

# Pricing Recovery Risk in Bonds and Swaps

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Travelers Research Symposia  
April 26, 2019  
The Peter J. Tobin College of Business  
St. John's University  
Queens, NY



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# Single-Name Default Models

Single-name default models typically fall into one of three main categories:

- **Structural Models.** Attempts to explain default in terms of fundamental properties, such as the firms balance sheet and economic conditions (Merton 1974, Black-Cox 1976, Leland 1994, etc..)
- An excellent resource on structural models in corporate finance is the lecture series by Hayne Leland, including "Lecture 1: Pros and Cons of Structural Models: An Introduction."
- **Reduced Form (Intensity) Models.** Directly postulates a model for the instantaneous probability of default via an exogenous process  $\lambda_t$  (Jarrow-Turnbul 1995, Duffie-Singleton 1999, etc..) via

$$\mathbb{P}[\tau \in [t, t + dt) | \mathcal{F}_t] = \lambda_t dt. \quad (1)$$

- **Hybrid Models.** Incorporates features from structural and reduced-form models by postulating that the default intensity is a function of the stock or of firm value (Madan-Unal 2000, Atlan-Leblanc 2005, Carr-Linetsky 2006, etc..)



# Simplest Framework for Structural Models

- The idea pioneered by Merton <sup>1</sup> is to model equity  $E_{t,T}$  at time  $t$  as a call option on the assets  $A$  of the firm at expiry  $T$  of the zero-coupon, notional  $N$  bond issued by the firm :
  - Assume Modigliani-Miller Theorem holds: The value of the firm is invariant to its capital structure (debt  $B$  to equity  $E$ ):

$$A_t = E_{t,T} + B_{t,T}. \quad (2)$$

- Equity is a call option on assets with notional  $N$  as strike:

$$E_{t,T}^{\text{Merton}} = \tilde{\mathbb{E}} \left[ e^{-r(T-t)} (A_T - N)_+ \mid A_t = A \right]. \quad (3)$$

- With a tractable debt structure, investors are able to compute default probability, bond price, credit spreads, recovery rates, etc..

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<sup>1</sup>**R. Merton**, *On the Pricing of Corporate Debt: the Risk Structure of Interest Rates*, Journal of Finance 29, 1974, pp. 449-470.



# Merton Model Asset Evolution

Assume a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ :

- Underlying asset is modeled as GBM under a *physical measure*  $\mathbb{P}$ :

$$dA_t = \mu_A A_t dt + \sigma_A A_t dW_t^A. \quad (4)$$

- Default is implicitly assumed to coincide with the event  $\{A_T \geq N\}^c$ :

$$\tau_{\text{Merton}} = T \mathbf{1}_{\{A_T < N\}} + \infty \mathbf{1}_{\{A_T \geq N\}}. \quad (5)$$

- This results in a turnover of the company's assets to bondholders if assets are worth less than the total value of bond outstanding.
- At maturity, the bond value (payoff) is

$$B_{T,T}^{\text{Merton}} = A_T \mathbf{1}_{\{A_T < N\}} + N \mathbf{1}_{\{A_T \geq N\}}. \quad (6)$$

- For pricing purposes, assume a risk-neutral probability measure  $\tilde{\mathbb{P}}$ .



# Merton Model Bond Price

Consequently, using the Feynmann-Kac approach to solution via expectation, we obtain

$$\begin{aligned} B_{t,T}^{\text{Merton}} &= e^{-r(T-t)} \tilde{\mathbb{E}}_t[B_{T,T}^{\text{Merton}}] = e^{-r(T-t)} \tilde{\mathbb{E}}_t[A_T \mathbf{1}_{\{A_T < N\}} + N \mathbf{1}_{\{A_T \geq N\}}] \\ &= e^{-r(T-t)} \left[ \int_0^N A \cdot \tilde{\mathbb{P}}_t[A_T \in dA] + N \int_N^\infty \tilde{\mathbb{P}}_t[A_T \in dA] \right] \\ &= e^{-r(T-t)} \left[ \tilde{\mathbb{E}}_t[A_T \mid A_T < N] \cdot \tilde{\mathbb{P}}_t[A_T < N] + N \cdot \tilde{\mathbb{P}}_t[A_T \geq N] \right] \\ &= Ne^{-r(T-t)} \left[ 1 - \tilde{\mathbb{E}}_t \left[ \left( \frac{N - A_T}{N} \right) \mid A_T < N \right] \cdot \tilde{\mathbb{P}}_t[A_T < N] \right] \\ &:= Ne^{-r(T-t)} \left[ 1 - \tilde{\mathbb{E}}_t[\mathbf{Loss} \mid \mathbf{Default}] \cdot \tilde{\mathbb{P}}_t[\mathbf{Default}] \right]. \end{aligned}$$



# Merton Model PD and LGD

$$B_{t,T}^{\text{Merton}} = Ne^{-r(T-t)}\Phi(d_0) + A_t\Phi(-d_1)$$

$$\text{PD}_{t,T}^{\text{Merton}} = \tilde{\mathbb{P}}_t[A_T < N] = \Phi(-d_0)$$

$$\text{LGD}_{t,T}^{\text{Merton}} = \tilde{\mathbb{E}}_t \left[ \left( \frac{N - A_T}{N} \right) \mid A_T < N \right] = 1 - e^{r(T-t)} \frac{A_t}{N} \frac{\Phi(-d_1)}{\Phi(-d_0)}$$

$$d_1 = \frac{\ln(A_t/N) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} = d_+ \quad (8)$$

$$d_0 = \frac{\ln(A_t/N) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} = d_-$$

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du.$$



# Merton Model Credit Spread

- The yield-to-maturity credit spread  $Y_{t,T}$  is defined as the spread over the risk free rate  $r$  which prices the bond as

$$B_{t,T} = Ne^{-(r+Y_{t,T})(T-t)}. \quad (9)$$

- Solving for  $Y$  yields

$$Y_{t,T} = \frac{1}{T-t} \ln \left( \frac{N}{B_{t,T}} \right) - r. \quad (10)$$

- For the Merton Model,

$$Y_{t,T}^{\text{Merton}} = -\frac{1}{T-t} \ln \left[ \Phi(d_0) + \frac{A_t}{N} e^{r(T-t)} \Phi(-d_1) \right]. \quad (11)$$

- Assumes only one possible time for default.





# Black-Cox Model (Bondholder Covenant)

- Same asset dynamics as in Merton model:

$$dA_t = \mu_A A_t dt + \sigma_A A_t dW_t^A.$$

- Default can happen at times other than maturity, in particular when firm assets fall below a prescribed default point (barrier)  $K$ :

$$\begin{aligned}\{\tau_{BC} > T\} &= \{\tau_K > T, \tau_{\text{Merton}} > T\} \\ \tau_K &= \inf\{t \geq 0 : A_t \leq K\} \\ \tau_{\text{Merton}} &= T \mathbf{1}_{\{A_T \geq N\}}^c + \infty \mathbf{1}_{\{A_T \geq N\}}.\end{aligned}\tag{12}$$



# Black-Cox Model (Bondholder Covenant)

- In the original Black-Cox paper <sup>2</sup>,  $K$  was taken to be  $K(t) = Ke^{-\alpha(T-t)}$ . We assume that  $\alpha = 0$ .
- Stochastic Approach: At times  $t \leq T$ , we define and solve

$$\begin{aligned} B_{T,T}^{\text{BC}} &= N1_{\{\tau_K > T, A_T \geq N\}} + A_T 1_{\{\tau_K > T, A_T \geq N\}^c} \\ B_{t,T}^{\text{BC}} &= e^{-r(T-t)} \tilde{\mathbb{E}}_t[B_{T,T}^{\text{BC}}]. \end{aligned} \tag{13}$$

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<sup>2</sup>F. Black & Cox, J.C., *Valuing Corporate Securities: Some Effects of Bond Indenture Provisions*, *Journal of Finance* 31, 1976, 351-367.

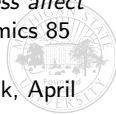


- Information gap arises from estimating  $(A, \sigma_A)$ , leading to surprises in default time (Default Risk) and recovery amount (Recovery Risk.)
- Single factor model combines Recovery Risk with Default Risk.
- As there is only one source of risk ( $A$ ) for both debt and equity, this model assumes that debt and equity are perfectly correlated. Same for PD-LGD correlation.
- Credit Risk is often misestimated <sup>3</sup>.
- Would be nice to decouple the default and recovery drivers <sup>4</sup>.
- What do empirical recovery rate-to-time plots look like?

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<sup>3</sup>**V.V. Acharya, T.S. Bharath, & A. Srinivasan**, *Does industry-wide distress affect defaulted firms? Evidence from creditor recoveries*, *Journal of Financial Economics* 85 (2007) 787-821.

<sup>4</sup>**G. Giese**, *The Impact of PD/LGD Correlations on Credit Risk Capital*, *Risk*, April 2005, pp 79-85.



# Bond Price under Correlated A-R Model

Define the quantities:

- $A_t$ , the asset value at time  $t > 0$
- $R_t$ , the recovery amount at time  $t > 0$ . Recovery amount evolves in a manner that is correlated to asset dynamics.

The dynamics for the asset and recovery <sup>5</sup> are, under a risk neutral measure  $\tilde{\mathbb{P}}$ , modeled to be

$$\begin{aligned}\frac{dA_t}{A_t} &= rdt + \sigma_A dW_t^A \\ \frac{dR_t}{R_t} &= rdt + \sigma_R dW_t^R \\ dW_t^A dW_t^R &= \rho_{A,R} dt.\end{aligned}\tag{14}$$

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<sup>5</sup> **A. Levy & A. Hu**, *Incorporating Systematic Risk in Recovery: Theory and Evidence, Modeling Methodology*, Moody's KMV, 2007.



# Stochastic Recovery Black-Cox Model (SRBC)

- Let  $K$  define our (level) default point written into the bondholder covenant on the asset  $A$ , and define  $\tau_K$  as the first time  $A$  reaches this default point.
- The default time  $\tau$  can also be defined, as in the Black-Cox model, via the event  $\{\mathbf{Default}\} \in \mathcal{F}_T$ :

$$\{\mathbf{Default}\} = \{\tau_K > T, A_T \geq N\}^c. \quad (15)$$

- An important quantity is  $\gamma := \rho_{A,R} \frac{\sigma_R}{\sigma_A}$ , which factors heavily into our SRBC bond price. Beyond the usual measure of sensitivity of recovery relative to the underlying asset,  $\gamma$  reflects the extra risk inherent in decoupling loss given default from probability of default.



# Stochastic Recovery Black-Cox Model (SRBC)

- This leads to a barrier-option interpretation of debt, one that carries over into a decoupled Stochastic Recovery Black-Cox (SRBC) Model that allows for partial information to play a role in pricing and estimating credit risk while allowing for default before expiry.
- Stochastic Approach: At times  $t \leq T$ , we define

$$\begin{aligned} B_{T,T}^{\text{SRBC}} &= N1_{\{\tau_K > T, A_T \geq N\}} + R_T 1_{\{\tau_K > T, A_T \geq N\}^c} \\ B_{t,T}^{\text{SRBC}} &= e^{-r(T-t)} \tilde{\mathbb{E}}_t[B_{T,T}^{\text{SRBC}}]. \end{aligned} \tag{16}$$



# Main Theorem: Bond Price under SRBC model.

## Theorem

(Bond Price under Stochastic Recovery Black Cox Model) <sup>a</sup>

I. Weak Covenant Case. If  $K \leq N$  the price of a zero-coupon bond is given by

$$\begin{aligned} B_{t,T}^{\text{SRBC}}(K) &= Ne^{-r(T-t)} \left[ \Phi(d_0^w) - \left( \frac{K}{A_t} \right)^{\frac{2r}{\sigma_A^2} - 1} \Phi(x_0^w) \right] \\ &+ R_t \left[ \Phi(-d_\gamma^w) + \left( \frac{K}{A_t} \right)^{\frac{2r}{\sigma_A^2} + 2\gamma - 1} \Phi(x_\gamma^w) \right] \\ B_{t,T}^{\text{SRM}} &= B_{t,T}^{\text{SRBC}}(0) = Ne^{-r(T-t)} \Phi(d_0^w) + R_t \Phi(-d_\gamma^w). \end{aligned} \tag{17}$$

<sup>a</sup>**A. C. & N. Costanzino** *Bond and CDS Pricing via the Stochastic Recovery Black-Cox Model*, **Risks**. Vol. 5, No. 26. 2017.

In the above closed form solutions for the SRBC bond prices, we use

$$\begin{aligned}d_0^w &= \frac{\ln\left(\frac{A_t}{N}\right) + \left(r - \frac{1}{2}\sigma_A^2\right)(T - t)}{\sigma_A\sqrt{T - t}} \\d_0^s &= \frac{\ln\left(\frac{A_t}{K}\right) + \left(r - \frac{1}{2}\sigma_A^2\right)(T - t)}{\sigma_A\sqrt{T - t}} \\x_0^w &= \frac{\ln\left(\frac{K^2}{NA_t}\right) + \left(r - \frac{1}{2}\sigma_A^2\right)(T - t)}{\sigma_A\sqrt{T - t}} \\x_0^s &= \frac{\ln\left(\frac{K}{A_t}\right) + \left(r - \frac{1}{2}\sigma_A^2\right)(T - t)}{\sigma_A\sqrt{T - t}}.\end{aligned}\tag{18}$$





Adjusting for recovery, we also have

$$\begin{aligned}d_{\gamma}^w &= d_0^w + \gamma\sigma_A\sqrt{T-t} \\d_{\gamma}^s &= d_0^s + \gamma\sigma_A\sqrt{T-t} \\x_{\gamma}^w &= x_0^w + \gamma\sigma_A\sqrt{T-t} \\x_{\gamma}^s &= x_0^s + \gamma\sigma_A\sqrt{T-t}.\end{aligned}\tag{19}$$



# Distance-to-Default Under Partial Information

- The *risk-adjusted* SRBC distances-to-default  $d_{\gamma, x_{\gamma}}$  in (19) reduce to the standard distances-to-default  $(d_0, x_0)$  and  $(d_1, x_1)$  of the BC model if  $\gamma = 0$  or  $\gamma = 1$ , respectively.
- This reflects the uncertainty of the firm manager in the partial information setting of what the recoverable value of the firm's assets truly are, and affects only the recovery term.
- The probability of default in the SRBC model is the same as in the BC model.



Extending the BC CDS <sup>6</sup> premium to include stochastic recovery results in

$$P_{t,T}^{\text{SRBC}} = \frac{\tilde{\mathbb{E}}_t \left[ D(t, \tau_{\text{BC}}) \left( 1 - \frac{R_{\tau_{\text{BC}}}}{N} \right) \mathbf{1}_{\{\tau_{\text{BC}} \leq T\}} \right]}{\int_t^T D(t, s) \tilde{\mathbb{P}}_t[\tau_{\text{BC}} > s] ds + \frac{1}{2} \int_t^T D(t, s) \tilde{\mathbb{P}}_t[\tau_{\text{BC}} \in ds]}. \quad (20)$$

The quantities in (20) are also computed in closed form <sup>7</sup>

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<sup>6</sup>I.H. Gökgoz, Ö. Uğur, & Y. Okur, *On the single name CDS price under structural modeling*, Journal of Computational and Applied Mathematics, Volume 259, Part B, 15 March 2014, pp 406-412.

<sup>7</sup>A. C. & N. Costanzino *Bond and CDS Pricing via the Stochastic Recovery Black-Cox Model*, **Risks**. Vol. 5, No. 2. 2017.



# Recovery Risk and Default Risk under Change of Measure

- In order to incorporate recovery risk into structural models and understand its effect on the credit spreads, a coupled stochastic recovery risk driver  $R$  was added to the classical one-dimensional structural models of Merton and Black-Cox.
- To enable this extension, the asset process  $A$  becomes the joint process of  $(A, R)$  which lives in a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  satisfying the usual conditions.
- In this setting, we assume that  $\mathbb{P}$  is a risk-neutral measure.



If we further define the *default event* as a set  $D \in \mathcal{F}_T^A \subseteq \mathcal{F}_T \subseteq \mathcal{F}$ , then the price  $B_{t,T}^{\text{NR}}$  of a bond is the exchange of asset for notional in the traditional structural models without stochastic recovery (NR):

$$B_{t,T}^{\text{NR}} = e^{-r(T-t)} \mathbb{E}^{\mathbb{P}} [A_T 1_{\{D\}} + N 1_{\{D^c\}} \mid \mathcal{F}_t^A]. \quad (21)$$



# Recovery Risk and Default Risk under Change of Measure

Under the stochastic recovery framework (SR), specifically for Merton and Black-Cox models, the price of a defaultable bond was shown to be similar in nature to the traditional structural models without stochastic recovery:

$$B_{t,T}^{\text{SR}} = e^{-r(T-t)} \mathbb{E}^{\mathbb{P}} [R_T 1_{\{D\}} + N 1_{\{D^c\}} \mid \mathcal{F}_t]. \quad (22)$$

By appealing to barrier option theory, Ishizaka & Takaoka<sup>8</sup> define a new measure  $\mathbb{Q}$  with Radon-Nikodym density process defined by

$$Z := \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-r(T-t)} \frac{A_T}{A_t}. \quad (23)$$

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<sup>8</sup>M. ISHIZAKA & K. TAKAOKA, *On the Pricing of Defaultable Bonds Using the Framework of Barrier Options*, Asia-Pacific Financial Markets, Volume 10, 2003, pp 151-162.



We set the notation above as

$$\begin{aligned}\mathbb{P}_t[\cdot] &:= \mathbb{P}[\cdot \mid \mathcal{F}_t^A] \\ \mathbb{Q}_t[\cdot] &:= \mathbb{Q}[\cdot \mid \mathcal{F}_t^A].\end{aligned}\tag{24}$$

It follows that the bond price without stochastic recovery can be written as

$$B_{t,T}^{\text{NR}} = e^{-r(T-t)} N \cdot \mathbb{P}_t[D^c] + A_t \cdot \mathbb{Q}_t[D].\tag{25}$$



# Recovery Risk and Default Risk under Change of Measure

This line of reasoning, that of changing numeraire, can be carried over into the case where *recovery* is used as the numeraire, to build a new measure  $\mathbb{Q}^*$  to estimate probability of default, with Radon-Nikodym density process and conditional probability defined by

$$\begin{aligned} Z^* &:= \frac{d\mathbb{Q}^*}{d\mathbb{P}} = e^{-r(T-t)} \frac{R_T}{R_t} \\ \bar{\mathbb{P}}_t[\cdot] &:= \mathbb{P}[\cdot \mid \mathcal{F}_t] \\ \mathbb{Q}_t^*[\cdot] &= \mathbb{Q}^*[\cdot \mid \mathcal{F}_t]. \end{aligned} \tag{26}$$





# Recovery Risk and Default Risk under Change of Measure

Armed with this measure, we use recovery  $R$  to estimate the probability of default triggered by asset value  $A$ . We presented closed form solutions in the Merton and Black-Cox cases, which can now be shown to equal

$$\begin{aligned} B_{t,T}^{SR} &= e^{-r(T-t)} N \cdot \bar{\mathbb{P}}_t[D^c] + R_t \cdot \mathbb{Q}_t^*[D]. \\ &= e^{-r(T-t)} N \cdot \mathbb{P}_t[D^c] + R_t \cdot \mathbb{Q}_t^*[D]. \end{aligned} \tag{27}$$



# Connection with Distance-to-Default

- Note that recovery  $R$  is only *partially informed* by asset level  $A$ , and so using  $R$  as the numeraire reflects a risk-adjustment in estimating the probability of default.
- We also point out that since the default event (trigger) remains in the smaller filtration  $\mathcal{F}_t^A \subseteq \mathcal{F}_t$ , the default-free part of the bond price (27) remains the same as it was in the structural model without recovery (22).
- The effect of incorporating stochastic recovery into the model is reflected in the new estimate  $\mathbb{Q}_t^*[D]$  for probability of default.



## Definition

We define the **Partial Information Transform (PIT)** of  $\mathbb{P}_t[\tau \leq T]$  for a model where assets  $A$  evolve via a continuous Geometric Brownian Motion (GBM) and default is a set  $D \in \mathcal{F}_T^A$ , to be the transform which maps  $\mathbb{P}_t[\tau \leq T] \rightarrow \mathbb{Q}_t^*[\tau \leq T]^a$ .

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<sup>a</sup>**A. C. and Nick Costanzino (2017)** *A General Framework for Incorporating Stochastic Recovery in Structural Models of Credit Risk*, **Risks**. Vol. 5, No. 26. 2017.



# Partial Information Transform

For the two-factor *Asset Recovery* model, we can compute the **PIT** via the following Lemma:

## Lemma

$$Q_t^*[D] = Q_t^*[\tau \leq T] := \frac{e^{[\frac{1}{2}\gamma(1-\gamma)\sigma_A^2 - \gamma r](T-t)}}{A_t^\gamma} \mathbb{E}^\mathbb{P} \left[ A_T^\gamma \mathbf{1}_{\{\tau \leq T\}} \mid \mathcal{F}_t^A \right]. \quad (28)$$

- The quantity (28) also represents the solution to a two-factor bond-pricing PDE, and is highly dependent on the boundary conditions representing default covenants (i.e. the set  $D \in \mathcal{F}_T^A$ .)
- The parameter  $\gamma$ , to be estimated from market data, is the only term that relates to post-default recovery  $R$  in the Partial Information Transform.
- Note that when
  - $\gamma = 1$ , we return the value  $Q_t^*[\tau \leq T] = Q_t[\tau \leq T]$ .
  - $\gamma = 0$ , we return the value  $Q_t^*[\tau \leq T] = \mathbb{P}_t[\tau \leq T]$ .



# Merton Revisited

For example, in the Merton Case the default set  $D := \{A_T < N\}$  and so using the result  $e^{-r(T-t)}\mathbb{E}_t [R_T 1_{\{A_T < N\}}] = R_t \Phi(-d_\gamma)$ , we can compute the asset and recovery numeraire defined probability of default as

$$\begin{aligned} \left( \mathbb{P}_t^{\text{Merton},r}[D], \mathbb{Q}_t^{\text{Merton},r}[D] \right) &= (\Phi(-d_0), \Phi(-d_1)) \\ \mathbb{Q}_t^*[D] &= \mathbb{Q}_t^*[A_T < N] = e^{-r(T-t)}\mathbb{E}_t \left[ \frac{R_T}{R_t} 1_{\{A_T < N\}} \right] \\ &= \frac{1}{R_t} e^{-r(T-t)}\mathbb{E}_t [R_T 1_{\{A_T < N\}}] \\ &= \frac{1}{R_t} R_t \Phi(-d_\gamma) = \Phi(-d_\gamma). \end{aligned} \tag{29}$$

In terms of transformations, we can write

$$\begin{aligned} \mathbb{Q}_t^*[D] &= \Phi \left( \Phi^{-1} \left( \mathbb{P}_t^{\text{Merton},r}[D] \right) - \gamma \sigma_A \sqrt{T-t} \right) \\ &= \Phi \left( \Phi^{-1} \left( \mathbb{Q}_t^{\text{Merton},r}[D] \right) + (1-\gamma) \sigma_A \sqrt{T-t} \right). \end{aligned}$$



# Matching Equity

- In the Stochastic Recovery Merton framework, we assume that equity is a call on *recovery* and not assets:

$$E_{t,T}^{\text{SRM}} := e^{-r(T-t)} \tilde{\mathbb{E}}_t [(R_T - N)_+]. \quad (31)$$

- The next step is to calibrate  $R_t$  and  $\sigma_R$ . Since in the model these are market defined quantities, they must be calibrated using market data.
- Hence, we use the equity and equity volatility to set up two equations for the two unknowns:

$$\begin{aligned} E_t^{\text{Market}} &= E_t^{\text{SRM}}(R, \sigma_R) = R_t \Phi(d_1^R) - Ne^{-r(T-t)} \Phi(d_0^R) \\ \sigma_E^{\text{Market}} E_t^{\text{Market}} &= \sigma_R R_t \Phi(d_1^R) \end{aligned} \quad (32)$$

where  $\sigma_E^{\text{Market}}$  and  $E_t^{\text{Market}}$  are the equity and volatility observed directly from the market. This procedure yields calibrated values  $R_t$  and  $\sigma_R$ .



- Finally, we set up two equations for the two remaining unknowns  $(d_0, \rho_{A,R})$ . The two coupled equations are,

$$\begin{aligned} B_{t,T}^{\text{Market}} &= B_{t,T}^{\text{SRM}}(A, \sigma_A, \rho_{A,R}, R_t, \sigma_R) \\ &= Ne^{-r(T-t)}\Phi(d_0) + R_t\Phi(-d_\gamma) \\ \left(\sigma_B^{\text{Market}}\right)^2 &= \left(\sigma_B^{\text{SRM}}(A, \sigma_A, \rho_{A,R}, R_t, \sigma_R)\right)^2 \\ &= \Omega_A^2\sigma_A^2 + \Omega_R^2\sigma_R^2 + 2\Omega_A\Omega_R\rho_{A,R}\sigma_A\sigma_R. \end{aligned} \tag{33}$$

- Our flow in the algorithm is thus

$$(N) \rightarrow (R, \sigma_R) \rightarrow (d_0, \rho_{A,R}). \tag{34}$$



# Physical vs Risk-Neutral Quantities

As equity is modeled as a call option on assets, we have that the Sharpe ratios of the equity and recovery are the same. It can be shown that

$$\mu_B - r = \Omega_A(\mu_A - r) + \Omega_R(\mu_R - r) \quad (35)$$

and so we can estimate directly from market data:

$$\begin{aligned} \frac{\mu_R - r}{\sigma_R} &= \frac{\mu_E - r}{\sigma_E} \\ \frac{\mu_A - r}{\sigma_A} &= \frac{(\mu_B - r) - \Omega_R \sigma_R \frac{\mu_E - r}{\sigma_E}}{\Omega_A \sigma_A}. \end{aligned} \quad (36)$$





# Physical vs Risk-Neutral Quantities

The Sharpe ratio for assets  $A$  also enables us to convert from risk-neutral distance-to-default to physical distance-to-default via

$$\begin{aligned}d_0^{\mu A} &= d_0 + \frac{\mu_A - r}{\sigma_A} \sqrt{T - t} \\d_\gamma^{\mu A} &= d_\gamma + \frac{\mu_A - r}{\sigma_A} \sqrt{T - t} \\&= d_0 + \rho_{A,R} \hat{\sigma}_R \sqrt{T - t} + \frac{\mu_A - r}{\sigma_A} \sqrt{T - t}.\end{aligned}\tag{37}$$

With all of this information, we can also calculate the *physical* probability of default and expected recovery-given-default:

$$\begin{aligned}\mathbb{P}[\tau \leq T] &= \Phi(-d_0^{\mu A}) \\ \mathbb{E}\left[\frac{R_T}{N} \mid A_T < N\right] &= e^{\mu_R(T-t)} \frac{R_t}{N} \frac{\Phi(-d_\gamma^{\mu A})}{\Phi(-d_0^{\mu A})}.\end{aligned}\tag{38}$$



In the following analysis, using our stochastic recovery model, we can obtain data from balance sheet information and use this data to calibrate our model. However, there are many difficulties that our two-dimensional model encounters in such a calibration that the original Merton model doesn't:

- First, in using bond data we are faced with many different bond issues. In the original Merton model, we must only concern ourselves with the single estimate of equity and its volatility.
- Second, each bond issue usually has coupons attached to it, whereas our model is for zero-coupon bonds.



- To tackle these two issues, we input a corresponding zero-coupon notional  $N$  and an *average* time-to-maturity of more than one bond issue
- The calibration that follows is computed by Matlab code, using the built in **fsolve** command, designed by Dr. Aditya Viswanathan.



Some more notes regarding inputs to our calibration:

- $E_t^{\text{Mkt}}$  is the market capital obtained from balance sheet information.
- $\sigma_E^{\text{Mkt}}$  is the historical volatility of the stock returns estimated from historical prices.  $\mu_E^{\text{Mkt}}$  is also calculated using this historical data.
- $B_{t,T}^{\text{Mkt}}$  is the total debt obtained from balance sheet information.
- $\sigma_B^{\text{Mkt}}$  is the estimated historical volatility of bond returns.
- $\mu_B^{\text{Mkt}}$  is also calculated using this historical data.



# Market Inputs

- We begin with a benchmark test of the algorithm.
- Consider the following inputs to benchmark the algorithm:

Table: Market-based Input Values

$E_t^{Market}$	19.03931396852958
$\sigma_E^{Market}$	0.6109279750612985
$B_{t,T}^{Market}$	82.47066247641656
$\sigma_B^{Market}$	0.04713145769720459
$\mu_E^{Market}$	0.10
$\mu_B^{Market}$	0.05
$T - t$	7.84
$r$	0.0156
$N$	100



# Model Outputs

The algorithm returns calibrated values

Table: Algorithm Final Values

$R$	80.000000
$\sigma_R$	0.25000
$d_0$	1.000000
$\rho_{AR}$	0.400000.

Using this data, our default and recovery metrics are calculated to be

Table: Metrics

$\tilde{\mathbb{E}}_t \left[ \frac{R_T}{N} \mid D \right]$	$\mathbb{E}_t \left[ \frac{R_T}{N} \mid D \right]$	$\tilde{\mathbb{P}}_t^{\text{Annualized}} [D]$	$\mathbb{P}_t^{\text{Annualized}} [D]$
0.5713913	0.3755446	0.0217939	0.0000120703.



# Notional Sensitivity

Subsequent runs of the algorithm show that solution of this nonlinear system (32) - (33) is indeed sensitive to variation of the input parameters. We concentrate on the shifting of the input notional and the effect on the expected recovery and asset-recovery correlation:

- if the notional is lowered to 99.9, then the algorithm produces a higher physical and risk-neutral expected recovery pair of (59.1%, 39.7%), but asset-recovery correlation decreases to 37.7%.
- if the notional is increased to 100.1, then the algorithm produces a lower physical and risk-neutral expected recovery pair of (54.5%, 34.7%), but asset-recovery correlation increases to 43.2%.
- if the notional is further increased to 100.2, then the algorithm produces a much lower physical and risk-neutral expected recovery pair of (48.7%, 28.6%), but asset-recovery correlation increases significantly to 50.5%.



# Conclusions and Next Papers?

- Have extended one-factor models to decouple recovery risk from default risk, with closed form solutions (both Merton and Black-Cox.)
- Connections between default risk and recovery risk exist, via the notion of partial information.
- Can use with other products, such as Coupon Bonds.
- Recovered assets can exceed pre-default estimated assets. Historically possible (foreclosure / bank fees.)
- Can also include bounded recovery in this framework via Partial Information Transform.
- Calibration to market data for both CDS and Bond prices.





# Acknowledgments

- Joe Campolieti (WLU)
- Peter Carr (NYU)
- Nick Costanzino (co-author, Barclays Capital)
- Stephen J. Mildenhall (St. John's)
- J. Austin Murphy (Oakland)
- Harvey Stein (Bloomberg)
- Aditya Viswanathan (UM - Dearborn)

