= F \nu^n + F \cdot r \cdot a_{\overline{n}|j}
\]

\[
= F + F \cdot (r - j) \cdot a_{\overline{n}|j}
\]

\[
= F \nu^n + \frac{r}{j} \cdot (F - F \nu^n)
\]  

(93)

Note that Bond prices are usually listed, and the investor determines yield rate from listed price. To do so, must invert our Bond pricing formula. This requires some calculation.
Example 4.1

Price the following bond with face 100:

- Issue Date: March 1, 2004
- Maturity Date: Feb 28, 2006
- $r = 1.625\%$
- $j = 1.675\%$
- Term is 2 years (4 six month periods)
Example 4.1

Price the following bond with face 100:

- Issue Date: March 1, 2004
- Maturity Date: Feb 28, 2006
- \( r = 1.625\% \)
- \( j = 1.675\% \)
- Term is 2 years (4 six month periods)

\[
P = 100 \cdot 0.5 \cdot 0.01625 \cdot \left( \frac{1}{1.008375} + \ldots + \frac{1}{1.008375^4} \right) + \frac{100}{1.008375^4} = 99.02
\]

(94)
At time $t$, we define $P_t$ to be the price of the bond, *just after* the coupon payment. This is equal to the NPV of the remaining payments:

$$P_t = \frac{F}{(1+j)^{n-t}} + F \cdot r \cdot a_{n-t|j}$$  \hspace{1cm} (95)

Now, since the initial price $P$ satisfies the relation $P - F = F \cdot (r - j) a_{\overline{n}|j}$, we say that a bond is bought at *premium* if $r > j$, at *par* if $r = j$, and at a *discount* if $r < j$. 
Let

\[ u := \frac{\text{the number of days since last coupon payment}}{\text{number of days in coupon period}} \]  \hspace{1cm} (96)

Then \( P_t = P_m \cdot (1 + j)^u - u \cdot F \cdot r \), where \( m \) is the number of coupon payments left and \( P_m \cdot (1 + j)^u \) is known as the price plus accrued.
Amortization of Bond

Sometimes need to determine the Book Value of a bond for tax or investor purposes. The Book Value at time $t$ is the outstanding balance at time $t$ right after the coupon payment. This is the present value of the remaining cash payments of that the bond holder is entitled to receive:

$$P_t = OB_t = F + F \cdot (r - j) \cdot a_{n-t|j} \quad (97)$$
Amortization of Bond

We can use the idea that we are loaning money to bond-issuer, and that we price this as a mortgage. So,

\[ P_t = OB_t = F + F \cdot (r - j) \cdot a_{n-t}^j \]

\[ K_t = F \cdot r \text{ for } t \in \{1, 2, 3, \ldots, n - 1\} \]

\[ K_n = F \cdot r + F \]

\[ I_t = K_t - PR_t \]

\[ PR_t = K_t - (OB_{t-1} - OB_t) \]

\[ = F \cdot (r - j) \cdot \nu^{n-t+1} \]
Lemma

Consider a bond with face $F$, coupon rate $r$, and term $T = 2n$, $n \in \mathbb{N}$, purchased by $A$ for a price $P_A$. Just after the $n^{th}$ coupon, $A$ sells the bond to $B$ who desires a yield $j$ and correspondingly pays $P_B$. If $P_B > P_A$ and

- $j < r$ or
- $j > r$ and $P_B > \frac{F + P_A}{2}$

then $i > j$, where $i$ is the non-zero yield rate calculated on $A$'s investment.
Bond Pricing Lemma: Proof 1

We prove each case separately, both of which reveal something about the situation described above.

**Case 1** First, if \( j < r \)

\[
P_A = F \cdot r \cdot a_{n \mid j} + \frac{P_B}{(1 + i)^n} \]

\[
P_B = F + F \cdot (r - j) \cdot a_{n \mid j} \]

We can see that this implies

\[
\frac{Fr}{i} < P_A < P_B = F + F \cdot (r - j) \cdot a_{n \mid j} < F + F \cdot \frac{r - j}{j} = Fr \]

and so \( i > j \)

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Proof.

**Case 2** If $j > r$ then by Makeham’s formula $F > P_B > P_A$ and

$$P_A = F \cdot r_{a|n|i} + \frac{P_B}{(1 + i)^n}$$

$$= F \cdot r_{a|n|i} + \frac{F}{(1 + i)^n} + \left[ \frac{P_B - F}{(1 + i)^n} \right]$$

(101)

$$P_B = F \cdot r_{a|n|j} + \frac{F}{(1 + j)^n}$$

For the sake of contradiction, assume $i \leq j$. Then it follows by our assumptions that

$$F - P_B < P_B - P_A \leq \frac{F - P_B}{(1 + i)^n}$$

(102)

which is impossible.
Callable Bonds: Bond can be recalled by issuer over a window of redemption dates. Price investor pays is the minimum of these possible prices. So, if the bond was purchased at a premium, then price at earliest possible date. If purchased at a discount, price at latest possible redemption date.
Serial Bonds: Face amount $F$ is redeemed over a collection of dates \( \{t_1, \ldots, t_n\} \) with staggered payments $F_1 + \ldots + F_n = F$, coupon rates $r_1, \ldots, r_n$, tailored to yield rates $j_1, \ldots, j_n$. One can view this as the sum of $m$ bonds with the corresponding payments, and can use Makehams formula to price this

\[
P = \sum_{m=1}^{n} P_m
\]

\[
P_t = \frac{F_t}{(1 + j)^{n_t}} + \frac{r_t}{j_t} \cdot \left( F_t - \frac{F_t}{(1 + j_t)^{n_t}} \right)
\]

(103)
Measuring The Rate of Return Of An Investment

- Yield rate can be derived from cash-flows from an investment
- When presented with more than one investment opportunity, would like to be able to quantitatively measure their value to the investor
- There are many ways to do this some work in situations that others dont
In an investment, at any time may have cash flows out and in say payments to buy a license for a taxicab, with monthly maintenance fees versus fare payments in.

Notation is such that $A_k$ is a payment received and $B_k$ is a payment made out, each at the same time $t_k$. The net payment is thus $C_k$.

$$f(j) := \sum_{k=1}^{n} \frac{C_k}{(1+j)^{t_k}}$$  \hspace{1cm} (104)

The **Internal Rate of Return** $j$ is one that balances out all net payments so that their net present value is 0.
Example 5.1

Smith buys 1000 shares of stock at 5.00 per share and pays a commission of 2%. Six months later he receives a cash-dividend of 0.20 per share, which he immediately reinvests commission free in shares at a price of 4.00 per share. Six months after that he buys another 500 shares at a price 4.50 per share, along with a commission of 2%. Six months after that he receives another cash dividend of 0.25 per share and sells his existing shares at 5.00 per share, again paying a 2% commission. Find Smiths internal rate of return for the entire transaction in the form $i^{(2)}$
Example 5.1

Let 0 represent the time of the original share purchase, \( t = 1 \) at 6 months, \( t = 2 \) at 12 months and \( t = 3 \) at 18 months after the original purchase. Then

\[
(A_0, B_0) = (0, 5000 + [0.02 \times 5000]) = (0, 5100)
\]
\[
(A_1, B_1) = (200, 200)
\]
\[
(A_2, B_2) = (0, [1.02 \times 4.50 \times 500]) = (0, 2295)
\]
\[
(A_3, B_3) = ([0.25 + 0.98 \times 5] \times 1550, 0)
\]
\[
= (7982.50, 0)
\]

\[
\therefore (C_0, C_1, C_2, C_3) = (-5100, 0, -2295, 7982.5)
\]

\[
\Rightarrow 0 = -5100 + 0 \cdot \nu - 2295\nu^2 + 7982.5\nu^3
\]

Solving this polynomial, we obtain

\[
\nu = \frac{1}{1 + j} \Rightarrow i^{(2)} = 2j = 0.0649
\]
If $C_0 < 0$ and $C_j > 0$ for all $j \in \{1, 2, \ldots, n\}$, then by the Intermediate Value theorem and simple calculus, we have a unique solution for the internal return rate $j$. (i.e. Mortgages or other loans paid off by periodic payments)

BUT, we can have situations where the above isn't satisfied, and we compute more than one solution for $j$, or worse, no real solution.

What can we use to measure the value of an investment in this case?
Assume that among all possible investment alternatives, all have same measure of risk and \textbf{fixed} interest rate \( i \) that investor proposes (Utility theory!)

In this incomplete case, use \( i \), also known as \textit{cost of capital}, to find present value of cash flows of each investment on the table

The one with the largest present value, given investors rate \( i \), is most preferable. In symbols, for investments \( a, b, c \), with NPV \( f_a(i), f_b(i), f_c(i) \), choose the biggest

\textbf{HW:} Can NPV curves ever intersect for different values of \( i \)?

Of course, if \( i = j = \text{yield rate} \), then \( NPV = 0 \)
Again, if the investor **proposes his own interest rate** \( i \), then we can use another measure, the Profitability Index:

\[
I = \frac{PV \text{ [ Cash Flows In]}}{PV \text{ [ Cash Flows Out]}} \tag{107}
\]

Note that if \( i = j = IRR \), then \( I = 1 \) by design, as yield rate is what we use to balance out flows with in flows.
Profitability Index: Example

Compare

- (a) lending 1000 and being repaid 250 per year for 5 years with \( i = 0.05 \) and
- (b) lending 1000 and being repaid 140 per year for 10 years with \( i = 0.05 \)
Profitability Index: Example

Compare

- (a) lending 1000 and being repaid 250 per year for 5 years with $i = 0.05$ and
- (b) lending 1000 and being repaid 140 per year for 10 years with $i = 0.05$

We compute

$$I_a = \frac{250a_{5|0.05}}{1000} = 1.0824$$

$$I_b = \frac{140a_{10|0.05}}{1000} = 1.0810$$
Goal: Find $j$ such that Accumulated Value at end of project of payments out at rate $j$ matches Accumulated Value of payments received at investor’s cost of capital $i$.

Notation:

- $A_k$ is the payment in at time $k$
- $B_k$ is the payment out at time $k$
Goal: Find $j$ such that Accumulated Value at end of project of payments out at rate $j$ matches Accumulated Value of payments received at investor’s cost of capital $i$.

Notation:

- $A_k$ is the payment in at time $k$
- $B_k$ is the payment out at time $k$

Then $j$ solves the equation

$$\sum_{n=0}^{N} A_n \cdot (1 + i)^n = \sum_{m=0}^{N} B_m \cdot (1 + j)^m$$  \hspace{1cm} (109)
Compare

- (a) lending 1000 and being repaid 250 per year for 5 years with
  \( i = 0.05 \) and
- (b) lending 1000 and being repaid 140 per year for 10 years with
  \( i = 0.05 \)
MIRR: Example

Compare

- (a) lending 1000 and being repaid 250 per year for 5 years with \( i = 0.05 \) and
- (b) lending 1000 and being repaid 140 per year for 10 years with \( i = 0.05 \)

We compute

\[
1000(1 + j_a)^5 = 250s_{5|0.05} = 1381.41
\]

\Rightarrow j_a = 0.0668

\[
1000(1 + j_b)^{10} = 140s_{10|0.05} = 1760.90
\]

\Rightarrow j_b = 0.0582

(110)
Project Return Rate

There are times during a project where investor is borrower and times when she is lender. Cost of financing at rate $i$, and project return rate is $j$ when she is net lender. Solving $j$ in terms of $i$ would give minimum return rate needed for project to break even if set NPV to 0 at end of project.

Goal: Set up an equation of value such that Net Project Value = 0 when $t = N$. If the value at $t = n$ is $> 0$, then earn at rate $j$. If balance $< 0$, then ”earn” (pay) at rate $i$ (cost of capital). As an example, consider

\[
S_0 = 1 \rightarrow S_{1-} = (1 + j) \\
S_1 = 1 + j - 2.3 \rightarrow S_{2-} = (1 + j - 2.3)(1 + i) \\
S_2 = (1 + j - 2.3)(1 + i) + 1.33 = 0
\]

\[\Rightarrow j = 1.3 - \frac{1.33}{1 + i}\]
Goal: Calculate measure of return over 1 year.
Goal: Calculate measure of return over 1 year. Use Simple Interest for fraction of years. Define *interest gained* as the difference between the total amount paid out during year to term and the total amount paid in during year to term.

\[
id_{\text{dollar weighted}} = \frac{\text{Total Interest Gained}}{\text{Average Amount on deposit during year}}
\] (112)
A Pension fund receives contributions and pays benefits from time to time. The fund began the year 2005 with a balance of 1,000,000. There were contributions to the fund of 200,000 at the end of February and again at the end of August. There was a benefit of 500,000 paid out of the fund at the end of October. The balance remaining at the start of the year 2006 was 1,100,000. Find the dollar weighted return on the fund, assuming each month is $\frac{1}{12}$ of a year.
Example 5.3

\[ 1,100,000 = 1,000,000(1 + i) + 200,000 \cdot \left( 1 + \frac{10}{12}i \right) \]
\[ + 200,000 \cdot \left( 1 + \frac{4}{12}i \right) - 500,000 \cdot \left( 1 + \frac{2}{12}i \right) \]

\[ T.I.G. = 1,100,000 + 500,000 - 1,000,000 \]
\[ - 200,000 - 200,000 = 200,000 \]

\[ A.A.o.d.d.y = 1,000,000 + 200,000 \cdot \frac{10}{12} \]
\[ + 200,000 \cdot \frac{4}{12} - 500,000 \cdot \frac{2}{12} = 1,150,000 \]

\[ \Rightarrow i = \frac{200,000}{1,150,000} = 0.1739 \]
Even More Methods of Valuation

- **Time Weighted Rate of Return**: Compound over fraction of years, but the time length doesn't matter.
- **Portfolio Method**: Contributions to a fund, once opened, are segregated from initial investment and accrue interest at a different rate for a fixed term.
- **Interest Preference Rates for Borrowing and Lending**: If accumulated amount falls below zero, pay interest at a different rate than interest credited on a positive balance. This is useful to compare two alternatives when one may have a negative balance for some time.
Continuous Withdrawal and Investment

Begin with the equation for accumulated value

\[ F(t_2) = F(t_1) \cdot (1+i)^{t_2-t_1} + \sum_{k=1}^{n} c_{t_k^*} (1+i)^{t_2-t_k^*} + \int_{t_1}^{t_2} \bar{c}(t)(1+i)^{t_2-t} \, dt \] (114)
Continuous Withdrawal and Investment

Begin with the equation for accumulated value

\[ F(t_2) = F(t_1) \cdot (1+i)^{t_2-t_1} + \sum_{k=1}^{n} c_{t_k^*}(1+i)^{t_2-t_k^*} + \int_{t_1}^{t_2} \bar{c}(t)(1+i)^{t_2-t} \, dt \quad \text{(114)} \]

As an example, consider \( t_1 = 0, \ t_2 = 1, \ t_k^* = \frac{k}{n}, \ c_{t_k^*} = 1, \ \bar{c}(t) = N - n \). Then

\[ F(1) = F(0) \cdot (1+i) + \sum_{k=1}^{n} (1+i)^{1-\frac{k}{n}} + (N-n) \cdot \int_{0}^{1} (1+i)^{1-t} \, dt \]

\[ = F(0) \cdot (1+i) + \frac{i}{(1+i)^{\frac{1}{n}} - 1} + \frac{(N-n) \cdot i}{\ln (1+i)} \quad \text{(115)} \]
Continuous Withdrawal and Investment

Begin with the equation for accumulated value

\[
F(t_2) = F(t_1) \cdot (1+i)^{t_2-t_1} + \sum_{k=1}^{n} c_{t_k^*} (1+i)^{t_2-t_k^*} + \int_{t_1}^{t_2} \bar{c}(t)(1+i)^{t_2-t} dt \tag{114}
\]

As an example, consider \( t_1 = 0, t_2 = 1, t_k^* = \frac{k}{n}, c_{t_k^*} = 1, \bar{c}(t) = N - n \).

Then

\[
F(1) = F(0) \cdot (1 + i) + \sum_{k=1}^{n} (1 + i)^{1 - \frac{k}{n}} + (N - n) \cdot \int_{0}^{1} (1 + i)^{1-t} dt
\]

\[
= F(0) \cdot (1 + i) + \frac{i}{(1 + i)^{\frac{1}{n}} - 1} + \frac{(N - n) \cdot i}{\ln (1 + i)} \tag{115}
\]

Now, given \( F(0), F(1), n, N \), solve for \( i \).
The relationship of the yield of a bond (or other fixed income product) to its time to maturity is the **Term Structure**.

The graph of this relationship is the Yield Curve.

Term Structure is dynamic; reflects new information gained as markets evolve.

A yield curve that corresponds to **Normal Term Structure** is one that sees yield increase when the time to maturity increases. When the yield decreases as term increases, we say the yield curve is **Inverted**, and if it is roughly constant, we say it is a **Flat** curve.
Bond Prices are linear. If we have a coupon paying bond, it must have the same value as the sum of the prices of each coupon as an individual zero coupon bond, as well as the lump sum paid at maturity.

If this does not hold, then we have observed a case of arbitrage. This means one could construct a portfolio that consists of purchasing individual pieces of a coupon bond while selling the equivalent whole bond, making a net profit with no risk involved.
Example 6.1

Suppose that the current term structure has the following yields on zero-coupon bonds, where all yields are nominal annual rates of interest compounded semi-annually.

<table>
<thead>
<tr>
<th>Term (Years)</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Coupon Bond Rate</td>
<td>8%</td>
<td>9%</td>
<td>10%</td>
<td>11%</td>
</tr>
</tbody>
</table>

Find the price per 100 face amount and yield to maturity of each of the following 2-year bonds (with semi-annual coupons):

- a.) Zero coupon bond
- b.) 5% annual coupon rate
- c.) 10% annual coupon rate
Example 6.1

- \( a. \) \( PV = \frac{100}{(1+\frac{0.11}{2})^4} = 80.72 \), which implies \( j^{(2)} = 0.11 \)

- \( b. \)

\[
PV = \frac{2.5}{1.04} + \frac{2.5}{1.045^2} + \frac{2.5}{1.05^3} + \frac{102.5}{1.055^4} = 89.59
\]

\[
= \frac{2.5}{1 + j^{(2)}/2} + \frac{2.5}{(1 + j^{(2)}/2)^2} + \frac{2.5}{(1 + j^{(2)}/2)^3} + \frac{102.5}{(1 + j^{(2)}/2)^4}
\]

\( \Rightarrow j^{(2)} = 0.109354 \)
Example 6.1

• c.) Similarly,

\[
PV = \frac{5}{1.04} + \frac{5}{1.045^2} + \frac{5}{1.05^3} + \frac{105}{1.055^4}
\]

\[
= 98.46
\]

\[
= \frac{5}{1 + \frac{j^{(2)}}{2}} + \frac{5}{\left(1 + \frac{j^{(2)}}{2}\right)^2} + \frac{5}{\left(1 + \frac{j^{(2)}}{2}\right)^3} + \frac{105}{\left(1 + \frac{j^{(2)}}{2}\right)^4}
\]

(117)

and so \( j^{(2)} = 0.108775 \)
Spot Rates

- The Spot Rate $s_t$ is the effective annual interest rate for a zero coupon bond maturing $t$ years from now.
- There is a relationship between the spot rate and the yield to maturity of a bond.
- If term structure of spot rates is increasing, then yield for bonds with same maturity decreases as coupons increase.
Spot Rates and Yield Rates

The general equation of value here to determine the yield rate $y_r$ on a bond with face 1 is

$$PV = r \cdot \left( \frac{1}{1 + s_1} + \frac{1}{(1 + s_2)^2} + \ldots + \frac{1}{(1 + s_n)^n} \right) + \frac{1}{(1 + s_n)^n}$$

$$= r \cdot \left( \frac{1}{1 + y_r} + \frac{1}{(1 + y_r)^2} + \ldots + \frac{1}{(1 + y_r)^n} \right) + \frac{1}{(1 + y_r)^n} \tag{118}$$
Solving for $r$, we obtain

$$r = -\frac{\sum_{k=1}^{n} \frac{1}{(1+y_r)^k} - \sum_{k=1}^{n} \frac{1}{(1+s_n)^k}}{\sum_{k=1}^{n} \frac{1}{(1+y_r)^k} - \sum_{k=1}^{n} \frac{1}{(1+s_k)^k}}$$

(119)
Q: Is there anything we can say about $\frac{dy_r}{dr}$ given a normal term structure?
Q: Is there anything we can say about \( \frac{dy_r}{dr} \) given a normal term structure?

A: Implicit differentiation leads to

\[
\frac{dy_r}{dr} = \frac{1}{r} \cdot \frac{1}{(1+s_n)^n} - \frac{1}{(1+y_r)^n} - r \sum_{k=1}^{n} \frac{k}{(1+y_r)^{k+1}} + \frac{n}{(1+y_r)^{n+1}} < 0
\]

(120)

if we have a normal term structure as we expect \( s_n > y_r \) and so

\[
\frac{1}{(1+s_n)^n} < \frac{1}{(1+y_r)^n}.
\]
In keeping with Law of One Price (No Arbitrage), can set up the relationship between spot rates and forward rates.

- Forward Rate is the interest rate to be charged starting at time $t$ for a fixed period.
- We will talk about 1–year Forward Rates, i.e. interest rate charged from $t$ to $t+1$. 
If we set up a deal such that 1 is to be invested at time 0, and the 1–year spot rate $s_1 = 0.08$, and the 2–year spot rate is $s_2 = 0.09$, then at $t = 1$, if the rate to borrow and lend is the same, the rate from year 1 to year 2, $i_{1,2}$, should be defined via

$$1.09^2 = (1 + i_{1,2}) \cdot 1.08$$

$$i_{1,2} = 0.1001$$
General Relationship between Spot Rates and Forward Rates

\[ 1 + s_1 = 1 + i_{0,1} \]

\[ (1 + s_2)^2 = (1 + i_{0,1})(1 + i_{1,2}) \]

\[ \vdots = \vdots \]

\[ (1 + s_n)^n = \prod_{k=1}^{n}(1 + i_{k-1,k}) \]

Inverting this relationship, we obtain

\[ i_{k-1,k} = \frac{(1 + s_k)^k}{(1 + s_{k-1})^{k-1}} - 1 \]
General Relationship between Spot Rates and Forward Rates

\[ 1 + s_1 = 1 + i_{0,1} \]
\[ (1 + s_2)^2 = (1 + i_{0,1})(1 + i_{1,2}) \]
\[ \vdots = \vdots \]
\[ (1 + s_n)^n = \prod_{k=1}^{n}(1 + i_{k-1,k}) \]  

Inverting this relationship, we obtain

\[ i_{k-1,k} = \frac{(1 + s_k)^k}{(1 + s_{k-1})^{k-1}} - 1 \]  

Q: For a normal term structure where \( s_k > s_{k-1} \), what can we say about the relationship between \( s_k \) and \( i_{k-1,k} \)?
General Relationship between Spot Rates and Forward Rates

From the previous slide, we can compute the following table:

Table: Example 6.3: Term Structure

<table>
<thead>
<tr>
<th>Spot Rate</th>
<th>s_1 = 0.05</th>
<th>s_2 = 0.1</th>
<th>s_3 = 0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward Rate</td>
<td>i_{0,1} = 0.05</td>
<td>i_{1,2} = 0.1524</td>
<td>i_{2,3} = 0.2596</td>
</tr>
</tbody>
</table>
Define $\alpha_t$ as the continuously compounded yield rate. Then

$$PV(t = 0) = Fe^{-t\alpha_t}$$

$$= Fe^{-\int_0^t \delta_s ds}$$

$$\Rightarrow t\alpha_t = \int_0^t \delta_s ds$$

$$\Rightarrow \alpha_t + t\frac{d\alpha_t}{dt} = \delta_t$$

(124)
Summarizing, we have the dual relationship

\[ \alpha_t = \frac{1}{t} \int_0^t \delta_s ds \]

\[ \delta_t = \alpha_t + t \frac{d\alpha_t}{dt} \]
Recall that in the discrete case,

\[ 1 + s_n = [(i + i_{0,1}) \cdot (1 + i_{1,2}) \cdot \ldots \cdot (1 + i_{n-1,n})]^\frac{1}{n} \]  

(126)
Force of Interest vs YTM

Recall that in the discrete case,

\[ 1 + s_n = [(i + i_{0,1}) \cdot (1 + i_{1,2}) \cdot \ldots \cdot (1 + i_{n-1,n})]^{\frac{1}{n}} \] (126)

Hence, \( 1 + s_n \) is the **geometric mean** of the forward rates.
Recall that in the discrete case,

\[
1 + s_n = [(i + i_{0,1}) \cdot (1 + i_{1,2}) \cdot \ldots \cdot (1 + i_{n-1,n})]^\frac{1}{n}
\]  

(126)

Hence, \(1 + s_n\) is the **geometric mean** of the forward rates.

In the continuous case, the YTM \(\alpha_t\) is the arithmetic mean of the force of interest over the interval \([0, t]\). It follows that \(\delta_t\) can be seen as a forward rate in the continuous case.
Recall that in the discrete case,

\[ 1 + s_n = [(i + i_{0,1}) \cdot (1 + i_{1,2}) \cdot \ldots \cdot (1 + i_{n-1,n})]^{\frac{1}{n}} \tag{126} \]

Hence, \( 1 + s_n \) is the geometrical mean of the forward rates.

In the continuous case, the YTM \( \alpha_t \) is the arithmetic mean of the force of interest over the interval \([0, t]\). It follows that \( \delta_t \) can be seen as a forward rate in the continuous case.

Q: Again, if the term structure is increasing, i.e. if \( \frac{d\alpha_t}{dt} > 0 \), then how does \( \alpha_t \) relate to \( \delta_t \)?
Example 6.5

Suppose that the yield to maturity for a zero-coupon bond maturing at time $t$ is $\alpha_t = 0.09 - 0.08 \cdot 0.94^t$, a continuously compounded rate.

Forward Rate is the interest rate to be charged starting at time $t$ for a fixed period

- (a) Find the related forward rate
- (b) A borrower plans to borrow 1000 in one year and repay the loan with a single payment at the end of the second year. Determine the amount that will have to be paid back based on the stated term structure
Example 6.5

• **a.** We find that
  \[
  \delta_t = \alpha_t + t \frac{d\alpha_t}{dt} = 0.09 - 0.08 \cdot 0.94^t - 0.08 \cdot 0.94^t \cdot \ln (0.94) \cdot t \quad (127)
  \]

• **b.** We find that
  \[
  \text{Value} = 1000e^{\int_1^2 \delta_t dt} = 1024.11
  \]
Example 6.5

a.) We find that

\[ \delta_t = \alpha_t + t \frac{d\alpha_t}{dt} = 0.09 - 0.08 \cdot 0.94^t - 0.08 \cdot 0.94^t \cdot \ln(0.94) \cdot t \]  

(127)

b.) We find that

\[ \text{Value} = 1000e^{\int_1^2 \delta_t \, dt} = 1024.11 = 1000e^{2\alpha_2 - \alpha_1} \]  

(128)
At Par Yield

- Measure of Bond Yield: Find the Coupon rate $r_t$ such that the Bond Yield is also $r_t$.
- In this case, the Bond price is at par.
Measure of Bond Yield: Find the Coupon rate $r_t$ such that the Bond Yield is also $r_t$.

In this case, the Bond price is at par.

For an $n$–year bond,

$$1 = r_n \cdot \left( \sum_{k=1}^{n} (1 + s_k)^{-k} \right) + \frac{1}{(1 + s_n)^n}$$

$$\Rightarrow r_n = \frac{1 - \frac{1}{(1+s_n)^n}}{\sum_{k=1}^{n} \frac{1}{(1+s_k)^k}}$$
Interest Rate Swaps

- Used to convert Floating Rate Liability to Fixed Rate Liability
- Used between two borrowers that have access to different interest markets
- Example

\[
(FixedRate(A), FloatingRate(A)) = (0.08, Prime + 0.01)
\]

\[
(FixedRate(B), FloatingRate(B)) = (0.09, Prime + 0.015)
\] 

(130)
Swap: A borrows at FixedRate(A), and loans to B at rate 0.085. B borrows at FloatingRate(B), and loans to A at rate prime + 0.0125.

Net effect:

- A pays \((0.08 - 0.085 + \text{prime} + 0.0125) = \text{prime} + 0.0075\)
- B pays \((\text{prime} + 0.015 - (\text{prime} + 0.0125) + 0.085) = 0.0875\)

Notice both of these are smaller than what they could have gotten on their own. Usually, there is a question of default risk, so an intermediary can be brought in, for a small fee, depending on the spread between available rates for A and B.
• In general, a **swap** is a series of coupons exchanged upon delivery.

• Imagine a bond with face value $F$, and term structure (implied forward rates) for period $k - 1$ to $k$: $\{i_{k-1,k}\}_{k=1}^n$.

• Then the **implied swap rate** $R_n$ for term $n$ is the rate that equates the present value all the coupon exchanges, forward rate to constant $R_n$, to 0:

$$0 = \sum_{k=1}^{n} \frac{(R_n - i_{k-1,k})F}{(1 + s_k)^k} \quad (131)$$
We can interpret $R_n$ in two ways. The first is to look at $R_n$ as an average forward rate:

$$R_n = \sum_{j=1}^{n} p_j \times i_{j-1, j}$$  \hspace{1cm} (132)

where

$$p_j = \frac{(1 + s_j)^{-j}}{\sum_{m=1}^{n}(1 + s_m)^{-m}}$$

Note that $0 < p_j < 1$ and $\sum_{j=1}^{m} p_j = 1$ for positive spot rates.
The second way to interpret $R_n$ is to look at $R_n$ as an at par yield rate:

$$R_n = \sum_{j=1}^{n} \left( \frac{(1 + s_j)^{-j}}{\sum_{m=1}^{n}(1 + s_m)^{-m}} \right) \times \left( \frac{(1 + s_j)^{j}}{(1 + s_{j-1})^{j-1} - 1} \right)$$

$$= \frac{\sum_{j=1}^{n} \left( \frac{1}{(1 + s_{j-1})^{j-1}} - \frac{1}{(1 + s_j)^j} \right)}{\sum_{m=1}^{n}(1 + s_m)^{-m}} = \frac{1 - (1 + s_n)^{-n}}{\sum_{m=1}^{n}(1 + s_m)^{-m}}$$

$$= r_n$$

as the numerator above is a telescoping series.
Keeping with the analysis above, we have a **Deferred Swap** beginning in $k$ periods with value

$$R_n^{(k)} = \sum_{j=k}^{n} p_j^{(k)} \times i_{j-1,j}$$

$$p_j^{(k)} = \frac{(1 + s_j)^{-j}}{\sum_{m=k}^{n} (1 + s_m)^{-m}}.$$
Deferred and Non-Level Notional Interest Rate Swaps

Similarly, if the notional used to calculate the coupon varies with time, and is defined as $F_j$ at time $j$, then

$$0 = \sum_{k=1}^{n} \frac{(R_n - i_{k-1,k})F_k}{(1 + s_k)^k}$$

(135)

then

$$\tilde{R}_n = \sum_{j=k}^{n} q_j \times i_{j-1,j}$$

(136)

$$q_j = \frac{F_j \times (1 + s_j)^{-j}}{\sum_{m=k}^{n} F_m \times (1 + s_m)^{-m}}.$$
Example: Calculating the Swap Curve

Recall the table from Example 6.3:

<table>
<thead>
<tr>
<th>Spot Rate</th>
<th>$s_1 = 0.05$</th>
<th>$s_2 = 0.1$</th>
<th>$s_3 = 0.15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward Rate</td>
<td>$i_{0,1} = 0.05$</td>
<td>$i_{1,2} = 0.1524$</td>
<td>$i_{2,3} = 0.2596$</td>
</tr>
</tbody>
</table>
Example: Calculating the Swap Curve

We can now add to this table:

Table: Example 6.3: Term Structure

<table>
<thead>
<tr>
<th>Spot</th>
<th>$s_1 = 0.05$</th>
<th>$s_2 = 0.1$</th>
<th>$s_3 = 0.15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fwd</td>
<td>$i_{0,1} = 0.05$</td>
<td>$i_{1,2} = 0.1524$</td>
<td>$i_{2,3} = 0.2596$</td>
</tr>
<tr>
<td>Prob</td>
<td>$p_1 = \frac{\frac{1}{1.05}}{\frac{1}{1.05} + \frac{1}{1.10^2} + \frac{1}{1.15^3}}$</td>
<td>$p_2 = \frac{\frac{1}{1.10^2}}{\frac{1}{1.05} + \frac{1}{1.10^2} + \frac{1}{1.15^3}}$</td>
<td>$p_3 = \frac{\frac{1}{1.15^3}}{\frac{1}{1.05} + \frac{1}{1.10^2} + \frac{1}{1.15^3}}$</td>
</tr>
</tbody>
</table>

\[
\therefore r_3 = \frac{1}{\frac{1}{1.05} + \frac{1}{1.10^2} + \frac{1}{1.15^3}} \times 0.05 + \frac{1}{\frac{1}{1.05} + \frac{1}{1.10^2} + \frac{1}{1.15^3}} \times 0.1524 + \frac{1}{\frac{1}{1.05} + \frac{1}{1.10^2} + \frac{1}{1.15^3}} \times 0.2596
\]

\[
= 0.1413.
\]

**HW:** Compute $r_1$, $r_2$. Can you set up a spreadsheet to do this?
How does the PV of the cashflow change as the yield rate (term structure) changes? (1\textsuperscript{st} order changes)

Once quantified, this can be used as a measure of risk associated with investing in this cashflow.
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Consider the case where we have continuous compounding with a force of interest $\delta$. For a stream of payments, we have the present value, or up front price, determined by $P(\delta)$. The only risk factor that could change the value then is $\delta$. If we were to equate this value with a zero-coupon bond of unknown \textbf{Duration} $D$, then

\[ P(\delta) = P(0)e^{-\delta D} \]
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$$D = -\frac{1}{P} \frac{dP}{d\delta}$$

(138)
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$$P(\delta) = P(0)e^{-\delta D}$$

$$D = -\frac{1}{P} \frac{dP}{d\delta}$$

(138)

This of course implies that $\frac{dP}{d\delta} < 0$. 

If the environment is one of annual compounding, then $\delta = \ln (1 + i)$ and we define

$$DM := \lim_{h \to 0} \frac{1}{h} \cdot \frac{P(i) - P(i + h)}{P(i)}$$

$$= -\frac{d}{di} \ln (P(i))$$

$$D := (1 + i) \cdot DM$$

For $P(i) = (1 + i)^{-n}$, we obtain

$$DM = \frac{n}{1 + i}$$

$$D = n$$
Increasing yield decreases present value. Thus, we take the negative of derivative w.r.t. yield.

The Modified Duration $DM$ measures the ratio of rate of change of $P$ wrt $i$ to the actual value $P$.

The Macaulay Duration $D := (1 + i) \cdot DM$ says that the longer the time maturity $n$, the more susceptible it is to yield change. For an $n$–year zero coupon bond, $D = n$. Hence, the name duration

Can apply the linearization formula from calculus:

$$P(i + h) \approx P(i) + P'(i) \cdot h = P(i) - DM \cdot P(i) \cdot h$$

$$= P(i) - \frac{D}{1 + i} \cdot P(i) \cdot h$$

(141)
Duration of General Cashflows

\[ P = \sum_{m=1}^{n} \frac{K_m}{(1 + i)^m} \]

\[ DM = - \frac{d}{di} \ln (P(i)) = - \frac{1}{P} \frac{dP}{di} = \frac{\sum_{m=1}^{n} m \cdot K_m}{\sum_{m=1}^{n} K_m (1 + i)^m} \]

\[ D = (1 + i) \cdot DM = \sum_{m=1}^{n} m \cdot p_m = \mathbb{E}[m] \]

\[ p_m = \frac{K_m}{(1+i)^m} \frac{1}{\sum_{l=1}^{n} K_l (1+i)^l} \]
Note also that

\[
\frac{dD}{di} = \frac{d}{di} \left( \sum_{m=1}^{n} m \cdot \frac{K_m}{(1+i)^m} \cdot \frac{1}{\sum_{l=1}^{n} K_l (1+i)^l} \right)
\]
Note also that

\[
\frac{dD}{di} = \frac{d}{di} \left( \sum_{m=1}^{n} m \cdot \frac{K_{m}}{(1+i)^{m}} \right) \sum_{l=1}^{n} \frac{K_{l}}{(1+i)^{l}} \\
= -\frac{1}{1+i} \left( \sum_{m=1}^{n} m^{2} \cdot p_{m} - \left( \sum_{m=1}^{n} m \cdot p_{m} \right)^{2} \right) \\
= -\frac{\text{Var} \ [m]}{1+i} < 0
\]  

(143)
Duration of a Perpetuity

Even though it has no finite term, we can still find the duration of a perpetuity. In this case

\[ D = -(1 + i) \frac{1}{P} \frac{dP}{di} \]

\[ = -(1 + i) \cdot \frac{1}{i} \cdot \frac{d}{di} \left( \frac{1}{i} \right) \]

\[ = 1 + \frac{1}{i} \]

(144)
Duration of a Perpetuity

Even though it has no finite term, we can still find the duration of a perpetuity. In this case

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\]

\[
= 1 + \frac{1}{i}
\]

An equivalent way to calculate this is to see that

\[
D = \frac{\sum_{k=1}^{\infty} \frac{k}{(1+i)^k}}{\sum_{k=1}^{\infty} \frac{1}{(1+i)^k}} = \frac{(la)_{\infty}}{a_{\infty}}
\]

\[
= \frac{1+i}{i^2} = 1 + \frac{1}{i}
\]
Example 7.3 Duration of a Coupon Bond

In the case of a coupon bond, we have \( K_1 = K_2 = \ldots = K_{n-1} = F \cdot r \), and \( K_n = F + F \cdot r \), and so

\[
D = \frac{\sum_{m=1}^{n} \frac{m \cdot F \cdot r}{(1+i)^m} + \frac{n \cdot F}{(1+i)^n}}{\sum_{m=1}^{n} \frac{F \cdot r}{(1+i)^m} + \frac{F}{(1+i)^n}}
\]

\[
= \frac{r \cdot \left( \frac{(1+i) a_{n|i}}{i} - \frac{n}{(1+i)^n} \right) + \frac{n}{(1+i)^n}}{r \cdot a_{n|i} + \frac{1}{(1+i)^n}}
\]

(146)
A bond with Face $F$ will pay a coupon of $F \cdot r$ at the end of each of the next three years. It will also pay the face value of $F$ at the end of the three-year period. The bond’s duration (Macaulay duration) when valued using an annual effective interest rate of 20% is $X$. Calculate $X$. 
Recall that the Macaulay Duration over $n$ time periods is defined as an expectation of sorts:

$$D = \frac{\sum_{i=1}^{n} t_i PV(i)}{\sum_{j=1}^{n} PV(j)}$$

(147)

where the $PV(i)$ are the present values of the revenue received at time $t_i$.

For our present case, we have

$$D = \frac{1 \cdot F \cdot r_{1.2} + 2 \cdot F \cdot r_{(1.2)^2} + 3 \cdot F \cdot (1+r)_{(1.2)^3}}{F \cdot r_{1.2} + F \cdot r_{(1.2)^2} + F \cdot (1+r)_{(1.2)^3}}$$

(148)

$$= \frac{1 \cdot r_{1.2} + 2 \cdot r_{(1.2)^2} + 3 \cdot (1+r)_{(1.2)^3}}{r_{1.2} + r_{(1.2)^2} + (1+r)_{(1.2)^3}}$$
Portfolio Duration

Imagine that $P(i)$ is the price of a portfolio of income streams: $P(i) = \sum_{k=1}^{n} P_k(i)$. Then

$$D = -(1 + i) \cdot \frac{P'(i)}{P(i)} = -(1 + i) \cdot \frac{\sum_{k=1}^{n} P'_k(i)}{P(i)}$$

$$= -(1 + i) \cdot \sum_{k=1}^{n} \frac{P'_k(i)}{P(i)} = -(1 + i) \cdot \sum_{k=1}^{n} \frac{P_k(i)}{P(i)} \cdot \frac{P'_k(i)}{P_k(i)}$$

$$= \sum_{k=1}^{n} D_k \cdot \tilde{q}_k$$

$$(D_k, \tilde{q}_k) = \left( -(1 + i) \cdot \frac{P'_k(i)}{P_k(i)}, \frac{P_k(i)}{P(i)} \right)$$

and so the portfolio duration is the **weighted average** of the individual durations.
Smith holds a portfolio of a 20-year 100 bond bought at par and with duration of 8, a stock purchased for 50 with duration 7 and an 30-year annuity purchased for 50 with duration 5. What is the portfolio duration?

\[
D = 8 \cdot \frac{100}{200} + 7 \cdot \frac{50}{200} + 5 \cdot \frac{50}{200} = 7
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\[
D = 8 \cdot \frac{100}{200} + 7 \cdot \frac{50}{200} + 5 \cdot \frac{50}{200} = 7
\]
An institution such as a bank or insurance company will have both assets and liabilities which may come due at different times. These may be structured so that the net present value of the institution is zero. Hence, for a given $i$,

$$P(i) \approx 0$$  \hspace{1cm} (151)

If the rate $i$ is perturbed slightly, the institution may find itself with a negative NPV. To counter this risk, the institution may structure its assets and liabilities such that $P$ is a local minimum at $i$. 
To explain this further, define the **Convexity**

\[ \Gamma := \frac{P''(i)}{P(i)} \]  

(152)

and so

\[ \frac{P(i + h) - P(i)}{P(i)} \approx -DM \cdot h + \Gamma \cdot h^2 \]  

(153)

If \( P'(i) = 0 \) and \( \Gamma(i) > 0 \), then \( DM = 0 \) and we have a local minimum
Suppose a bank has promised its investors a one-year 5% return on deposits of 100. The bank has 50 per investor on hand, is able to purchase perpetuities that pay one per year and one year zero-coupon bonds that yield 10%. How should the bank structure its assets so they are immunized from interest rate risk?
Cashflow Immunization Example

At rate $i$, we have the NPV of $x$ units perpetuity and $y$ units of bond minus the NPV of the liability of the one-year withdrawal as

\begin{align*}
P(i) &= 50 + \frac{x}{i} + \frac{1.1 \cdot y}{1+i} - \frac{105}{1+i} \\
P'(i) &= -\frac{x}{i^2} - \frac{1.1 \cdot y - 105}{(1+i)^2} \quad (154) \\
P''(i) &= 2 \cdot \frac{x}{i^3} + 2 \cdot \frac{1.1 \cdot y - 105}{(1+i)^3}
\end{align*}

For whatever rate $i$ we fix, if we set

\begin{align*}
x &= 50i^2 \\
y &= \frac{105 - 50 \cdot (1+i)^2}{1.1} \quad (155)
\end{align*}

then $P(i) = P'(i) = 0$ and $P''(i) = \frac{100}{i \cdot (1+i)} > 0$ so we have a local minimum.
Some Comments

- Book, or Amortized, value is calculated using fixed rate.
- Market value determined via market term structure.
- The spot rate used in this term structure varies with time.
- This leads to difference between Book and Market value of cashflow.
- These values converge as time approaches maturity; investor can hold on to his investment to realize original yield rate.
- Before maturity, however, the cashflow is valued at market rate.
- Previous analysis is for flat term structure. To generalize to normal or inverted term structure, require new definitions and multivariable calculus.
As we have seen, interest rate fluctuation can lead to possible shortfalls that banks may wish to immunize their portfolio against. **Currency fluctuation** also leads to operational risk, and should factor into planning. For example, should a band tour Europe or America this summer? Click *here* for an insightful article by Neil Shah in the *Wall Street Journal*™, with comment from the manager of a very prominent rock band.

Interest rates are only one risk factor. Another very real factor is known as **longevity risk**, which is due to the possibility that a pensioner may live longer than expected. Hedging against such a possibility is extremely important, and a topic we hope to cover in STT 455 – 456. In the meantime, please consult the paper by Tsai, Tzeng, and Wang on *Hedging Longevity Risk When Interest Rates Are Uncertain*. 
Let us define an event as a point $\omega$ in the set of all possible outcomes $\Omega$. This includes the events "The stock doubled in price over two trading periods" or "the average stock price over ten years was 10 dollars".
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\[
S_1(H) = uS_0, \quad S_1(T) = dS_0
\]  

(156)
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$$S_1(H) = uS_0, \quad S_1(T) = dS_0$$

(156)

with $d < 1 < u$. Hence, a stock increases or decreases in price, according to the flip of a coin.
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with $d < 1 < u$. Hence, a stock increases or decreases in price, according to the flip of a coin.

- Let $P$ be the probability measure associated with these events:

$$P[H] = p = 1 - P[T]$$  \hspace{1cm} (157)
Arbitrage

Assume that $S_0(1 + r) > uS_0$

- Why would anyone hold a bank account (zero-coupon bond)?

Lemma

Arbitrage free $\Rightarrow d < 1 + r < u$
Arbitrage

- Assume that $S_0(1 + r) > uS_0$
- Where is the risk involved with investing in the asset $S$?
Arbitrage

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- Why would anyone hold a bank account (zero-coupon bond)?
Assume that $S_0(1 + r) > uS_0$

Where is the risk involved with investing in the asset $S$?

Assume that $S_0(1 + r) < dS_0$

Why would anyone hold a bank account (zero-coupon bond)?

Lemma Arbitrage free $\Rightarrow d < 1 + r < u$
Let $S_1(\omega)$ be the price of an underlying asset at time 1. Define the following instruments:

- **Zero-Coupon Bond**: $V_B^0 = 1$, $V_B^1(\omega) = 1 + r$,
- **Forward Contract**: $V_F^0 = 0$, $V_F^1(\omega) = S_1(\omega) - F$,
- **Call Option**: $V_C^1(\omega) = \max(S_1(\omega) - K, 0)$,
- **Put Option**: $V_P^1(\omega) = \max(K - S_1(\omega), 0)$.

In both the Call and Put option, $K$ is known as the *Strike*. Once again, a Forward Contract is a deal that is locked in at time 0 for initial price 0, but requires at time 1 the buyer to purchase the asset for price $F$. What is the value $V_0$ of the above put and call options?
Let $S_1(\omega)$ be the price of an underlying asset at time 1. Define the following instruments:

- **Zero-Coupon Bond**: $V^B_0 = \frac{1}{1+r}$, $V^B_1(\omega) = 1$
- **Forward Contract**: $V^F_0 = 0$, $V^F_1(\omega) = S_1(\omega) - F$
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- **Zero-Coupon Bond**: $V_0^B = \frac{1}{1+r}$, $V_1^B(\omega) = 1$
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- **Call Option**: $V_1^C(\omega) = \max(S_1(\omega) - K, 0)$
- **Put Option**: $V_1^P(\omega) = \max(K - S_1(\omega), 0)$

In both the Call and Put option, $K$ is known as the *Strike*. Once again, a *Forward Contract* is a deal that is locked in at time 0 for initial price 0, but requires at time 1 the buyer to purchase the asset for price $F$. 
Let $S_1(\omega)$ be the price of an underlying asset at time 1. Define the following instruments:

- Zero-Coupon Bond: $V_0^B = \frac{1}{1+r}$, $V_1^B(\omega) = 1$
- Forward Contract: $V_0^F = 0$, $V_1^F = S_1(\omega) - F$
- Call Option: $V_1^C(\omega) = \max(S_1(\omega) - K, 0)$
- Put Option: $V_1^P(\omega) = \max(K - S_1(\omega), 0)$

In both the Call and Put option, $K$ is known as the Strike.

Once again, a Forward Contract is a deal that is locked in at time 0 for initial price 0, but requires at time 1 the buyer to purchase the asset for price $F$.

- What is the value $V_0$ of the above put and call options?
Can we *replicate* a forward contract using zero coupon bonds and put and call options?

Yes: The final value of a replicating strategy $X$ has value $V_C - V_P + (K - F) = S_1 - F = X_1(\omega)$ (158) 

This is achieved (replicated) by:

- Purchasing one call option
- Selling one put option
- Purchasing $K - F$ zero coupon bonds

Since this strategy must have zero initial value, we obtain $V_C - V_P = F - K + r$ (159)

Question: How would this change in a multi-period model?
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all at time 0.
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all at time 0.

Since this strategy must have zero initial value, we obtain

$$V_0^C - V_0^P = \frac{F - K}{1 + r} \quad (159)$$
Put-Call Parity

Can we *replicate* a forward contract using zero coupon bonds and put and call options?
Yes: The final value of a replicating strategy $X$ has value

$$V_1^C - V_1^P + (K - F) = S_1 - F = X_1(\omega)$$

(158)

This is achieved (replicated) by

- Purchasing one call option
- Selling one put option
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all at time 0.

Since this strategy must have zero initial value, we obtain

$$V_0^C - V_0^P = \frac{F - K}{1 + r}$$

(159)

Question: How would this change in a multi-period model?
If we begin with some initial capital $X_0$, then we end with $X_1(\omega)$. To price a derivative, we need to match

$$X_1(\omega) = V_1(\omega) \quad \forall \ \omega \in \Omega$$

(160)

to have $X_0 = V_0$, the price of the derivative we seek.

- A strategy by the pair $(X_0, \Delta_0)$ wherein
- $X_0$ is the initial capital
- $\Delta_0$ is the initial number of shares (units of underlying asset.)
- What does the sign of $\Delta_0$ indicate?
Replicating Strategy

Initial holding in bond (bank account) is $X_0 - \Delta_0 S_0$

Value of portfolio at maturity is $X_1(\omega) = (X_0 - \Delta_0 S_0)(1 + r) + \Delta_0 S_1(\omega)$ (161)

Pathwise, we compute $V_1(H) = (X_0 - \Delta_0 S_0)(1 + r) + \Delta_0 uS_0$

$V_1(T) = (X_0 - \Delta_0 S_0)(1 + r) + \Delta_0 dS_0$

Algebra yields $\Delta_0 = V_1(H) - V_1(T)(u - d)S_0$ (162)
**Replicating Strategy**

- Initial holding in bond (bank account) is \(X_0 - \Delta_0 S_0\)
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Algebra yields

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{(u - d) S_0} \quad (162)$$
Let us assume the existence of a pair \((\tilde{p}, \tilde{q})\) of positive numbers, and use these to multiply our pricing equation(s):

\[
\tilde{p} V_1(H) = \tilde{p}(X_0 - \Delta_0 S_0)(1 + r) + \tilde{p}\Delta_0 uS_0
\]

\[
\tilde{q} V_1(T) = \tilde{q}(X_0 - \Delta_0 S_0)(1 + r) + \tilde{q}\Delta_0 dS_0
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Let us assume the existence of a pair \((\tilde{p}, \tilde{q})\) of positive numbers, and use these to multiply our pricing equation(s):

\[
\begin{align*}
\tilde{p} V_1(H) &= \tilde{p}(X_0 - \Delta_0 S_0)(1 + r) + \tilde{p}\Delta_0 u S_0 \\
\tilde{q} V_1(T) &= \tilde{q}(X_0 - \Delta_0 S_0)(1 + r) + \tilde{q}\Delta_0 d S_0
\end{align*}
\]

Addition yields

\[
X_0(1 + r) + \Delta_0 S_0(\tilde{p}u + \tilde{q}d - (1 + r)) = \tilde{p} V_1(H) + \tilde{q} V_1(T) \quad (163)
\]
If we constrain

\[ 0 = \tilde{p}u + \tilde{q}d - (1 + r) \]
\[ 1 = \tilde{p} + \tilde{q} \]
\[ 0 \leq \tilde{p} \]
\[ 0 \leq \tilde{q} \]
If we constrain

\[
0 = \tilde{p}u + \tilde{q}d - (1 + r) \\
1 = \tilde{p} + \tilde{q} \\
0 \leq \tilde{p} \\
0 \leq \tilde{q}
\]

then we have a risk neutral probability $\tilde{\mathbb{P}}$ where

\[
V_0 = X_0 = \frac{1}{1 + r} \mathbb{E}_0[V_1] = \frac{\tilde{p}V_1(H) + \tilde{q}V_1(T)}{1 + r}
\]

with

\[
\tilde{p} = \frac{1 + r - d}{u - d} = \frac{u - (1 + r)}{u - d}
\]
Example: Pricing a forward contract

Consider the case of a stock with

- $S_0 = 100$
- $u = 1.2$
- $d = 0.8$
- $r = 0.05$

Then the forward price is computed via
Example: Pricing a forward contract

Consider the case of a stock with

- $S_0 = 100$
- $u = 1.2$
- $d = 0.8$
- $r = 0.05$

Then the forward price is computed via

$$0 = \frac{1}{1 + r} \tilde{E}[S_1 - F] \Rightarrow F = \tilde{E}[S_1]$$

(165)
This leads to the explicit price

\[ F = \tilde{p}uS_0 + \tilde{q}dS_0 \]
\[ = (0.625)(1.2)(100) + (0.375)(0.8)(100) = 105 \]
This leads to the explicit price

\[ F = \tilde{\beta} u S_0 + \tilde{\beta} d S_0 \]
\[ = (0.625)(1.2)(100) + (0.375)(0.8)(100) = 105 \]

Homework Question: What is the price of a call option in the case above, with strike \( K = 95 \)?
We define a finite set of outcomes $\Omega \equiv \{\omega_1, \omega_2, ..., \omega_n\}$ and any subcollection of outcomes $A \subset \Omega$ an event.
We define a finite set of outcomes $\Omega \equiv \{\omega_1, \omega_2, \ldots, \omega_n\}$ and any subcollection of outcomes $A \subset \Omega$ an event. Furthermore, we define a probability measure $\tilde{P}$, not necessarily the physical measure $P$ to be risk neutral if

- $\tilde{P}[\omega] > 0 \ \forall \ \omega \in \Omega$
- $X_0 = \frac{1}{1+r} \tilde{E}[X_1]$

for all strategies $X$. 
The measure is indifferent to investing in a zero-coupon bond, or a risky asset \( X \).
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The same initial capital $X_0$ in both cases produces the same "‘average’" return after one period.
General one period risk neutral measure

- The measure is indifferent to investing in a zero-coupon bond, or a risky asset $X$.
- The same initial capital $X_0$ in both cases produces the same "average" return after one period.
- Not the physical measure attached by observation, experts, etc.
The measure is indifferent to investing in a zero-coupon bond, or a risky asset $X$.

The same initial capital $X_0$ in both cases produces the same "‘average’" return after one period.

Not the physical measure attached by observation, experts, etc..

In fact, physical measure has no impact on pricing.
Example: Risk Neutral measure for trinomial case

Assume that $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with

\[
S_1(\omega_1) = uS_0 \\
S_1(\omega_2) = S_0 \\
S_1(\omega_3) = dS_0
\]
Example: Risk Neutral measure for trinomial case

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Given a payoff $V_1(\omega)$ to replicate, are we assured that a replicating strategy exists?
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S_1(\omega_3) = dS_0
\]

Given a payoff $V_1(\omega)$ to replicate, are we assured that a replicating strategy exists?

Homework: Try our first example with

$S_0 = 100$

$r = 0.05$, $u = 1.2$, $d = 0.8$

$V_1(\omega) = \chi (S_1(\omega) > 90)$
Existence of Risk Neutral measure

Let $\tilde{P}$ be a probability measure on a finite space $\Omega$. The following are equivalent:

Proof: Homework (Hint: One direction is much easier than others. Also, strategies are linear in the underlying asset.)
Existence of Risk Neutral measure

Let \( \tilde{P} \) be a probability measure on a finite space \( \Omega \). The following are equivalent:

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Existence of Risk Neutral measure

Let $\tilde{P}$ be a probability measure on a finite space $\Omega$. The following are equivalent:

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- For all traded securities $S^i$, $S_0^i = \frac{1}{1+r} \tilde{E} [S_1^i]$
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Proof: Homework (Hint: One direction is much easier than others. Also, strategies are linear in the underlying asset.)
A market is *complete* if it is arbitrage free and every non-traded asset can be replicated.
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- Fundamental Theorem of Asset Pricing 1: A market is *arbitrage free* iff there exists a risk neutral measure.
- Fundamental Theorem of Asset Pricing 2: A market is *complete* iff there exists exactly one risk neutral measure.
Consider the case

\[ r = 0.05, \ S_0 = 100 \]
\[ S_1(H) = 1.2S_0 \]
\[ S_1(T) = 0.8S_0 \]
\[ S_2(HH) = 1.2S_1(H) \]
\[ S_2(HT) = 0.8S_1(H) \]
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Now price a digital option that has payoff \( V_2 := \chi(S_2 \geq 100) \)
We can do this for 2-period problems
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- Case by Case, or..
- by developing a general theory for multi-period asset pricing
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- Case by Case, or..
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In the latter method, we need a general framework to carry out our computations
Again, we define the finite set of outcomes $\Omega$ and any subcollection of outcomes $A \subset \Omega$ an event.
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Some Examples
σ—algebras

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Some Examples

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$
- $\mathcal{F}_1 = \{\emptyset, \Omega, \{HH, HT\}, \{TT, TH\}\}$
- $\mathcal{F}_2 = \{\emptyset, \Omega, \{HH\}, \{HT\}, \{TT\}, \{TH\}, \ldots\}$

$\mathcal{F}_2$ is completed by taking all unions of $\emptyset, \Omega, \{HH\}, \{HT\}, \{TT\}, \{TH\}$.
Notice that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$.

Correspondingly, given an $\Omega$, we define a
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Given a pair $(\Omega, \mathcal{F})$, we define a **Random Variable** $X(\omega)$ as a mapping $X : \Omega \to \mathbb{R}$.
Given a pair $\Omega, \mathcal{F}$ and random variable $X$,

- $\sigma(X) = \{\omega \in \Omega \mid X(\omega) \in A \subset \mathbb{R}\}$
Given a pair \((\Omega, \mathcal{F})\) and random variable \(X\),

- \(\sigma(X) = \) the collection of all sets \(\{\omega \in \Omega \mid X(\omega) \in A \subset \mathbb{R}\}\)
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A **Probability Space** for us will be the triple \((\Omega, \mathcal{F}, \mathbb{P})\), where
\[ \mathbb{P} : \mathcal{F} \to [0, 1] \]
\[ \mathbb{P}[\emptyset] = 0 \]
For any countable disjoint sets \( A_1, A_2, \ldots \in \mathcal{F} \)
\[ \mathbb{P} \left[ \bigcup_{n=1}^{\infty} A_n \right] = \sum_{n=1}^{\infty} \mathbb{P}[A_n] \]
And so

- $P[A] := \sum_{\omega \in A} P[\omega]$
- $E[X] := \sum_{\omega} X(\omega)P[\omega] = \sum_{k=1}^{n} x_k P[\{X(\omega) = x_k\}]$

with Variance $\text{Variance} := E \left[ (X - E[X])^2 \right]$
Let us return to the two flip model.
Conditional Expectation

Let us return to the two flip model.

If we are given a probability measure $\tilde{P}$, and know the value $S_1$, can we estimate the value $S_2$ given (conditional on) this information?

An Example:

$S_0 = 100$, $S_1(H) = 120$, $S_1(T) = 80$, $\tilde{p} = 0.4$, $\tilde{q} = 0.6$ for each flip $\omega = (\omega_1, \omega_2)$ - each flip is independent, and a path $\omega$ is the total path generated by both flips.

Given this set-up, compute $\tilde{E}[S_2 | S_1](\omega_1)$ (166)
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Given this set-up, compute

\[
\tilde{\mathbb{E}} [S_2 \mid S_1](\omega_1)
\] (166)
This means if we take $X = S_1$, for any $A \in \sigma(S_1)$

$$\sum_{\omega \in A} S_2(\omega)\tilde{P}[\omega] = \sum_{\omega \in A} \tilde{E}[S_2 | S_1](\omega)\tilde{P}[\omega]$$  \hspace{1cm} (167)

Can extend this to any sub $\sigma-$algebra of $\sigma(S_2)$
Assume $X \in \mathcal{F}$ and $G \subset \mathcal{F}$

- **Tower Property:** If $G_1 \subset G_2$, then

  $$
  \tilde{E} [X \mid G_1] = \tilde{E} \left[ \tilde{E} [X \mid G_2] \mid G_1 \right]
  $$

- If $X$ is $G-$measurable, then $\tilde{E} [X \mid G] = X$
• **Jensen’s Inequality**: If $f : \mathbb{R} \to \mathbb{R}$ is convex, and $\tilde{E} [|X|] < \infty$, then

$$\tilde{E} [f(X) | G] \geq f \left( \tilde{E} [X | G] \right) \quad (169)$$

• If $\tilde{P} \{X \geq 0\} = 1$, then $\tilde{E} [X | G] \geq 0$

• **Linearity**

• **Independence**: If $X$ does not depend on the information contained in $G$, then

$$\mathbb{E} [X | G] = \mathbb{E} [X] \quad (170)$$
● Taking out what is known: If \( X \) is dependent only on the information contained in \( G \), then

\[
\tilde{E}[XY \mid G] = X\tilde{E}[Y \mid G]
\] (171)

● Example: If \( G = \mathcal{F}_1 \), and \( X = S_1(\omega_1) \), \( Y = S_2 \), then

\[
\tilde{E}[S_1(\omega_1)S_2 \mid \mathcal{F}_1] = S_1(\omega_1)\tilde{E}[S_2 \mid \mathcal{F}_1]
\] (172)
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• Example: What is $\tilde{E}[S_2 \mid S_0]$?
• Taking out what is known: If $X$ is dependent only on the information contained in $G$, then

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(171)

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$$\mathbb{E}[S_1(\omega_1)S_2 | \mathcal{F}_1] = S_1(\omega_1) \mathbb{E}[S_2 | \mathcal{F}_1]$$  

(172)

• Example: What is $\mathbb{E}[S_2 | S_0]$?

• Definition: When conditioning on the $\sigma$–algebra $\mathcal{F}_n$, we take the notation

$$\mathbb{E}[X | \mathcal{F}_n] = \mathbb{E}_n[X]$$  

(173)

This is true, of course, for any measure we use.
Martingales

Observe a random process $M_n$ that depends only on the first $n$ coin flips. The history of the random variable is encapsulated in the filtration $\mathcal{F}_n$ it generates.
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By the definition and the Tower property above, we have for all $k \geq 0$: 

$$M_n = \mathbb{E}_n [M_{n+k}]$$
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if $M_n$ is a Martingale
**Martingale Pricing Theorem** Under our model with risk neutral probabilities \( \tilde{p} = \frac{1+r-d}{u-d}, \tilde{q} = \frac{u-1-r}{u-d} \), the process
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\[
M_n := \frac{S_n}{(1+r)^n}
\]  

is a martingale

Proof

We use the Tower property again:

\[
\tilde{E}_n \left[ S_{n+1} (1+r)^{n+1} \right] = \tilde{E}_n \left[ S_n (1+r)^n \tilde{E}_n \left[ \frac{1}{1+r} S_{n+1} S_n \right] \right] = S_n \tilde{p} + \tilde{q} = S_n
\]

QED
**Martingale Pricing Theorem** Under our model with risk neutral probabilities \( \tilde{p} = \frac{1 + r - d}{u - d} \), \( \tilde{q} = \frac{u - 1 - r}{u - d} \), the process

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\]

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**Proof**

We use the Tower property again:

\[
\tilde{E}_n \left[ \frac{S_{n+1}}{(1+r)^{n+1}} \right] = \tilde{E}_n \left[ \frac{S_n}{(1+r)^n} \frac{1}{1+r} \frac{S_{n+1}}{S_n} \right] \\
= \frac{S_n}{(1+r)^n} \tilde{E}_n \left[ \frac{1}{1+r} \frac{S_{n+1}}{S_n} \right] \\
= \frac{S_n}{(1+r)^n} \frac{\tilde{p}u + \tilde{q}d}{1+r} \\
= \frac{S_n}{(1+r)^n}
\]
Let us return to the multiperiod model. For each time $n$, we will dynamically redistribute our wealth $X_n$ by deciding to hold $\Delta_n$ shares of $S_n$ and invest the rest in the bank account at rate $r$. At time $n+1$, this means we have the value
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$$X_{n+1} = (X_n - \Delta_n S_n)(1 + r) + \Delta_n S_{n+1}$$

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If we discount this recursive process to form

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then $M_n$ is a martingale
Assume now that we have the regular assumptions on our coin flip space, and that at time $N$ we are asked to deliver a path dependent derivative value $V_N$. Then for times $0 \leq n \leq N$, the value of this derivative is computed via

$$V_n = \tilde{E}_n \left[ V_{n+1} \left( 1 + r \right) \right]$$

and so

$$V_0 = \tilde{E}_0 \left[ V_N \left( 1 + r \right)^N \right]$$
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Risk Neutral Pricing Formula

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(178)

and so

$$V_0 = \tilde{E}_0 \left[ \frac{V_N}{(1 + r)^N} \right]$$

(179)
Markov Processes

If we use the above approach for a more exotic option, say a lookback option that pays the maximum over the term of a stock, then we find this approach lacking. There is not enough information in the tree or the distinct values for $S_3$ as stated. We need more. Consider our general multi-period binomial model under $\tilde{\mathbb{P}}$

**Definition** We say that a process $X$ is **adapted** if it depends only on the flips $\omega_1, \ldots, \omega_n$

**Definition** We say that an adapted process $X$ is **Markov** if for every $0 \leq n \leq N - 1$ and every function $f(x)$ there exists another function $g(x)$ such that

$$\tilde{\mathbb{E}}_n[f(X_{n+1})] = g(X_n) \quad (180)$$
This notion of Markovity is essential to our state-dependent pricing algorithm. Indeed, since our stock process evolves from time \( n \) to time \( n + 1 \), using only the information in \( S_n \), we can in fact say that for every \( f(s) \) there exists a \( g(s) \) such that

\[
g(s) = \tilde{\mathbb{E}}_n [f(S_{n+1}) \mid S_n = s] \]

\[
g(s) = \frac{\tilde{p}f(2s) + \tilde{q}f(0.5s)}{1 + r} \quad (181)
\]

So, for any \( f(s) := V_N(s) \), we can work our recursive algorithm backwards to find the \( g_n(s) := V_n(s) \) for all \( 0 \leq n \leq N - 1 \).
Returning to our example of a lookback option, we see that the problem was that \( M_n := \max_{0 \leq i \leq n} S_i \) is not Markov by itself, but the pair \( (M_n, S_n) \) is. Why?

Let’s generate the tree!

**Homework** Can you think of any other processes that are not Markov? That are Martingales, but not Markov?
Consider the following scenario: After graduating, you go on the job market, and have 4 possible job interviews with 4 different companies. So sure of your prospects that you know that each company will make an offer, with an identically, independently distributed probability attached to the 4 possible salary offers -
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\[
\begin{align*}
\Pr[\text{Salary Offer} = x_1] &= p_1 \\
\Pr[\text{Salary Offer} = x_2] &= p_2 \\
\Pr[\text{Salary Offer} = x_3] &= p_3 \\
\Pr[\text{Salary Offer} = x_4] &= p_4 \\
p_1 + p_2 + p_3 + p_4 &= 1
\end{align*}
\]
How should you interview?

Specifically, when should you accept an offer and cancel the remaining interviews?

How does your strategy change if you can interview as many times as you like, but the distribution of offers remains the same as above?
The Interview Process

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- Specifically, when should you accept an offer and cancel the remaining interviews?
- How does your strategy change if you can interview as many times as you like, but the distribution of offers remains the same as above?
Let’s review the basic contracts we can write:

1. **Forward Contract**: Initial value is 0, because both buyer and seller may have to pay a balance at maturity.

2. **(European) Put/Call Option**: Initial value is > 0, because the option holder is the one who must pay the balance at maturity.

3. **(European) “Exotic” Option**: Initial value is > 0, because the option holder is the one who must pay the balance at maturity.

During the term of the contract, can the value of the contract ever fall below the intrinsic value of the payoff? Symbolically, does it ever occur that

\[ v_n(s) < g(s) \]  

(183)

where \( g(s) \) is of the form of

\[ g(S) := \max\{S - K, 0\} \]

in the case of a Call option, for example.
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where $g(s)$ is of the form of $g(S) := \max \{S - K, 0\}$, in the case of a Call option, for example.
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- Charge more than we would for a European contract that is exercised only at the term \( N \)
- Hedge our replicating strategy \( X \) differently, to allow for the possibility of early exercise
- Extend the notion of the replicating strategy as a Martingale, when properly discounted
Some examples:

- "American Bond:" $g(s) = 1$
- "American Digital Option:" $g(s) = 1_{6 \leq s \leq 10}$ and

$$\tilde{p} = \frac{1}{2} = \tilde{q}$$

$$r = \frac{1}{4}$$

$S_0 = 4$, $u = 2$, $d = \frac{1}{2}$
Using the state-variable version of our Risk-Neutral pricing algorithm, we were able to compute the fair price of the European option as $V_0$. How would we compute this $V_0$ for the corresponding American option? What would our tree look like? What would our strategy be? Certainly, for any strategy $X$ we enact, we must have $X_n \geq \max\{K - S_n, 0\}$ (185) to reflect the early payoff the option holder can take.
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to reflect the early payoff the option holder can take.
The question of when exactly an investor will choose to "prune the tree" and take her payoff must be asked. In precise language, we define a **Stopping Time** $\tau$ as an adapted random variable on our discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_k\}_{k=0}^{N}$

$$\tau : \Omega \rightarrow \{0, 1, 2, \ldots, N\}$$

$$\{\omega \in \Omega \mid \tau(\omega) = k\} \in \mathcal{F}_k \ \forall k = 0, 1, 2, \ldots, N$$

(186)
Stopping Times: Example

Consider the case of

\[ \tilde{p} = \frac{1}{2} = \tilde{q} \]
\[ r = \frac{1}{4} \]

\[ S_0 = 4, \ u = 2, \ d = \frac{1}{2}, \ N = 2 \]

\[ V_2 := \max \{ K - S_2, 0 \} \]

Then for \( \tau := \min \{ m \mid v_m(S_m) = \{ K - S_m, 0 \} \} \), we have

\[ \{ \omega \mid \tau(\omega) = 0 \} = \phi \in \mathcal{F}_0 \]
\[ \{ \omega \mid \tau(\omega) = 1 \} = \{ TH, TT \} \in \mathcal{F}_1 \]
\[ \{ \omega \mid \tau(\omega) = 2 \} = \{ HH, HT \} \in \mathcal{F}_2 \]

and so \( \tau \) is a stopping time. Can you come up with a random time that is not a stopping time?