Copyright Acknowledgment

Many examples and theorem proofs in these slides, and on in class exam preparation slides, are taken from our textbook "Actuarial Mathematics for Life Contingent Risks" by Dickson, Hardy, and Waters.

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Also, we will from time-to-time look at problems from released previous Exams MLC by the SOA. All such questions belong in copyright to the Society of Actuaries, and we make no claim on them. It is of course an honor to be able to present analysis of such examples here.
An insurance policy is a contract where the policyholder pays a *premium* to the insurer in return for a *benefit* or payment later.

The contract specifies what event the payment is *contingent* on. This event may be random in nature.

Assume that interest rates are deterministic, *for now*.

Consider the case where an insurance company provides a benefit upon death of the policyholder. This time is unknown, and so the issuer requires, at least, a model of human mortality.
Define \((x)\) as a human at age \(x\). Also, define that person’s future lifetime as the continuous random variable \(T_x\). This means that \(x + T_x\) represents that person’s age at death.

Define the *lifetime distribution*

\[
F_x(t) = \mathbb{P}[T_x \leq t]
\]

the probability that \((x)\) does not survive beyond age \(x + t\) years, and it’s complement, the *survival function* \(S_x(t) = 1 - F_x(t)\).
Conditional Equivalence

We have an important *conditional relationship*

\[ P[T_x \leq t] = P[T_0 \leq x + t | T_0 > x] \]
Conditional Equivalence

We have an important *conditional relationship*

\[
P[T_x \leq t] = P[T_0 \leq x + t \mid T_0 > x] \\
= \frac{P[x < T_0 \leq x + t]}{P[T_0 > x]} \tag{2}
\]

and so

\[
F_x(t) = \frac{F_0(x + t) - F_0(x)}{1 - F_0(x)}
\]
Conditional Equivalence

We have an important *conditional relationship*

\[ P[T_x \leq t] = P[T_0 \leq x + t \mid T_0 > x] \]

\[ = \frac{P[x < T_0 \leq x + t]}{P[T_0 > x]} \quad \text{(2)} \]

and so

\[ F_x(t) = \frac{F_0(x + t) - F_0(x)}{1 - F_0(x)} \]

\[ S_x(t) = \frac{S_0(x + t)}{S_0(x)} \quad \text{(3)} \]

In general we can extend this to

\[ S_x(t + u) = S_x(t)S_{x+t}(u) \quad \text{(4)} \]
Conditions and Assumptions

Conditions on $S_x(t)$

- $S_x(0) = 1$
- $\lim_{t \to \infty} S_x(t) = 0$ for all $x \geq 0$
- $S_x(t_1) \geq S_x(t_2)$ for all $t_1 \leq t_2$ and $x \geq 0$

Assumptions on $S_x(t)$

- $\frac{d}{dt} S_x(t)$ exists $\forall t \in \mathbb{R}_+$
- $\lim_{t \to \infty} t \cdot S_x(t) = 0$ for all $x \geq 0$
- $\lim_{t \to \infty} t^2 \cdot S_x(t) = 0$ for all $x \geq 0$

The last two conditions ensure that $\mathbb{E}[T_x]$ and $\mathbb{E}[T_x^2]$ exist, respectively.
Example 2.1

Assume that $F_0(t) = 1 - \left(1 - \frac{t}{120}\right)^{\frac{1}{6}}$ for $0 \leq t \leq 120$. Calculate the probability that

- (0) survives beyond age 30
- (30) dies before age 50
- (40) survives beyond age 65
Example 2.1

Assume that $F_0(t) = 1 - \left(1 - \frac{t}{120}\right)^{\frac{1}{6}}$ for $0 \leq t \leq 120$. Calculate the probability that

- (0) survives beyond age 30
- (30) dies before age 50
- (40) survives beyond age 65

\[
\mathbb{P}[(0) \text{ survives beyond age } 30] = S_0(30) = 1 - F_0(30) = \left(1 - \frac{30}{120}\right)^{\frac{1}{6}} = 0.9532
\]

\[
\mathbb{P}[(30) \text{ dies before age } 50] = F_{30}(20) = F_0(50) - F_0(30) = 0.0410
\]

\[
\mathbb{P}[(40) \text{ survives beyond age } 65] = S_{40}(25) = \frac{S_0(65)}{S_0(40)} = 0.9395
\]
Recall from basic probability that the density of $F_x(t)$ is defined as 
$$f_x(t) := \frac{d}{dt} F_x(t).$$
Recall from basic probability that the density of $F_x(t)$ is defined as $f_x(t) := \frac{d}{dt} F_x(t)$.

It follows that

\[
f_0(x) := \frac{d}{dx} F_0(x) = \lim_{dx \to 0^+} \frac{F_0(x + dx) - F_0(x)}{dx} = \lim_{dx \to 0^+} \frac{\mathbb{P}[x < T_0 \leq x + dx]}{dx}
\]  
(6)
The Force of Mortality

However, we can find the **conditional** density, also known as the *Force of Mortality* via

\[
\mu_x = \lim_{dx \to 0^+} \frac{\mathbb{P}[x < T_0 \leq x + dx \mid T_0 > x]}{dx}
\]

\[
= \lim_{dx \to 0^+} \frac{\mathbb{P}[T_x \leq dx]}{dx} = \lim_{dx \to 0^+} \frac{1 - S_x(dx)}{dx}
\]

\[
= \lim_{dx \to 0^+} \frac{1 - S_x(dx)}{dx}
\]

\[
= \lim_{dx \to 0^+} \frac{1 - \frac{S_0(x+dx)}{S_0(x)}}{dx} = \frac{1}{S_0(x)} \lim_{dx \to 0^+} \frac{S_0(x) - S_0(x + dx)}{dx}
\]

\[
= - \frac{1}{S_0(x)} \frac{d}{dx} S_0(x) = \frac{f_0(x)}{S_0(x)}
\]
The Force of Mortality

In general, we can show

\[ \mu_{x+t} = -\frac{1}{S_x(t)} \frac{d}{dt} S_x(t) = \frac{f_x(t)}{S_x(t)} \]  

(8)

and integration of this relation leads to

\[ S_x(t) = \frac{S_0(x + t)}{S_0(x)} \]

\[ = \frac{e^{-\int_0^{x+t} \mu_s ds}}{e^{-\int_0^{x} \mu_s ds}} \]

\[ = e^{-\int_x^{x+t} \mu_s ds} \]

\[ = e^{-\int_0^t \mu_{x+s} ds} \]  

(9)
Example 2.2

Assume that $F_0(t) = 1 - \left(1 - \frac{t}{120}\right)^{\frac{1}{6}}$ for $0 \leq t \leq 120$. Calculate $\mu_x$. 
Example 2.2

Assume that $F_0(t) = 1 - \left(1 - \frac{t}{120}\right)^{\frac{1}{6}}$ for $0 \leq t \leq 120$. Calculate $\mu_x$

\[
\frac{d}{dx} S_0(x) = \frac{1}{6} \cdot \left(1 - \frac{x}{120}\right)^{-\frac{5}{6}} \cdot \left(-\frac{1}{120}\right)
\]

\[\therefore \mu_x = \frac{1}{\left(1 - \frac{x}{120}\right)^{\frac{1}{6}}} \cdot \left(\frac{1}{6} \cdot \left(1 - \frac{x}{120}\right)^{-\frac{5}{6}} \cdot \left(-\frac{1}{120}\right)\right)\]

\[= \frac{1}{720 - 6x}\]
One model of human mortality, postulated by Gompertz, is \( \mu_x = Bc^x \), where \((B, c) \in (0, 1) \times (1, \infty)\). This is based on the assumption that mortality is age dependent, and that the growth rate for mortality is proportional to its own value. Makeham proposed that there should also be an age independent component, and so Makeham’s Law is

\[
\mu_x = A + Bc^x
\]  

(11)
Of course, when $A = 0$, this reduces back to Gompertz’ Law.
Of course, when \( A = 0 \), this reduces back to Gompertz’ Law.

By definition,

\[
S_x(t) = e^{- \int_x^{x+t} \mu_s ds} = e^{- \int_x^{x+t} (A + Bc^s) ds} = e^{-At - \frac{B}{\ln c} c^x (c^t - 1)}
\]

Keep in mind that this is a multivariable function of \((x, t) \in \mathbb{R}^2_+\).

**Some online resources:**

Comparison with US Gov’t data

Figure 2. Percent surviving by age, race, and sex: United States, 2000

Figure 3. Percent surviving by age: Death-registration States, 1900–1902, and United States, 1949–51 and 2000
Actuarial Notation

Actuaries make the notational conventions

\[
\begin{align*}
  t p_x &= \mathbb{P}[T_x > t] = S_x(t) \\
  t q_x &= \mathbb{P}[T_x \leq t] = F_x(t) \\
  u \| t q_x &= \mathbb{P}[u < T_x \leq u + t] = S_x(u) - S_x(u + t)
\end{align*}
\]

(u\|t q_x), also known as the deferred mortality probability, is the probability that (x) survives u years, and then dies in the subsequent t years.

Another convention is that \( p_x := 1 p_x \) and \( q_x := 1 q_x \).
Consequently,

\[ t p_x + t q_x = 1 \]

\[ u \mid t q_x = u p_x - u + t p_x \]  

\[ t + u p_x = t p_x \cdot u p_x + t \]  

\[ \mu_x = -\frac{1}{x p_0} \frac{d}{dx} (x p_0) \]

Similarly,

\[ \mu_{x+t} = -\frac{1}{t p_x} \frac{d}{dt} t p_x \Rightarrow \frac{d}{dt} t p_x = \mu_{x+t} \cdot t p_x \]

\[ \mu_{x+t} = \frac{f_x(t)}{S_x(t)} \Rightarrow f_x(t) = \mu_{x+t} \cdot t p_x \]

\[ t p_x = e^{-\int_0^t \mu_{x+s} ds} \]
Also, since $F_x(t) = \int_0^t f_x(s)ds$, we have as a linear approximation

$$t q_x = \int_0^t s p_x \cdot \mu_x + s ds$$
Also, since $F_x(t) = \int_0^t f_x(s)ds$, we have as a linear approximation

$$
\begin{align*}
t q_x &= \int_0^t s p_x \cdot \mu_{x+s} ds \\
q_x &= \int_0^1 s p_x \cdot \mu_{x+s} ds \\
&= \int_0^1 e^{-\int_0^s \mu_{x+v} dv} \cdot \mu_{x+s} ds \\
&\approx \int_0^1 \mu_{x+s} ds \\
&\approx \mu_{x+\frac{1}{2}}
\end{align*}
$$

(16)
Mean and Standard Deviation of $T_x$

Actuaries make the notational definition $\hat{e}_x := \mathbb{E}[T_x]$, also known as the *complete expectation of life*. Recall $f_x(t) = t p_x \cdot \mu_{x+t} = -\frac{d}{dt} t p_x$, and

$$\hat{e}_x = \int_0^\infty t \cdot f_x(t) dt$$

$$= \int_0^\infty t \cdot t p_x \cdot \mu_{x+t} dt$$

$$= \int_0^\infty t \cdot -\frac{d}{dt} t p_x dt$$
Mean and Standard Deviation of $T_x$

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\[
\hat{e}_x = \int_0^\infty t \cdot f_x(t) dt \\
= \int_0^\infty t \cdot t p_x \cdot \mu_{x+t} dt \\
= \int_0^\infty t \cdot \frac{d}{dt} t p_x dt \\
= \int_0^\infty t p_x dt
\]

$E[T_x^2] := \mathbb{E}[T_x^2]$
Actuaries make the notational definition $\hat{e}_x := \mathbb{E}[T_x]$, also known as the complete expectation of life. Recall $f_x(t) = t p_x \cdot \mu_{x+t} = -\frac{d}{dt} t p_x$, and

\[
\hat{e}_x = \int_0^\infty t \cdot f_x(t) dt = \int_0^\infty t \cdot t p_x \cdot \mu_{x+t} dt = \int_0^\infty t \cdot -\frac{d}{dt} t p_x dt = \int_0^\infty t p_x dt
\]

\[
\mathbb{E}[T_x^2] = \int_0^\infty t^2 \cdot f_x(t) dt = \int_0^\infty 2t \cdot t p_x dt
\]
Mean and Standard Deviation of $T_x$

Actuaries make the notational definition $\hat{e}_x := \mathbb{E}[T_x]$, also known as the *complete expectation of life*. Recall $f_x(t) = t p_x \cdot \mu_{x+t} = -\frac{d}{dt} t p_x$, and

\[
\hat{e}_x = \int_0^\infty t \cdot f_x(t) \, dt
\]

\[
= \int_0^\infty t \cdot t p_x \cdot \mu_{x+t} \, dt
\]

\[
= \int_0^\infty t \cdot -\frac{d}{dt} t p_x \, dt
\]

\[
= \int_0^\infty t p_x \, dt
\]

\[
E[T_x^2] = \int_0^\infty t^2 \cdot f_x(t) \, dt = \int_0^\infty 2t \cdot t p_x \, dt
\]

\[
V[T_x] := E[T_x^2] - (\hat{e}_x)^2
\]
Assume that $F_0(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}}$ for $0 \leq x \leq 120$. Calculate $\hat{e}_x$, $V[T_x]$ for $a.) x = 30$ and $b.) x = 80$. 
Example 2.6

Assume that $F_0(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}}$ for $0 \leq x \leq 120$. Calculate $\hat{e}_x$, $V[T_x]$ for $a.) x = 30$ and $b.) x = 80$.

Since $S_0(x) = \left(1 - \frac{x}{120}\right)^{\frac{1}{6}}$, it follows that in keeping with the model where survival is constrained to be less than 120,

$$t p_x = \frac{S_0(x + t)}{S_0(x)} = \begin{cases} 
\left(1 - \frac{t}{120-x}\right)^{\frac{1}{6}} : x + t \leq 120 \\
0 : x + t > 120 
\end{cases}$$
Example 2.6

So,

\[ \hat{e}_x = \int_0^{120-x} \left( 1 - \frac{t}{120-x} \right)^{\frac{1}{6}} dt = \frac{6}{7} \cdot (120 - x) \]

\[ \mathbb{E}[T_x^2] = \int_0^{120-x} 2t \cdot \left( 1 - \frac{t}{120-x} \right)^{\frac{1}{6}} dt \]

\[ = \left( \frac{6}{7} - \frac{6}{13} \right) \cdot 2(120 - x)^2 \]  

(18)

and

\( (\hat{e}_{30}, \hat{e}_{80}) = (77.143, 34.286) \)

\( (V[T_{30}], V[T_{80}]) = ((21.396)^2, (9.509)^2) \)  

(19)
Kevin and Kira excel at the newest video game at the local arcade, “Reversion”. The arcade has only one station for it. Kevin is playing. Kira is next in line. You are given:

- (i) Kevin will play until his parents call him to come home.
- (ii) Kira will leave when her parents call her. She will start playing as soon as Kevin leaves if he is called first.
- (iii) Each child is subject to a constant force of being called: 0.7 per hour for Kevin; 0.6 per hour for Kira.
- (iv) Calls are independent.
- (v) If Kira gets to play, she will score points at a rate of 100,000 per hour.
Calculate the expected number of points Kira will score before she leaves.

- (A) 77,000
- (B) 80,000
- (C) 84,000
- (D) 87,000
- (E) 90,000
Define

\[ tp_x = \mathbb{P}[\text{Kevin still there}] \]
\[ tp_y = \mathbb{P}[\text{Kira still there}] \]  \hspace{1cm} (20)

and so

\[ \mathbb{E}[\text{Kira’s playing time}] = \int_0^\infty (1 - tp_x) \cdot tp_y \, dt \]
\[ = \int_0^\infty (1 - e^{-0.7t}) \cdot e^{-0.6t} \, dt \]  \hspace{1cm} (21)
\[ = \int_0^\infty (e^{-0.6t} - e^{-1.3t}) \, dt \]
\[ = \frac{1}{0.6} - \frac{1}{1.3} \approx 0.89744 \text{ hrs.} \]
It follows that

\[
\mathbb{E} \text{[Kira’s winnings]} = 100000 \frac{\$}{hr} \cdot \mathbb{E} \text{[Kira’s playing time]} \]

\[= \$89744.\]  

Hence, we choose \((E)\).
Numerical Considerations for $T_x$

In general, computations for the mean and SD for $T_x$ will require numerical integration. For example,

**Table: 2.1: Gompertz Model Statistics: $(B, c) = (0.0003, 1.07)$**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\hat{e}_x$</th>
<th>$SD[T_x]$</th>
<th>$x + \hat{e}_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>71.938</td>
<td>18.074</td>
<td>71.938</td>
</tr>
<tr>
<td>10</td>
<td>62.223</td>
<td>17.579</td>
<td>72.223</td>
</tr>
<tr>
<td>20</td>
<td>52.703</td>
<td>16.857</td>
<td>72.703</td>
</tr>
<tr>
<td>30</td>
<td>43.492</td>
<td>15.841</td>
<td>73.492</td>
</tr>
<tr>
<td>40</td>
<td>34.252</td>
<td>14.477</td>
<td>74.752</td>
</tr>
<tr>
<td>50</td>
<td>26.691</td>
<td>12.746</td>
<td>76.691</td>
</tr>
<tr>
<td>60</td>
<td>19.550</td>
<td>10.693</td>
<td>79.550</td>
</tr>
<tr>
<td>70</td>
<td>13.555</td>
<td>8.449</td>
<td>83.555</td>
</tr>
<tr>
<td>80</td>
<td>8.848</td>
<td>6.224</td>
<td>88.848</td>
</tr>
<tr>
<td>90</td>
<td>5.433</td>
<td>4.246</td>
<td>95.433</td>
</tr>
<tr>
<td>100</td>
<td>3.152</td>
<td>2.682</td>
<td>103.152</td>
</tr>
</tbody>
</table>
Define

\[ K_x := \lfloor T_x \rfloor \]  \hspace{1cm} (23)

and so

\[
P[K_x = k] = P[k \leq T_x < k + 1] = k \cdot q_x
\]

\[
= k \cdot p_x - (k+1) \cdot p_x
\]

\[
= k \cdot p_x - k \cdot p_x \cdot p_{x+k}
\]

\[
= k \cdot p_x \cdot q_{x+k}
\]  \hspace{1cm} (24)
Curtate Future Lifetime

\[ \mathbb{E}[K_x] := e_x = \sum_{k=0}^{\infty} k \cdot \mathbb{P}[K_x = k] \]

\[ = \sum_{k=0}^{\infty} k \cdot (k p_x - (k+1) p_x) \]

\[ = \sum_{k=1}^{\infty} k p_x \text{ by telescoping series.} \]
Curtate Future Lifetime

\[ \mathbb{E}[K_x] := e_x = \sum_{k=0}^{\infty} k \cdot P[K_x = k] \]

\[ = \sum_{k=0}^{\infty} k \cdot (k p_x - (k+1)p_x) \]

\[ = \sum_{k=1}^{\infty} k p_x \] by telescoping series.

(25)

\[ \mathbb{E}[K_x^2] = \sum_{k=0}^{\infty} k^2 \cdot P[K_x = k] \]

\[ = 2 \cdot \sum_{k=1}^{\infty} k \cdot k p_x - \sum_{k=1}^{\infty} k p_x \]

\[ = 2 \cdot \sum_{k=1}^{\infty} k \cdot k p_x - e_x \]
Recall

\[ \hat{e}_x = \int_0^\infty t p_x \, dt = \sum_{j=0}^{\infty} \int_j^{j+1} t p_x \, dt \]  

(26)

By trapezoid rule for numerical integration, we obtain

\[ \int_j^{j+1} t p_x \, dt \approx \frac{1}{2} (j p_x + (j+1) p_x), \]  

and so

\[ \hat{e}_x \approx \sum_{j=0}^{\infty} \frac{1}{2} (j p_x + (j+1) p_x) \]

(27)

\[ = \frac{1}{2} + \sum_{j=1}^{\infty} j p_x = \frac{1}{2} + e_x \]

As with all numerical schemes, this approximation can be refined when necessary.
Comparison of $\hat{e}_x$ and $e_x$

Approximation matches well for small values of $x$

Table: 2.2: Gompertz Model Statistics: $(B, c) = (0.0003, 1.07)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$e_x$</th>
<th>$\hat{e}_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>71.438</td>
<td>71.938</td>
</tr>
<tr>
<td>10</td>
<td>61.723</td>
<td>62.223</td>
</tr>
<tr>
<td>20</td>
<td>52.203</td>
<td>52.703</td>
</tr>
<tr>
<td>30</td>
<td>42.992</td>
<td>43.492</td>
</tr>
<tr>
<td>40</td>
<td>34.252</td>
<td>34.752</td>
</tr>
<tr>
<td>50</td>
<td>26.192</td>
<td>26.691</td>
</tr>
<tr>
<td>60</td>
<td>19.052</td>
<td>19.550</td>
</tr>
<tr>
<td>70</td>
<td>13.058</td>
<td>13.55</td>
</tr>
<tr>
<td>80</td>
<td>8.354</td>
<td>8.848</td>
</tr>
<tr>
<td>90</td>
<td>4.944</td>
<td>5.433</td>
</tr>
<tr>
<td>100</td>
<td>2.673</td>
<td>3.152</td>
</tr>
</tbody>
</table>
An extension of Gompertz - Makeham Laws is the $GM(r, s)$ formula

$$\mu_x = h_r^1(x) + e^{h_s^2(x)}$$

where $h_r^1(x), h_s^2(x)$ are polynomials of degree $r$ and $s$, respectively.

Hazard rate in survival analysis and failure rate in reliability theory is the same as what actuaries call force of mortality.
HW: 2.1, 2.2, 2.5, 2.6, 2.10, 2.13, 2.14, 2.15
Define for a model with maximum age $\omega$ and initial age $x_0$ the *radix* $l_{x_0}$, where

$$l_{x_0 + t} = l_{x_0} \cdot t \cdot p_{x_0}$$

(28)
It follows that

\[ l_{x+t} = l_{x_0} \cdot x+t-x_0 \cdot p_{x_0} \]

\[ = l_{x_0} \cdot x-x_0 \cdot p_{x_0} \cdot t \cdot p_x \]

\[ = l_x \cdot t \cdot p_x \]
It follows that

\[ l_{x+t} = l_{x_0} \cdot x+t-x_0 \cdot p_{x_0} \]
\[ = l_{x_0} \cdot x-x_0 \cdot p_{x_0} \cdot t \cdot p_x \]
\[ = l_x \cdot t \cdot p_x \]

We assume a binomial model where \( L_t \) is the number of survivors to age \( x + t \).
So, if there are $l_x$ independent individuals aged $x$ with probability $tp_x$ of survival to age $x + t$, then we interpret $l_{x+t}$ as the expected number of survivors to age $x + t$ out of $l_x$ independent individuals aged $x$. Symbolically,

$$\mathbb{E}[L_t \mid L_0 = l_x] = l_{x+t} = l_x \cdot tp_x$$

(30)
So, if there are \( l_x \) independent individuals aged \( x \) with probability \( t p_x \) of survival to age \( x + t \), then we interpret \( l_{x+t} \) as the expected number of survivors to age \( x + t \) out of \( l_x \) independent individuals aged \( x \). Symbolically,

\[
\mathbb{E}[L_t | L_0 = l_x] = l_{x+t} = l_x \cdot t p_x
\] (30)

Also, define the expected number of deaths from year \( x \) to year \( x + 1 \) as

\[
d_x := l_x - l_{x+1} = l_x \cdot \left(1 - \frac{l_{x+1}}{l_x}\right) = l_x \cdot (1 - p_x) = l_x q_x
\] (31)
Example 3.1

Table: 3.1: Extract from a life table

<table>
<thead>
<tr>
<th>x</th>
<th>$l_x$</th>
<th>$d_x$</th>
</tr>
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<td>72.99</td>
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<tr>
<td>39</td>
<td>9534.08</td>
<td>80.11</td>
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</table>
Example 3.1

Calculate:

- a.) $l_{40}$
- b.) $10p_{30}$
- c.) $q_{35}$
- d.) $5q_{30}$
- e.) $\mathbb{P}[(30) \text{ dies between age 35 and 36}]$
Calculate:

- a.) \( l_{40} = l_{39} - d_{39} = 9453.97 \)
Example 3.1

Calculate:

- a.) \( l_{40} = l_{39} - d_{39} = 9453.97 \)
- b.) \( 10p_{30} = \frac{l_{40}}{l_{30}} = 0.94540 \)
Example 3.1

Calculate:

- a.) $l_{40} = l_{39} - d_{39} = 9453.97$
- b.) $10p_{30} = \frac{l_{40}}{l_{30}} = 0.94540$
- c.) $q_{35} = \frac{d_{35}}{l_{35}} = 0.00564$
Example 3.1

Calculate:

a.) \( l_{40} = l_{39} - d_{39} = 9453.97 \)

b.) \( 10p_{30} = \frac{l_{40}}{l_{30}} = 0.94540 \)

c.) \( q_{35} = \frac{d_{35}}{l_{35}} = 0.00564 \)

d.) \( 5q_{30} = \frac{l_{30} - l_{35}}{l_{30}} = 0.02107 \)
Example 3.1

Calculate:

a.) \( l_{40} = l_{39} - d_{39} = 9453.97 \)
b.) \( 10p_{30} = \frac{l_{40}}{l_{30}} = 0.94540 \)
c.) \( q_{35} = \frac{d_{35}}{l_{35}} = 0.00564 \)
d.) \( 5q_{30} = \frac{l_{30} - l_{35}}{l_{30}} = 0.02107 \)
e.) \( P[(30) \text{ dies between age 35 and 36}] = \frac{l_{35} - l_{36}}{l_{30}} = 0.00552 \)
So far, the life table approach has mirrored the survival distribution method we encountered in the previous lecture. However, in detailing the life table, no information is presented on the cohort in between whole years. To account for this, we must make some fractional age assumptions. The following are equivalent:
Fractional Age Assumptions

So far, the life table approach has mirrored the survival distribution method we encountered in the previous lecture. However, in detailing the life table, no information is presented on the cohort in between whole years. To account for this, we must make some **fractional age assumptions**. The following are equivalent:

- **UDD1** For all \((x, s) \in \mathbb{N} \times [0, 1)\), we assume that \(s q_x = s \cdot q_x\)
- **UDD2** For all \(x \in \mathbb{N}\), we assume
  - \(R_x := T_x - K_x \sim U(0, 1)\)
  - \(R_x\) is independent of \(K_x\).
Proof of Equivalence

Proof:

**UDD1 ⇒ UDD2**: Assume for all \((x, s) \in \mathbb{N} \times [0, 1)\), we assume that \(s q_x = s \cdot q_x\). Then

\[
\begin{align*}
\mathbb{P} [R_x \leq s] &= \sum_{k=0}^{\infty} \mathbb{P} [R_x \leq s, K_x = k] \\
&= \sum_{k=0}^{\infty} \mathbb{P} [k \leq T_x \leq k + s] \\
&= \sum_{k=0}^{\infty} k p_x \cdot s q_{x+k} = \sum_{k=0}^{\infty} k p_x \cdot s \cdot q_{x+k} \\
&= s \cdot \sum_{k=0}^{\infty} k p_x \cdot q_{x+k} = s \cdot \sum_{k=0}^{\infty} \mathbb{P} [K_x = k] = s
\end{align*}
\]

and so \(R_x \sim U(0, 1)\).
### Proof of Equivalence

To show independence of $R_x$ and $K_x$,

\[
P[R_x \leq s, K_x = k] = P[k \leq T_x \leq k + s]
\]

\[
= k p_x \cdot s q_{x+k}
\]

\[
= s \cdot k p_x \cdot q_{x+k}
\]

\[
= P[R_x \leq s] \cdot P[K_x = k]
\]

\[
\text{(33)}
\]

**UDD2 ⇒ UDD1**: Assuming **UDD2** is true, then for $(x, s) \in \mathbb{N} \times [0, 1)$ we have

\[
s q_x = P[T_x \leq s]
\]

\[
= P[K_x = 0, R_x \leq s]
\]

\[
= P[R_x \leq s] \cdot P[K_x = 0]
\]

\[
= s \cdot q_x
\]

\[
\text{(34)}
\]

**QED**
Recall that \( sq_x = \frac{l_x - l_{x+s}}{l_x} \). It follows now that

\[
sq_x = sq_x = s \frac{d_x}{l_x} = \frac{l_x - l_{x+s}}{l_x}
\]
Corollary

Recall that \( sq_x = \frac{l_x - l_{x+s}}{l_x} \). It follows now that

\[
 sq_x = sq_x = s \frac{d_x}{l_x} = \frac{l_x - l_{x+s}}{l_x}
\]

\( l_{x+s} = l_x - s \cdot d_x \)

which is a linear decreasing function of \( s \in [0, 1) \)
Recall that \( s q_x = \frac{l_x - l_{x+s}}{l_x} \). It follows now that

\[
\begin{align*}
    s q_x &= s q_x = s \frac{d_x}{l_x} = \frac{l_x - l_{x+s}}{l_x} \\
    l_{x+s} &= l_x - s \cdot d_x
\end{align*}
\]

which is a linear decreasing function of \( s \in [0, 1) \).

\[
q_x = \frac{d}{ds} [s q_x] = f_x(s) = s p_x \cdot \mu_{x+s}
\]

But, since \( q_x \) is constant in \( s \), we have \( f_x(s) \) is constant for \( s \in [0, 1) \).

Read over Examples 3.2 – 3.5
For all \((x, s) \in \mathbb{N} \times [0, 1)\), we assume that \(\mu_{x+s}\) does not depend on \(s\), and we denote \(\mu_{x+s} := \mu_x^*\). It follows that

\[
\begin{align*}
  p_x &= e^{-\int_0^1 \mu_{x+s} ds} = e^{-\mu_x^*} \\
  sp_x &= e^{-\int_0^s \mu_x^* du} = e^{-\mu_x^* s} = (p_x)^s \\
  sp_{x+t} &= e^{-\int_0^s \mu_x^* du} = (p_x)^s \quad \text{when } t + s < 1 \\
  q_x &= 1 - e^{-\mu_x^*} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (\mu_x^*)^k}{k!} \approx \mu_x^* \\
  s q_x &= 1 - e^{-\mu_x^* t} \approx \mu_x^* t,
\end{align*}
\]

where the last two lines assume \(\mu_x^* \ll 1\).

Read Examples 3.6, 3.7 and Sections 3.4, 3.5, 3.6.
Homework Questions

HW: 3.1, 3.2, 3.4, 3.7, 3.8, 3.9, 3.10
Contingent Events

We have spent the previous two lectures on modeling human mortality. The need for such models in insurance pricing arises when designing contracts that are **event-contingent**. Such events include reaching retirement before the end of the underlying life \( (x) \).

However, one can also write contracts that are dependent on a life \( (x) \) being admitted to college (planning for school), and also on \( (x)'s \) external portfolios maintaining a minimal value over a time-interval (insuring external investments.)
Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and an event \(A \in \mathcal{F}\).

If we are working with a force of interest \(\delta_s(\omega)\) and the time of event \(W\) as \(\tau_W\), then we have under the stated probability measure \(\mathbb{P}\) the **Expected Present Value of a payoff** \(K(\omega)\) contingent upon \(W\)

\[
EPV = \mathbb{E}[K(\omega)e^{-\int_0^{\tau_W} \delta_s(\omega)ds}]
\]

Actuarial Encounters of the Third Kind !!
Some Initial Simplifying Assumptions

- $K(\omega) = 1$ for all $\omega \in \Omega$ (a.s.)
- $\delta_s(\omega) = \delta$ for all $\omega \in \Omega$ (a.s.)
- $W := \{\text{event that } (x) \text{ dies}\} \Rightarrow \tau_W := T_x$
- $\mathcal{P}$ is obtained via historical observation and is thus a physical measure. Specifically, we use $t p_x$ obtained from life tables or via models of human mortality
- We do not assume now that a unique risk-neutral pricing measure $\tilde{\mathcal{P}}$ exists.
- **Standard Ultimate Survival Model** with assumes Makeham’s law with $(A, B, c) = (0.00022, 2.7 \times 10^{-6}, 1.124)$
Recall...

- The equivalent interest rate $i := e^\delta - 1$ per year
- The discount factor $v := \frac{1}{1+i} = e^{-\delta}$ per year
- The nominal interest rate $i^{(p)} = p \cdot \left( (1 + i)^{\frac{1}{p}} - 1 \right)$ compounded $p$ times per year
- The effective rate of discount $d := 1 - v = i \cdot v = 1 - e^{-\delta}$ per year
- The nominal rate of discount $d^{(p)} := p \cdot \left( 1 - v^{\frac{1}{p}} \right)$ compounded $p$ times per year
Consider now the random variable

$$ Z = v^T_x = e^{-\delta T_x} $$

(38)

which represents the present value of a dollar upon death of \((x)\). We are interested in statistical measures of this quantity:

$$ \mathbb{E}[Z] = \bar{A}_x := \mathbb{E}[e^{-\delta T_x}] = \int_0^\infty e^{-\delta t} t p_x \mu_{x+t} dt $$

$$ \mathbb{E}[Z^2] = 2 \bar{A}_x := \mathbb{E}[e^{-2\delta T_x}] $$

$$ = \int_0^\infty e^{-2\delta t} t p_x \mu_{x+t} dt $$

(39)

$$ \text{Var}(Z) = 2 \bar{A}_x - (\bar{A}_x)^2 $$

$$ \mathbb{P}[Z \leq z] = \mathbb{P} \left[ T_x \geq \frac{-\ln(z)}{\delta} \right] $$
Whole Life Insurance: Yearly Case

Assuming payments are made at the end of the death year, our random variable is now $Z = v^{K_x+1} = e^{-\delta K_x-\delta}$ and so

$$E[Z] = A_x := E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} P[K_x = k]$$

$$= \sum_{k=0}^{\infty} v^{k+1} k \cdot q_x$$

$$E[Z^2] = \sum_{k=0}^{\infty} v^{2k+2} k \cdot q_x$$

$$Var(Z) = 2 A_x - (A_x)^2$$

$$P[Z \leq z] = P \left[ K_x \geq \frac{-\delta - \ln(z)}{\delta} \right]$$

(40)
Instead of only paying at the end of the last whole year lived, an insurance contract might specify payment upon the end of the last period lived. In this case, if we split a year into $m$ periods, and define

$$K_x^{(m)} = \frac{1}{m} \lfloor mT_x \rfloor$$  \hspace{1cm} (41)

For example, if $K_x = 19.78$, then

$$K_x^{(m)} = \begin{cases} 
19 & m = 1 \\
19\frac{1}{2} = 19.5 & m = 2 \\
19\frac{3}{4} = 19.75 & m = 4 \\
19\frac{9}{12} = 19.75 & m = 12
\end{cases}$$
Whole Life Insurance: \( \frac{1}{m} \) thly Case

It follows that we need \( \forall r \in \{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1, \frac{m+1}{m}, \ldots\} \)

\[
P[K_x^{(m)} = r] = P[r \leq T_x < r + \frac{1}{m}] = r \frac{1}{m} q_x \tag{42}
\]

to compute statistics for our random variable

\[
Z = \sqrt{K_x^{(m)} + \frac{1}{m}} \tag{43}
\]
Whole Life Insurance: $\frac{1}{m}^{thly}$ Case

$$
\mathbb{E}[Z] = A_x^{(m)} := \mathbb{E}[vK_x^{(m)} + \frac{1}{m}] = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} \frac{k}{m} \frac{1}{m} q_x
$$

$$
\mathbb{E}[Z^2] = 2A_x^{(m)} = \sum_{k=0}^{\infty} v^{\frac{2k+2}{m}} \frac{k}{m} \frac{1}{m} q_x
$$

$$
\text{Var}(Z) = 2A_x^{(m)} - \left( A_x^{(m)} \right)^2
$$

$$
\mathbb{P}[Z \leq z] = \mathbb{P} \left[ K_x^{(m)} \geq -\frac{\ln(z)}{\delta} - \frac{1}{m} \right]
$$
One of the computational tools we share directly with quantitative finance is the method of backwards-pricing. In option pricing, we assume the contract has a finite term. Here, we assume a finite lifetime maximum of \( \omega < \infty \). It follows that

\[
A_{\omega-1} = \mathbb{E} \left[ v^{K_{\omega-1}+1} \right] = \mathbb{E} \left[ v^1 \right] = v
\]

(45)
Recursion Method

One of the computational tools we share directly with quantitative finance is the method of backwards-pricing. In option pricing, we assume the contract has a finite term. Here, we assume a finite lifetime maximum of $\omega < \infty$. It follows that

$$A_{\omega-1} = E \left[ v^{K_{\omega-1}+1} \right] = E \left[ v^1 \right] = v \quad (45)$$

At age $\omega - 2$, we have $P[K_{\omega-2} = 0] = q_{\omega-2}$ and so

$$A_{\omega-2} = E \left[ v^{K_{\omega-2}+1} \right]$$

$$= q_{\omega-2} \cdot v + p_{\omega-2} \cdot E \left[ v^{(1+K_{\omega-1})+1} \right]$$

$$= q_{\omega-2} \cdot v + p_{\omega-2} \cdot v \cdot E \left[ v^{K_{\omega-1}+1} \right]$$

$$= q_{\omega-2} \cdot v + p_{\omega-2} \cdot v^2 \quad (46)$$
Recursion Method

In general, we have the recursion equation for a life ($x$) that satisfies

$$A_x = vq_x + vp_x A_{x+1}$$

(47)

$$A_{\omega-1} = v$$

in the whole life case, and

$$A^{(m)}_x = v \frac{1}{m} q_{x} + v \frac{1}{m} p_{x} A^{(m)}_{x+\frac{1}{m}}$$

(48)

$$A^{(m)}_{\omega-\frac{1}{m}} = v \frac{1}{m}$$

in the $\frac{1}{m}$thly case.
Recall that for Makeham’s law we have, respectively

\[
\frac{1}{m} p_x = e^{- \frac{A}{m} - \frac{Bc^x}{\ln(c)} \left( \frac{1}{c} - 1 \right)}
\]

\[
1p_x = e^{-A - \frac{Bc^x}{\ln(c)}(c-1)}
\]  

and for the power law of survival, \( S_0(x) = \left(1 - \frac{x}{\omega}\right)^a \) for some \( a, \omega > 0 \)

\[
\frac{1}{m} p_x = \left( \frac{\omega - x - \frac{1}{m}}{\omega - x} \right)^a
\]

\[
1p_x = \left( \frac{\omega - x - 1}{\omega - x} \right)^a
\]
For any positive integer $m$, it follows that for Makeham’s law:

\[ A_x^{(m)} = \nu \frac{1}{m} \left( 1 - e^{-\frac{A}{m} - \frac{Bc^x}{\ln(c)} \left( c \frac{1}{m} - 1 \right)} \right) \]

\[ + \nu \frac{1}{m} e^{-\frac{A}{m} - \frac{Bc^x}{\ln(c)} \left( c \frac{1}{m} - 1 \right)} A_{x+\frac{1}{m}}^{(m)} \]

\[ A_{\omega - \frac{1}{m}}^{(m)} = \nu \frac{1}{m} \]
Recursion Method

For any positive integer \( m \), it follows that for **Power law**:

\[
A_x^{(m)} = v^\frac{1}{m} \left( 1 - \left( \frac{\omega - x - \frac{1}{m}}{\omega - x} \right)^a \right)
\]

\[
+ v^\frac{1}{m} \left( \frac{\omega - x - \frac{1}{m}}{\omega - x} \right)^a A_{x+\frac{1}{m}}^{(m)}
\]

\[
A_{\omega - \frac{1}{m}}^{(m)} = v^\frac{1}{m}
\]

HW Project: For Power law with \( a = \frac{3}{5} \) and \( \omega = 101 \)

- Generate a spreadsheet like Table 4.1 in the text, *including* values for \( 2A_x^{(m)} \)
- Repeat Example 4.3 with the Power law model
Consider now the case where payment is made in the continuous case, and death benefit is payable to the policyholder only if $T_x \leq n$. Then, we are interested in the random variable

$$Z = e^{-\delta T_x} 1\{T_x \leq n\}$$  \hspace{1cm} (53)$$

and so

$$\bar{A}^1_{x:n} = \mathbb{E}[Z] = \int_0^n e^{-\delta t} t p_x \mu_x + t dt$$

$$2 \bar{A}^1_{x:n} = \mathbb{E}[Z^2] = \int_0^n e^{-2\delta t} t p_x \mu_x + t dt$$  \hspace{1cm} (54)$$
Term Insurance: $\frac{1}{m}^{thly}$ Case

Consider again the case where the death benefit is payable at the end of the $\frac{1}{m}^{thly}$ period in the death year to the policyholder only if $K_x^{(m)} + \frac{1}{m} \leq n$. Then, we are interested in the random variable

$$Z = e^{-\delta(K_x^{(m)} + \frac{1}{m})} \mathbb{1}\{K_x^{(m)} + \frac{1}{m} \leq n\}$$

and so

$$A^{(m)}_{1\ x:n} = \mathbb{E}[Z] = \sum_{k=0}^{mn-1} v^{k+1} \left(\frac{k+1}{m}\right) q_x$$

$$\mathbb{E}[Z^2] = \sum_{k=0}^{mn-1} v^{2k+2} \left(\frac{k+1}{m}\right) q_x$$
Pure Endowment

Pure endowment benefits depend on the survival policyholder \((x)\) until at least age \(x + n\). In such a contract, a fixed benefit of 1 is paid at time \(n\). This is expressed via

\[
Z = e^{-\delta n} 1\{T_x \geq n\}
\]

\[
A_{x: n}^1 = \mathbb{E}[Z] = v^n n p_x
\]

\[
= e^{-\delta n} n p_x
\]
Endowment Insurance

Endowment insurance is a combination of term insurance and pure endowment. In such a policy, the amount is paid upon death if it occurs with a fixed term \( n \). However, if \((x)\) survives beyond \( n\) years, the sum insured is payable at the end of the \( n^{th}\) year. The corresponding present value random variable is

\[
Z = e^{-\delta \min\{T_x, n\}}
\]

\[
\mathbb{E}[Z] = \bar{A}_{x:n}
\]

\[
= \int_{0}^{n} e^{-\delta t} t p_x \mu_x + t dt + e^{-\delta n} n p_x
\]

\[
= \bar{A}^{1}_{x:n} + A_{x:1}
\]
Endowment insurance is a combination of term insurance and pure endowment. In such a policy, the amount is paid upon death if it occurs with a fixed term $n$. However, if $(x)$ survives beyond $n$ years, the sum insured is payable at the end of the $n^{th}$ year. The corresponding present value random variable is

$$Z = e^{-\delta \min\{T_x, n\}}$$

$$\mathbb{E}[Z] = \bar{A}_{x:\overline{n}}$$

$$= \int_0^n e^{-\delta t} t p_x \mu_x + t \, dt + e^{-\delta n} np_x$$

$$= \bar{A}_{x:\overline{n}}^1 + A_{x:\overline{n}}^1$$

$$Z = e^{-\delta \min\{K_x + 1, n\}}$$

$$\Rightarrow A_{x:\overline{n}} = A_{x:\overline{n}}^1 + A_{x:\overline{n}}^1.$$
Endowment insurance is a combination of term insurance and pure endowment. In such a policy, the amount is paid upon death if it occurs with a fixed term $n$. However, if $(x)$ survives beyond $n$ years, the sum insured is payable at the end of the $n^{th}$ year. The corresponding present value random variable is

$$Z = e^{-\delta \min\{T_x, n\}}$$

$$\mathbb{E}[Z] = \bar{A}_{x:n}$$

$$= \int_0^n e^{-\delta t} p_x \mu_x t dt + e^{-\delta n} n p_x$$

$$= \bar{A}_{x+1:n} + A_{x:n}$$

$$Z = e^{-\delta \min\{K_x+1, n\}}$$

$$\Rightarrow A_{x:n} = A_{x+1:n} + A_{x:n}^{1}.$$

This can also be extended to the $\frac{1}{m}$thly case and for $\mathbb{E}[Z^2]$. 

(58)
Deferred Insurance Benefits

Suppose policyholder on a life \((x)\) receives benefit 1 if \(u \leq T_x < u + n\). Then

\[
Z = e^{-\delta T_x} 1_{\{u \leq T_x < u + n\}} \\
\mathbb{E}[Z] = u | \overline{A}_{x:u+n}^1 \\
= \int_{u}^{u+n} e^{-\delta t} t p_x \mu_{x+t} dt \\
= \int_{0}^{n} e^{-\delta(s+u)} s+u p_x \mu_{x+s+u} ds \\
= e^{-\delta u} \int_{0}^{n} e^{-\delta s} u p_x \cdot s p_x + u \mu_{x+s+u} ds \\
= e^{-\delta u} u p_x \overline{A}_{x+u:u+n}^1 \\
= \overline{A}_{x:u+n}^1 - \overline{A}_{x:u}^1
\]
By definition, we have

\[ A_x = A^1_{x:n} + n|A_x \]
\[ = A^1_{x:n} + n^n p_x A_{x+n} \]  

(60)
By definition, we have

\[ A_x = A_{x:n|} + n|A_x \]

\[ = A_{x:n|} + n \nu^n n p_x A_{x+n} \]  

(60)

What about relationship between \( \bar{A}_x \) and \( A_x \)?
Employing UDD: $\bar{A}_x$ vs $A_x$

If expected values are computed via information derived from life tables, then certainly $\bar{A}_x$ must be approximated using techniques from previous lecture.
If expected values are computed via information derived from life tables, then certainly \( \bar{A}_x \) must be approximated using techniques from previous lecture.

Recall that by the definition of \( sp_x \) and the UDD, we have

\[
sp_x \mu_x + s = f_x(s) = \frac{d}{ds} P[T_x \leq s] = \frac{d}{ds} (s q_x) = \frac{d}{ds} (s \cdot q_x) = q_x
\]  

(61)
Employing UDD: $\tilde{A}_x$ vs $A_x$

It follows that using the UDD approximation leads to

$$\tilde{A}_x = \int_0^\infty e^{-\delta t} t p_x \mu x+t dt = \sum_{k=0}^\infty \int_k^{k+1} e^{-\delta t} t p_x \mu x+t dt$$

$$= \sum_{k=0}^\infty k p_x v^{k+1} \cdot \int_0^1 e^\delta e^{-\delta s} s p_x+k \mu x+k+s ds$$

$$\approx \sum_{k=0}^\infty k p_x v^{k+1} q x+k \cdot \int_0^1 e^\delta e^{-\delta s} ds$$

$$= \sum_{k=0}^\infty v^{k+1} P[K_x = k] \cdot \int_0^1 e^\delta e^{-\delta s} ds = A_x \cdot \int_0^1 e^\delta e^{-\delta s} ds$$

$$= A_x \cdot \frac{i}{\delta}$$
Employing UDD: $\tilde{A}_x$ vs $A_x$

It follows that using the UDD approximation leads to

$$\tilde{A}_x = \int_0^\infty e^{-\delta t} p_x \mu_{x+t} dt = \sum_{k=0}^{\infty} \int_k^{k+1} e^{-\delta t} p_x \mu_{x+t} dt$$

$$= \sum_{k=0}^{\infty} kp_x v^{k+1} \cdot \int_0^1 e^\delta e^{-\delta s} p_{x+k} \mu_{x+k+s} ds$$

$$\approx \sum_{k=0}^{\infty} kp_x v^{k+1} q_{x+k} \cdot \int_0^1 e^\delta e^{-\delta s} ds$$

$$= \sum_{k=0}^{\infty} v^{k+1} P[K_x = k] \cdot \int_0^1 e^\delta e^{-\delta s} ds = A_x \cdot \int_0^1 e^\delta e^{-\delta s} ds$$

$$= A_x \cdot \frac{i}{\delta}$$

$$A_x^{(m)} \approx \frac{i}{i^{(m)}} A_x = \frac{i}{m \cdot \left( (1+i)^{\frac{1}{m}} - 1 \right)} \cdot A_x$$

(62)
Consider now a policy that pays the holder at the end of the $\frac{1}{m}$thly period of death. In this case, the benefit is paid at one of the times $r$ where

$$r \in \left\{ K_x + \frac{1}{m}, K_x + \frac{2}{m}, \ldots, K_x + \frac{m}{m} \right\}$$  \hspace{1cm} (63)$$

and so under the UDD,

$$\mathbb{E} \left[ T_{payment} \mid K_x = k \right] = \sum_{j=1}^{m} \frac{1}{m} \cdot \left( k + \frac{j}{m} \right) = k + \frac{m + 1}{2m}$$  \hspace{1cm} (64)$$
Claims Acceleration Approach: $A_x^{(m)}$ vs $A_x$

Once again, it follows using the UDD approximation

$$A_x^{(m)} = E[v^{K_x(m)} + \frac{1}{m}] = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} \left(1 + \frac{1}{m}\right) q_x$$

$$\approx \sum_{k=0}^{\infty} v^{E[T_{payment} | K_x = k]} \mathbb{P}[K_x = k]$$

$$= \sum_{k=0}^{\infty} v^{k + \frac{m+1}{2m}} q_x = (1 + i)^{\frac{m-1}{2m}} \cdot \sum_{k=0}^{\infty} v^{k+1} q_x$$

$$= (1 + i)^{\frac{m-1}{2m}} \cdot A_x \to (1 + i)^{\frac{1}{2}} \cdot A_x$$

as $m \to \infty$. Note that using the UDD approximation, both $\frac{\bar{A}_x}{A_x}$ and $\frac{A_x^{(m)}}{A_x}$ are independent of $x$. 

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Variable Insurance Benefits

Upon death, we have considered policies that pay the holder a fixed amount. What varied was the method and time of payment. If, however, the actual payoff amount depended on the time $T_x$ of death for $(x)$, then we term such a contract a **Variable Insurance Contract**.

Specifically, if the payoff amount dependent on $T_x$ is $h(T_x)$, then

$$Z = h(T_x)e^{-\delta T_x}$$

$$\mathbb{E}[Z] = \int_0^\infty h(t)e^{-\delta t} tp_x \mu_{x+t} dt$$

$$\bar{I}\bar{A}_x := \int_0^\infty t e^{-\delta t} tp_x \mu_{x+t} dt$$

$$(\bar{I}\bar{A})_{1:x:n} := \int_0^n t e^{-\delta t} tp_x \mu_{x+t} dt$$

(66)
Example 4.8

Consider an \( n \)-year term insurance issued to \((x)\) under which the death benefit is paid at the end of the year of death. The death benefit if death occurs between ages \( x + k \) and \( x + k + 1 \) is valued at \((1 + j)^k\). Hence, using the definition \( i^* := \frac{1+i}{1+j} - 1 \),

\[
Z = v^{K_x+1}(1 + j)^{K_x}
\]

\[
\mathbb{E}[Z] = \sum_{k=0}^{n-1} v^{k+1}(1 + j)^k \cdot q_x
\]

\[
= \frac{1}{1+j} \cdot \sum_{k=0}^{n-1} v^{k+1}(1 + j)^{k+1} \cdot q_x
\]

\[
= \frac{1}{1+j} \cdot \sum_{k=0}^{n-1} \left( \frac{k|q_x}{\left( \frac{1+i}{1+j} \right)^{k+1}} \right) = \frac{1}{1+j} \cdot A_{x:n}^1
\]
Homework Questions

HW: 4.1, 4.2, 4.3, 4.7, 4.9, 4.11, 4.12, 4.14, 4.15, 4.16, 4.17, 4.18
A **Life Annuity** refers to a series of payments to or from an individual as long as that person is still alive. For a fixed rate $i$ and term $n$, we recall the deterministic pricing theory:

\[
\ddot{a}_{n|i} = 1 + v + \ldots + v^{n-1} = \frac{1 - v^n}{d}
\]

\[
a_{n|i} = v + \ldots + v^n = \ddot{a}_{n|i} - 1 + v^n = \frac{1 - v^n}{i}
\]

\[
\ddot{a}_{n|i} = \int_0^n v^t \, dt = \frac{1 - v^n}{\delta}
\]

\[
\ddot{a}_{n|i}^{(m)} = \frac{1}{m} \cdot \left( 1 + v^{\frac{1}{m}} + \ldots + v^{n-\frac{1}{m}} \right) = \frac{1 - v^n}{d^{(m)}}
\]

\[
a_{n|i}^{(m)} = \frac{1}{m} \cdot \left( v^{\frac{1}{m}} + \ldots + v^{n-\frac{1}{m}} + v^n \right) = \frac{1 - v^n}{i^{(m)}}
\]
Whole Life Annuity Due

Consider the case where 1 is paid out at the beginning of every period until death. Our present random variable is now

\[ Y := \dd{a}_{K+1} = \frac{1 - v^{K+1}}{d} \tag{69} \]

and so

\[ \dd{a}_x = \mathbb{E}[Y] = \mathbb{E} \left[ \frac{1 - v^{K+1}}{d} \right] = \frac{1 - A_x}{d} \tag{70} \]

\[ V[Y] = V \left[ \frac{1 - v^{K+1}}{d} \right] = \frac{1}{d^2} V[1] + \frac{1}{d^2} V[v^{K+1}] \]

\[ = 0 + \frac{2A_x - A_x^2}{d^2} \tag{71} \]
The present value random variable can also be represented as

\[
Y = \sum_{k=0}^{\infty} v^k 1_{\{T_x > k\}}
\]

(72)

As \( P[T_x > k] = t p_x \), we have the alternate expression for \( \ddot{a}_x \)

\[
\ddot{a}_x = \mathbb{E}[Y] = \mathbb{E} \left[ \sum_{k=0}^{\infty} v^k 1_{\{T_x > k\}} \right]
\]

\[
= \sum_{k=0}^{\infty} \mathbb{E}[v^k 1_{\{T_x > k\}}]
\]

\[
= \sum_{k=0}^{\infty} v^k k p_x = \sum_{k=0}^{\infty} k q_x \ddot{a}_{x+k+1}
\]

(73)
Define the present value random variable

\[
Y = \begin{cases} 
\ddot{a}_{K_x+1} & : K_x \in \{0, 1, 2, \ldots, n - 1\} \\
\ddot{a}_n & : K_x \in \{n, n + 1, n + 2, \ldots\}
\end{cases}
\]

Another expression is

\[
Y = \dot{a}_{\min\{K_x+1, n\}} = \frac{1 - v^{\min\{K_x+1, n\}}}{d}
\] \tag{74}

and so

\[
\ddot{a}_{x:n} = \mathbb{E}[Y] = \frac{1 - \mathbb{E}[v^{\min\{K_x+1, n\}}]}{d} = \frac{1 - A_{x:n}}{d}
\] \tag{75}

\[
= \sum_{t=0}^{n-1} v^t t \cdot p_x = \sum_{k=0}^{n-1} k \cdot q_x \ddot{a}_{k+1} + np_x \ddot{a}_n
\]
Define $Y^* = \sum_{k=1}^{\infty} v^k 1\{T_x > k\}$. Then we have an annuity immediate that begins payment one unit of time from now. It follows that

$$a_x = \ddot{a}_x - 1$$

$$V[Y^*] = V[Y]$$

(76)

Also, if we define $Y = a_{\min\{K_x, n\}}$, then

$$a_{x:n} = \sum_{t=1}^{n} v^t t p_x = \ddot{a}_{x:n} - 1 + v^n n p_x$$

(77)
Whole Life Continuous Annuity

Define

\[ Y = \bar{a}_{T_x} = \frac{1 - v^{T_x}}{\delta} = \int_0^\infty e^{-\delta t} 1_{\{T_x > t\}} dt \]

\[ \bar{a}_x = \mathbb{E}[Y] = \frac{1 - A_x}{\delta} = \int_0^\infty e^{-\delta t} t p_x dt \]

(78)

Note that if \( \delta = 0 \), then \( \bar{a}_x = \hat{e}_x \)
Define $Y = \bar{a}_{\min \{T_x, n\}}$. Then

\[ Y = \frac{1 - v_{\min \{T_x, n\}}}{\delta} \]

\[ \bar{a}_{x:n} = \mathbb{E} [Y] = \frac{1 - \bar{A}_{x:n}}{\delta} \]

\[ = \int_0^n e^{-\delta t} t p_x dt \]
Deferred Annuity

Consider now the case of an annuity for $(x)$ that will pay 1 at the end of each year, beginning at age $x + u$ and will continue until death age $x + T_x$. We define $u|\ddot{a}_x$ to be the Expected Present Value of this policy. It should be apparent that

$$u|\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:u}$$

$$= \sum_{t=u}^{\infty} v^t \cdot t p_x$$

$$= v^u \cdot u p_x \cdot \sum_{t=0}^{\infty} v^t \cdot t p_{x+u}$$

$$= v^u \cdot u p_x \cdot \ddot{a}_{x+u}$$

holds in the discrete case, and similarly in the continuous case,

$$u|\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:u}$$

(80)
Term Deferred Annuity

For the cases of an annuity for \((x)\) that will pay 1 at the end of each year, beginning at age \(x + u\) and will continue until death age \(x + T_x\) up to a term of length \(n\), or annuity-due payable \(\frac{1}{m}\)thly. Then

\[
\begin{align*}
  u \mid a_{x: \overline{n}} & = v^u u p_x a_{x+u: \overline{n}} \\
  u \mid \ddot{a}_{x}^{(m)} & = v^u u p_x \ddot{a}_{x+u}^{(m)}
\end{align*}
\]

respectively.

These combine with the previous slide to reveal the useful formulae:

\[
\begin{align*}
  \ddot{a}_{x: \overline{n}} & = \ddot{a}_{x} - v^n np_x \ddot{a}_{x+n} \\
  \ddot{a}_{x}^{(m)} & = \ddot{a}_{x}^{(m)} - v^n np_x \ddot{a}_{x+n}^{(m)}
\end{align*}
\]
Define an annuity where the payments increase linearly at times $t = 0, 1, 2, \ldots$ provided that $(x)$ is alive at time $t$

\[
(l \ddot{a})_x = \sum_{t=0}^{\infty} (t + 1) \cdot v^t_t p_x
\]

\[
(l \ddot{a})_{x: \overline{m}} = \sum_{t=0}^{n-1} (t + 1) \cdot v^t_t p_x
\]

(84)
If then the annuity is payable continuously, with payments increasing by 1 at each year end and the rate of payment in the $t^{th}$ year constant and equal to $t$ for $t \in \{1, 2, \ldots, m, \ldots, n\}$, then $h(t) = (m + 1)1_{\{m \leq t < m+1\}}$, and the EPV is

$$\overline{(l \ddot{a})}_{x:m} = \sum_{m=0}^{n-1} (m + 1)m \overline{\ddot{a}}_{x:1}$$

(85)

If $h(t) = t$, then

$$\overline{(l \ddot{a})}_{x:m} = \int_{0}^{n} te^{-\delta t} t p_x \, dt.$$  

(86)
By recursion, we observe

\[ \ddot{a}_x = 1 + v p_x + v^2 p_{x+1} + v^3 p_{x+1} + \ldots \]

\[ = 1 + v p_x (1 + v p_{x+1} + v^2 p_{x+1} + v^3 p_{x+1} + \ldots) \]

\[ = 1 + v p_x \ddot{a}_{x+1} \]
By recursion, we observe

\[ \ddot{a}_x = 1 + \nu p_x + \nu^2 2p_x + \nu^3 3p_x + \ldots \]
\[ = 1 + \nu p_x \left( 1 + \nu p_{x+1} + \nu^2 2p_{x+1} + \nu^3 3p_{x+1} + \ldots \right) \]
\[ = 1 + \nu p_x \ddot{a}_{x+1} \]

\[ \dddot{a}_x^{(m)} = \frac{1}{m} + \nu \frac{1}{m} p_x \dddot{a}_x^{(m)} + \frac{1}{m} \]

(87)
By recursion, we observe

\[ \ddot{a}_x = 1 + v p_x + v^2 p_x + v^3 p_x + \ldots \]

\[ = 1 + v p_x (1 + v p_{x+1} + v^2 p_{x+1} + v^3 p_{x+1} + \ldots) \]

\[ = 1 + v p_x \ddot{a}_{x+1} \]

\[ \ddot{a}_x(m) = \frac{1}{m} + v \frac{1}{m} p_x \ddot{a}_{x+1}(m) \]

Consider the case where there is a maximum age in the model, and so

\[ \ddot{a}_{\omega - 1} = 1 \]

\[ \ddot{a}_{\omega - \frac{1}{m}} = \frac{1}{m} \]
Evaluating Annuities Using UDD

Recall that under the UDD assumption,

\[
A_x^{(m)} = \frac{i}{i^*(m)}A_x
\]

\[
\ddot{A}_x = \frac{i}{\delta}A_x
\]

and by definition,

\[
\ddot{a}_x = \frac{1 - A_x}{d}
\]

\[
\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}
\]

\[
\ddot{a}_x = \frac{1 - \ddot{A}_x}{\delta}
\]
It follows that

\[
\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}
\]

\[
= \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}}
\]

\[
= \frac{i^{(m)} - i A_x}{i^{(m)} d^{(m)}}
\]

\[
= \frac{i^{(m)} - i (1 - d \ddot{a}_x)}{i^{(m)} d^{(m)}}
\]

\[
= \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)} d^{(m)}}
\]

\[
:= \alpha(m) \ddot{a}_x - \beta(m)
\]

\[
\ddot{a}_x = \frac{id}{\delta^2} \ddot{a}_x - \frac{i - \delta}{\delta^2}
\]
For term annuities, we have

\[
\ddot{a}_{x: \bar{n}}^{(m)} = \ddot{a}_x^{(m)} - v^n_n p_x \ddot{a}_{x+n}^{(m)} \\
= \alpha(m) \ddot{a}_x - \beta(m) - v^n_n p_x \cdot (\alpha(m) \ddot{a}_{x+n} - \beta(m)) \\
= \alpha(m) \cdot \left( \ddot{a}_x - v^n_n p_x \ddot{a}_{x+n}^{(m)} \right) - \beta(m) \cdot (1 - v^n_n p_x) \\
= \alpha(m) \cdot \ddot{a}_{x: \bar{n}} - \beta(m) \cdot (1 - v^n_n p_x) \\
\approx \ddot{a}_{x: \bar{n}} - \frac{m - 1}{2m} \cdot (1 - v^n_n p_x)
\]
Guaranteed Annuities

There are instances where an age \((x)\) wishes to buy a policy where payments are guaranteed to continue upon death to a beneficiary. In this case, define the present random variable as \(Y = \dd{a}_n + Y_1\), where

\[
Y_1 = \begin{cases} 
0 & : K_x \in \{0, 1, 2, \ldots, n - 1\} \\
\dd{a}_{K_x + 1} - \dd{a}_n & : K_x \in \{n, n + 1, n + 2, \ldots\}
\end{cases}
\]

and so

\[
\mathbb{E}[Y_1] = \mathbb{E} \left[ \left( \dd{a}_{K_x + 1} - \dd{a}_n \right) 1\{K_x \geq n\} \right] = n\dd{a}_x = v^n n p_x \dd{a}_{x+n}
\]
Guaranteed Annuities

There are instances where an age \( x \) wishes to buy a policy where payments are guaranteed to continue upon death to a beneficiary. In this case, define the present random variable as \( Y = \ddot{a}_n + Y_1 \), where

\[
Y_1 = \begin{cases} 
0 & : K_x \in \{0, 1, 2, \ldots, n - 1\} \\
\ddot{a}_{K_x+1} - \ddot{a}_n & : K_x \in \{n, n + 1, n + 2, \ldots\}
\end{cases}
\]

and so

\[
\mathbb{E}[Y_1] = \mathbb{E} \left[ \left( \ddot{a}_{K_x+1} - \ddot{a}_n \right) 1_{\{K_x \geq n\}} \right] = n \ddot{a}_x = v^n n p_x \ddot{a}_{x+n}
\]

\[
\mathbb{E}[Y] := \ddot{a}_{x:n} = \ddot{a}_n + v^n n p_x \ddot{a}_{x+n}
\]

and \( \mathbb{E}[Y^{(m)}] := \ddot{a}_{x:n}^{(m)} = \ddot{a}_n^{(m)} + v^n n p_x \ddot{a}_{x+n}^{(m)} \)
Example 5.4

A pension plan member is entitled to a benefit of 1000 per month, in advance, for life from age 65, with no guarantee. She can opt to take a lower benefit with a 10–year guarantee. The revised benefit is calculated to have equal EPV at age 65 to the original benefit. Calculate the revised benefit using the Standard Ultimate Survival Model, with interest at 5% per year.
Let $B$ denote the revised monthly benefit. Then the two options are

- 12000 per year, paid per month with Present Value $Y_1$
- $12B$ per year, paid per month with Present Value $Y_2$

Hence $E[Y_1 - Y_2] = 0$ implies

$$12000\dd{12}{65} = 12B\dd{12}{65:10}$$
$$= 12B \cdot \left(\dd{12}{10} + v^{10} 10p_{65} \dd{12}{75}\right)$$
Example 5.4

Let $B$ denote the revised monthly benefit. Then the two options are

- 12000 per year, paid per month with Present Value $Y_1$
- $12B$ per year, paid per month with Present Value $Y_2$

Hence $\mathbb{E}[Y_1 - Y_2] = 0$ implies

$$12000 \ddot{a}_{65}^{(12)} = 12B \ddot{a}_{65}^{(12)}$$

$$= 12B \cdot \left( \ddot{a}_{10}^{(12)} + v^{10} \cdot 10p_{65}\ddot{a}_{75}^{(12)} \right)$$

$$\therefore B = 1000 \cdot \frac{\ddot{a}_{65}^{(12)}}{\ddot{a}_{10}^{(12)} + v^{10} \cdot 10p_{65}\ddot{a}_{75}^{(12)}}$$

$$= 1000 \cdot \frac{13.0870}{13.3791} = 978.17$$
Example 5.4

Let $B$ denote the revised monthly benefit. Then the two options are

- 12000 per year, paid per month with Present Value $Y_1$
- $12B$ per year, paid per month with Present Value $Y_2$

Hence $\mathbb{E}[Y_1 - Y_2] = 0$ implies

\[
12000 \ddot{a}_{65}^{(12)} = 12B \ddot{a}_{65:10}^{(12)}
\]

\[
= 12B \cdot \left( \ddot{a}_{10}^{(12)} + v^{10} 10p_{65} \ddot{a}_{75}^{(12)} \right)
\]

\[
\therefore B = 1000 \cdot \frac{\ddot{a}_{65}^{(12)}}{\ddot{a}_{10}^{(12)} + v^{10} 10p_{65} \ddot{a}_{75}^{(12)}}
\]

\[
= 1000 \cdot \frac{13.0870}{13.3791} = 978.17
\]

$V[Y_1 - Y_2] = 0?}$
Example 5.4 - Extended Cut!

- A pension plan member is entitled to a benefit of 12000 per year, in advance, for life from age 65, with no guarantee.
- She can opt to take a lower benefit with a 10–year guarantee, but then returns to 12000 per year for life from age 75 with no guarantee.
- The revised benefit is calculated to be equal to the EPV at age 65 of the original benefit.
- Calculate the revised benefit using the **MSU Model**:
  - \( A^{\frac{1}{65:10}} = v^{10} \cdot 10 \cdot p_{65} = 0.55 \)
  - \( \ddot{a}_{65}^{(12)} = 12.45 \)
  - \( \ddot{a}_{75}^{(12)} = 11.35 \)
  - \( \ddot{a}_{10}^{(12)} = 8.11 \)
- Note that \( \ddot{a}_{10}^{(12)} = 8.11 \) doesn’t depend on the mortality model used, only on \( i \).
Example 5.4 - Extended Cut!

Let \( \tilde{B} \) denote the revised monthly benefit. Then

\[
12000\tilde{a}_{65}^{(12)} = 12\tilde{B}\tilde{a}_{10}^{(12)} + v^{10}_{10}p_{65} \cdot 12000\tilde{a}_{75}^{(12)}.
\]  \hspace{1cm} (95)

It follows that

\[
\tilde{B} = 1000 \cdot \frac{\tilde{a}_{65}^{(12)} - v^{10}_{10}p_{65} \cdot \tilde{a}_{75}^{(12)}}{\tilde{a}_{10}^{(12)}}
\]

\[
= 1000 \cdot \frac{12.45 - 0.55 \cdot 11.35}{8.11}
\]

\[
= 765.41.
\]  \hspace{1cm} (96)

HW: Now calculate this revised benefit if the insurance company is allowed to keep a 10% commission of the EPV at age 65 of the original benefit.
Let $\tilde{B}$ denote the revised monthly benefit. Then

$$0.90 \times 12000 \dddot{a}_{65}^{(12)} = 12 \tilde{B} \dddot{a}_{10}^{(12)} + \nu_{10}^{10} \cdot 12000 \dddot{a}_{75}^{(12)}.$$  \hspace{1cm} (97)

It follows that

$$\tilde{B} = \frac{10800 \dddot{a}_{65}^{(12)} - 12000 \nu_{10}^{10} \cdot \dddot{a}_{75}^{(12)}}{12 \dddot{a}_{10}^{(12)}},$$

$$= \frac{(10800)(12.45) - (0.55)(12000)(11.35)}{(12)(8.11)},$$

$$= \frac{0.90 \cdot 12.45 - 0.55 \cdot 11.35}{8.11},$$

$$= 611.90.$$  \hspace{1cm} (98)
Consider a function $g : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t \to \infty} g(t) = 0$, then

$$\int_0^{\infty} g(t)dt = h \cdot \sum_{k=0}^{\infty} g(kh) - \frac{h}{2}g(0) + \frac{h^2}{12}g'(0) - \frac{h^4}{720}g''(0) + \ldots \quad (99)$$
Woolhouse’s Formula

Define

\[ g(t) = v^t t p_x \]

\[ \therefore g'(t) = -t p_x \delta e^{-\delta t} - v^t t p_x \mu_{x+t} \]

\[ \therefore g'(0) = -\delta - \mu_x \]

and so for \( h = 1 \),

\[ \bar{a}_x \approx \sum_{k=0}^{\infty} g(k) - \frac{1}{2} + \frac{1}{12} g'(0) \]

\[ = \sum_{k=0}^{\infty} v^k k p_x - \frac{1}{2} - \frac{1}{12} (\delta + \mu_x) \]

\[ = \bar{a}_x - \frac{1}{2} - \frac{1}{12} (\delta + \mu_x) \]
Correspondingly, for \( h = \frac{1}{m} \),

\[
\bar{a}_x \approx \frac{1}{m} \sum_{k=0}^{\infty} g\left( \frac{k}{m} \right) - \frac{1}{2m} + \frac{1}{12m^2} g'(0)
\]

\[
= \sum_{k=0}^{\infty} v^{\frac{k}{m}} \cdot \frac{k}{m} \cdot p_x - \frac{1}{2m} - \frac{1}{12m^2} (\delta + \mu_x)
\]

\[
= \ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2} (\delta + \mu_x)
\]
Woolhouse’s Formula

Equating the previous two approximations for $\bar{a}_x$, we obtain

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m - 1}{2m} - \frac{m^2 - 1}{12m^2}(\delta + \mu_x) \quad (103)$$
Woolhouse’s Formula

For term annuities, we obtain the approximation

\[ \bar{a}_x^{(m)} \approx \bar{a}_x^{(n)} - \frac{m - 1}{2m} (1 - \nu^n_n p_x) - \frac{m^2 - 1}{12m^2} (\delta + \mu_x - \nu^n_n p_x (\delta + \mu_{x+n})) \]  

(104)
Woolhouse’s Formula

Letting $m \to \infty$, we get

\[
\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x)
\]

\[
\bar{a}_{x:n} \approx \ddot{a}_{x:n} - \frac{1}{2}(1 - v^n p_x) - \frac{1}{12}(\delta + \mu_x - v^n p_x(\delta + \mu_{x+n}))
\]

(105)

For $\bar{a}_x$ with $\delta = 0$, the approximation above reduces further to

\[
\dot{e}_x \approx (e_x + 1) - \frac{1}{2} - \frac{1}{12}\mu_x
\]

(106)

NB: For life tables, we can compute these quantities using the approximation $\mu_x \approx -\frac{1}{2} [\ln (p_x) + \ln (p_{x+1})]$
Select and Ultimate Survival Models

Notation:

- **Aggregate Survival Models**: Models for a large population, where $t \rho_x$ depends only on the current age $x$.

- **Select (and Ultimate) Survival Models**: Models for a select group of individuals that depend on the current age $x$ and on the age at which the individual joined the group.
  - Future survival probabilities for an individual in the group depend on the individual’s current age and on the age at which the individual joined the group.
  - $\exists d > 0$ such that if an individual joined the group more than $d$ years ago, future survival probabilities depend only on current age. So, after $d$ years, the person is considered to be back in the aggregate population.

Ultimately, a **select** survival model includes another event upon which probabilities are conditional on.
Select and Ultimate Survival Models

Notation:

- $d$ is the **select period**
- The mortality applicable to lives after the select period is over is known as the **ultimate** mortality.

A **select** group should have a different mortality rate, as they have been offered (selected for) life insurance. A question, of course, is the effect on mortality by maintaining proper health insurance.
Example 3.8

Consider men who need to undergo surgery because they are suffering from a particular disease. The surgery is complicated and

\[ \mathbb{P}[\text{survive one year after surgery}] = 0.5 \]

\[ (d, l_{60}, l_{61}, l_{70}) = (1, 89777, 89015, 77946) \]  \hspace{1cm} (107)

Calculate \( \mathbb{P}[A], \mathbb{P}[B], \mathbb{P}[C] \), where

- \( A = \{ (60), \text{about to have surgery, will be alive at age 70} \} \)
- \( B = \{ (60), \text{had surgery at age 59, will be alive at age 70} \} \)
- \( C = \{ (60), \text{had surgery at age 58, will be alive at age 70} \} \)
Example 3.8

\[ P[A] = P[(60) \text{, about to have surgery, alive at age 61}] \cdot \frac{l_{70}}{l_{61}} \]

\[ = 0.5 \cdot \frac{77946}{89015} = 0.4378 \]

\[ P[B] = \frac{l_{70}}{l_{60}} = 0.8682 \]

\[ P[C] = \frac{l_{70}}{l_{60}} = 0.8682 \]
Select Survival Models

\[
S_{[x]+s}(t) = \mathbb{P}[(x + s) \text{ selected at } (x), \text{survives to}(x + s + t)]
\]
\[
tq_{[x]+s} = \mathbb{P}[(x + s) \text{ selected at } (x), \text{dies before}(x + s + t)]
\]
\[
\mu_{[x]+s} = \text{force of mortality at } (x + s) \text{ for select at } (x)
\]
\[
= \lim_{h \to 0^+} \left( \frac{1 - S_{[x]+s}(h)}{h} \right)
\]
\[
tp_{[x]+s} = 1 - tq_{[x]+s} = S_{[x]+s}(t)
\]
\[
= e^{-\int_0^t \mu_{[x]+s+u} du}
\]

For \( t < d \), we refer to to the above as part of the select model. For \( t \geq d \), they are part of the ultimate model. Please read through section on Select Life Tables.
Sometimes, we wish to compute values from life tables. Consider again a model where \( x \geq x_0 \), where \( x_0 \) is the initial age, and \( 0 \leq t \leq d \). Then

\[
l_{x+t} = d - t \cdot p_{x} + t \cdot l_{x} + t
\]  \hspace{1cm} (110)
Example 3.9

**Theorem**

Consider $y \geq x + d > x + s > x + t \geq x \geq x_0$. Then

\[
\begin{align*}
y - x - t p[x] + t &= \frac{l_y}{l[x] + t} \\

s - t p[x] + t &= \frac{l[x] + s}{l[x] + t}
\end{align*}
\] (111)
Example 3.9

Proof.

\[
y - x - t P[x] + t = y - x - d P[x] + d \cdot d - t P[x] + t
\]

\[
= y - x - d P[x] + d \cdot d - t P[x] + t
\]

\[
= \frac{I_y}{I_{x+d}} \frac{I_{x+d}}{I_{[x]+t}}
\]

\[
= \frac{I_y}{I_{[x]+t}}
\]

\[
= \frac{d - t P[x] + t}{d - s P[x] + s}
\]

\[
= \frac{I_{x+d}}{I_{[x]+t}} \frac{I_{[x]+s}}{I_{x+d}}
\]

\[
= \frac{I_{[x]+s}}{I_{[x]+t}}
\]

(112)
Example 3.11

A select survival model has a select period of three years. Its ultimate mortality is equivalent to the US Life Tables, 2002 Females of which an extract is shown below. Information given is that for all $x \geq 65,$

$$\left( p[x], p[x-1]+1, p[x-2]+2 \right) = (0.999, 0.998, 0.997). \quad (113)$$

Table: 3.5: Extract from US Life Tables, 2002 Females

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>80556</td>
</tr>
<tr>
<td>71</td>
<td>79026</td>
</tr>
<tr>
<td>72</td>
<td>77410</td>
</tr>
<tr>
<td>73</td>
<td>75666</td>
</tr>
<tr>
<td>74</td>
<td>73802</td>
</tr>
<tr>
<td>75</td>
<td>71800</td>
</tr>
</tbody>
</table>
Calculate the probability that a woman currently aged 70 will survive to age 75 given that

1. she was select at age 67:
2. she was select at age 68
3. she was select at age 69
4. she was select at age 70
Example 3.11

\[ 5p_{[70-3]+3} = 5p_{70} = \frac{l_{75}}{l_{70}} = 0.8913 \]

\[ 5p_{[70-2]+2} = \frac{l_{[68]+2}+5}{l_{[68]+2}} = \frac{l_{75}}{l_{[68]+2}} \]

\[ = \frac{l_{75}}{l_{[68]+3}} = \frac{l_{75}}{l_{71}} \cdot 1p_{[68]+2} \]

\[ = 4p_{71} \cdot 1p_{[68]+2} \]

\[ = \frac{71800}{79026} \cdot 0.997 = 0.9058 \]
Example 3.11

\[
5p_{[70-1]+1} = \frac{l_{[69]+1+5}}{l_{[69]+1}} = \frac{l_{75}}{l_{[69]+1}}
\]

\[
= \frac{l_{75}}{l_{[69]+3}} \cdot \frac{(1p_{[69]+1}) \cdot (1p_{[69]+2})}{(1p_{[69]+1}) \cdot (1p_{[69]+2})}
\]

\[
= \frac{l_{75}}{l_{72}} \cdot (1p_{[69]+1}) \cdot (1p_{[69]+2})
\]

\[
= \frac{71800}{77410} \cdot 0.997 \cdot 0.998 = 0.9229
\]

\[
5p_{[70]} = \frac{l_{75}}{l_{73}} \cdot (1p_{[70]}) \cdot (1p_{[70]+1}) \cdot (1p_{[70]+2})
\]

\[
= \frac{71800}{75666} \cdot 0.997 \cdot 0.998 \cdot 0.999 = 0.9432
\]
Example 3.12

Given a table of values for \( q[x] \), \( q[x-1]+1 \), \( q_x \) and the knowledge that the model incorporates a 2—year select period, compute

- \( 4p[70] \)
- \( 3q[60]+1 \)
Example 3.12

Given a table of values for $q_x$, $q_{x-1}+1$, $q_x$ and the knowledge that the model incorporates a 2-year select period, compute

- $4p_{[70]}$
- $3q_{[60]}+1$

\[
4p_{[70]} = p_{[70]} p_{[70]+1} p_{[70]+2} p_{[70]+3} = p_{[70]} p_{[70]+1} p_{72} p_{73}
= (1 - q_{[70]}) \cdot (1 - q_{[70]+1}) \cdot (1 - q_{72}) \cdot (1 - q_{73})
\]
Example 3.12

Given a table of values for $q_{[x]}$, $q_{[x-1]+1}$, $q_x$ and the knowledge that the model incorporates a 2–year select period, compute

- $4p_{[70]}$
- $3q_{[60]}+1$

\[
4p_{[70]} = p_{[70]}p_{[70]+1}p_{[70]+2}p_{[70]+3} = p_{[70]}p_{[70]+1}p_{72}p_{73} \\
= (1 - q_{[70]}) \cdot (1 - q_{[70]+1}) \cdot (1 - q_{72}) \cdot (1 - q_{73})
\]

\[
3q_{[60]}+1 = q_{[60]+1} + p_{[60]+1}q_{62} + p_{[60]+1}p_{62}q_{63} \\
= q_{[60]+1} + (1 - q_{[60]+1}) \cdot q_{62} \\
+ (1 - q_{[60]+1}) \cdot (1 - q_{62}) \cdot q_{63}
\]
Example 3.13

A select survival model has a two-year select period and is specified as follows. The ultimate part of the model follows Makeham’s law, where \((A, B, c) = (0.00022, 2.7 \times 10^{-6}, 1.124)\):

\[
\mu_x = A + B \cdot c^x
\] (117)

The select part of the model is such that for \(0 \leq s \leq 2\),

\[
\mu_{[x]+s} = 0.9^{2-s} \mu_{x+s}
\] (118)

and so for \(0 \leq t \leq 2\),

\[
t_{p[x]} = e^{- \int_0^t \mu_{[x]+s} ds} = \exp \left[ - \int_0^t \left( 0.9^{2-s} (A + B \cdot c^{x+s}) \right) ds \right]
\] (119)
Example 3.13

It follows that given an initial cohort at age $x_0$, that is given $l_{x_0}$, we can compute the entries of a select life table via

$$l_x = p_{x-1}l_{x-1}$$

$$l_{[x]+1} = \frac{l_{x+2}}{p_{[x]+1}}$$

$$l_{[x]} = \frac{l_{x+2}}{2p_{[x]}}$$

(120)
Example: Linear $\mu_x$ Transformation

A select group has $\mu[x] + s = \frac{1}{2} \cdot \mu_{x+s}$, for all $s \geq 0$. It follows that

- (a) $tp[x] = 0.5 \cdot tp_x$.
- (b) $tp[x] = (tp_x)^2$.
- (c) $tp[x] = \sqrt{tp_x}$.
- (d) $tp[x] = 2 \cdot tp_x$.
- (e) None of the above.
Example: Linear $\mu_x$ Transformation

A select group has $\mu_{[x]+s} = \frac{1}{2} \cdot \mu_{x+s}$, for all $s \geq 0$. It follows that

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- (d) $tp_{[x]} = 2 \cdot tp_x$.
- (e) None of the above.

Answer: (c)
Example: Linear $\mu_x$ Transformation

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- (a) $tp_{[x]} = 0.5 \cdot tp_x$.  
- (b) $tp_{[x]} = (tp_x)^2$.  
- (c) $tp_{[x]} = \sqrt{tp_x}$.  
- (d) $tp_{[x]} = 2 \cdot tp_x$.  
- (e) None of the above.

Answer: (c)

$$tp_{[x]} = e^{-\int_0^t \mu_{[x]+s} \, ds} = e^{-\frac{1}{2} \int_0^t \mu_{x+s} \, ds}$$

$$= (e^{-\int_0^t \mu_{x+s} \, ds})^{\frac{1}{2}} = \sqrt{tp_x}.$$
Example: Linear $p_x$ Transformation

For a 2-year select mortality model, you are given:

\[ p_{x+1} = 0.10 + 0.90 \times p_{x+1} \]

\[ l_{76} = 98000 \]

\[ l_{77} = 96000 \]

Solve for $l_{[75]+1}$. 

(122)
Example: Linear $p_x$ Transformation

Answer:
First note that

\[
q_{x+1} = 1 - p_{x+1} = 1 - (0.10 + 0.90 \times p_{x+1}) \\
= 0.90 - 0.90 \times p_{x+1} = 0.90q_{x+1}.
\]  

(123)
Example: Linear $p_x$ Transformation

**Answer:**

First note that

$$q_{x+1} = 1 - p_{x+1} = 1 - (0.10 + 0.90 \times p_{x+1})$$

$$= 0.90 - 0.90 \times p_{x+1} = 0.90q_{x+1}.$$

(123)

In general, if $q_{x+1} = \alpha \times q_{x+1}$, then
Example: Linear $p_x$ Transformation

**Answer:**
First note that

$$q_{x+1} = 1 - p_{x+1} = 1 - (0.10 + 0.90 \times p_{x+1})$$

$$= 0.90 - 0.90 \times p_{x+1} = 0.90q_{x+1}.$$

In general, if $q_{x+1} = \alpha \times q_{x+1}$, then

$$1 - \frac{l_{x+2}}{l_{x+1}} = \alpha \left(1 - \frac{l_{x+2}}{l_{x+1}}\right)$$

$$\Rightarrow l_{x+1} = \frac{l_{x+2}}{(1 - \alpha) + \alpha \frac{l_{x+2}}{l_{x+1}}}.$$
Example: Linear $p_x$ Transformation

**Answer:**
First note that

$$q_{x+1} = 1 - p_{x+1} = 1 - (0.10 + 0.90 \times p_{x+1})$$

$$= 0.90 - 0.90 \times p_{x+1} = 0.90q_{x+1}.\quad (123)$$

In general, if $q_{x+1} = \alpha \times q_{x+1}$, then

$$1 - \frac{l_{x+2}}{l_{x+1}} = \alpha \left(1 - \frac{l_{x+2}}{l_{x+1}}\right)$$

$$\Rightarrow l_{x+1} = \frac{l_{x+2}}{(1 - \alpha) + \alpha \frac{l_{x+2}}{l_{x+1}}}.\quad (124)$$

So, for our case, with $\alpha = 0.9$, it follows that

$$l_{[75]+1} = \frac{96000}{0.1 + 0.9 \cdot \frac{96000}{98000}} = 97796.2577963 \approx 97796.3.\quad (125)$$
Example: Martian Annuities!

You are given the extract from a 3–year select mortality table:

<table>
<thead>
<tr>
<th>[x]</th>
<th>( l_x )</th>
<th>( l_x + 1 )</th>
<th>( l_x + 2 )</th>
<th>( l_{x+3} )</th>
<th>x+3</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>10000</td>
<td>9985</td>
<td>9975</td>
<td>9965</td>
<td>53</td>
</tr>
<tr>
<td>51</td>
<td>9990</td>
<td>9975</td>
<td>9965</td>
<td>9955</td>
<td>54</td>
</tr>
<tr>
<td>52</td>
<td>9980</td>
<td>9965</td>
<td>9955</td>
<td>9945</td>
<td>55</td>
</tr>
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<td>53</td>
<td>9970</td>
<td>9955</td>
<td>9945</td>
<td>9935</td>
<td>56</td>
</tr>
<tr>
<td>54</td>
<td>9960</td>
<td>9945</td>
<td>9935</td>
<td>9925</td>
<td>57</td>
</tr>
<tr>
<td>55</td>
<td>9950</td>
<td>9935</td>
<td>9925</td>
<td>9915</td>
<td>58</td>
</tr>
</tbody>
</table>

Calculate, as polynomials in \( v \):

- \( a_{[50]:4} \).
- \( (l \ddot{a})_{[50]:4} \).
- \( v \cdot \frac{d}{dv} \left( a_{[50]:4} \right) \).
Example: Martian Insurance!

**Answer:**

\[ a_{[50]:4} = v \frac{l_{50}+1}{l_{50}} + v^2 \frac{l_{50}+2}{l_{50}} + v^3 \frac{l_{53}}{l_{50}} + v^4 \frac{l_{54}}{l_{50}} \]

\[ = v \frac{9985}{10000} + v^2 \frac{9975}{10000} + v^3 \frac{9965}{10000} + v^4 \frac{9955}{10000} \]

\[ = 0.9985v + 0.9975v^2 + 0.9965v^3 + 0.9955v^4. \]

\[ (I\dot{a})_{[50]:4} = 1 + 2v \frac{l_{50}+1}{l_{50}} + 3v^2 \frac{l_{50}+2}{l_{50}} + 4v^3 \frac{l_{53}}{l_{50}} \]

\[ = 1 + 1.997v + 2.9925v^2 + 3.986v^3. \]

\[ v \cdot \frac{d}{dv} a_{[50]:4} = v \cdot \frac{d}{dv} \left( v \frac{l_{50}+1}{l_{50}} + v^2 \frac{l_{50}+2}{l_{50}} + v^3 \frac{l_{53}}{l_{50}} + v^4 \frac{l_{54}}{l_{50}} \right) \]

\[ = v \frac{l_{50}+1}{l_{50}} + 2v^2 \frac{l_{50}+2}{l_{50}} + 3v^3 \frac{l_{53}}{l_{50}} + 4v^4 \frac{l_{54}}{l_{50}} =: (Ia)_{[50]:4} \]

(126)
HW: 3.1, 3.2, 3.4, 3.7, 3.8, 3.9, 3.10, 5.1, 5.3, 5.5, 5.6, 5.11, 5.14.
What is a Premium?

When entering into a contract, the financial obligations of all parties must be specified. In an insurance contract, the insurance company agrees to pay the policyholder benefits in return for premium payments. The premiums secure the benefits as well as pay the company for expenses attached to the administration of the policy.
A **Net Premium** does not explicitly allow for company’s expenses, while a **Office** or **Gross Premium** does. There may be a **Single Premium** or a series of payments that could even match with the policyholder’s salary frequency.

It is important to note that premiums are paid as soon as the contract is signed, otherwise the policyholder would attain coverage before paying for it with the first premium. This could be seen as an arbitrage opportunity - non-zero probability of gain with no money up front.
Premium Types

Premiums cease upon death of the policyholder. The premium paying term is the maximum length of time that premiums are required. Certainly, premium term can be fixed so that upon retirement, say, no more payments are required.

Also, the benefits can be secured in the future (deferred) by a single premium payment up front. For example, pay now to secure annuity payments upon retirement until death.
Assumptions

Recall the life model used in Example 3.13: The select survival model has a two-year select period and is specified as follows. The ultimate part of the model follows Makeham’s law, where $(A, B, c) = (0.00022, 2.7 \times 10^{-6}, 1.124)$:

$$\mu_x = 0.00022 + (2.7 \times 10^{-6}) \cdot (1.124)^x$$  \hspace{1cm} (127)

The select part of the model is such that for $0 \leq s \leq 2$,

$$\mu_{x+s} = 0.9^{2-s} \mu_{x+s}$$  \hspace{1cm} (128)

and so for $0 \leq t \leq 2$,

$$t p_{[x]} = e^{-\int_0^t \mu_{x+s} ds} = \exp \left[ 0.9^{2-t} \left( \frac{1 - 0.9^t}{\ln (0.9)} + \frac{c^t - 0.9^t}{\ln \left( \frac{0.9}{c} \right)} \right) \right]$$  \hspace{1cm} (129)
Furthermore, we can extend the recursion principle when using a select life model to obtain

\[
\ddot{a}_x = 1 + vp_x \ddot{a}_{x+1} \\
\ddot{a}_{[x]+1} = 1 + vp_{[x]+1} \ddot{a}_{x+2} \\
\ddot{a}_{[x]} = 1 + vp_{[x]} \ddot{a}_{[x]+1}
\]  (130)
In general, an insurance company can expect to have a total benefit paid out, along with expense loading and other related costs. We represent this total benefit as $Z$. Similarly, to fund $Z$, the company can expect the policyholder to make a single payment, or stream of payments, that has present value $P \cdot Y$. Here, $P$ represents the level premium $P$ and $Y$ represents the present value associated to a unit payment or payment stream.
For life contingent contracts, there is an outflow and inflow of money during the term of the agreement. The premium income is certain, but since the benefits are life contingent, the term and total income may not be certain up front. To account for this, we define the **Net Future Loss** $L^n_0$ (which includes expenses) and the **Gross Future Loss** $L^g_0$ (which does not include expenses) as

\[
L^n_0 = PV \left[ \text{benefit outgo} \right] - PV \left[ \text{net premium income} \right] \\
L^g_0 = PV \left[ \text{benefit outgo} \right] + PV \left[ \text{expenses} \right] - PV \left[ \text{gross premium income} \right]
\]  

(131)
Example 6.2

An insurer issues a whole life insurance to [60], with sum insured $S$ payable immediately upon death. Premiums are payable annually in advance, ceasing at 80 or on earlier death. The net annual premium is $P$. What is the net future loss random variable $L_0^n$ for this contract in terms of lifetime random variables for [60]?
Example 6.2

An insurer issues a whole life insurance to [60], with sum insured $S$ payable immediately upon death. Premiums are payable annually in advance, ceasing at 80 or on earlier death. The net annual premium is $P$. What is the net future loss random variable $L^n_0$ for this contract in terms of lifetime random variables for [60]?

$$L^n_0 = S v^{T_{[60]}} - P \bar{a} \min\{K_{[60]} + 1, 20\}$$  \hspace{1cm} (132)
Absent a risk-neutral type pricing measure, insurers price these event-contingent contracts by setting the average value of the loss to be zero. Symbolically, this is simply (for net premiums) find $P$ such that

$$E[L^n_0] = 0 \quad (133)$$

Note that this value $P$ does not necessarily set $Var[L^n_0] = 0$

Returning to our general set-up, we see that the equivalence pricing principle can be summarized as

$$P = \frac{E[Z]}{E[Y]} \quad (134)$$
As an introductory example, consider $\lambda > 0$ and a contract where (under no selection)

$$Z = v^{T_x}$$
$$Y = \bar{a}_{T_x}$$
$$t p_x = e^{-\lambda t}$$

(135)
As an introductory example, consider $\lambda > 0$ and a contract where (under no selection)

$$Z = v^Tx$$
$$Y = \bar{a}_{T_x\mid}$$
$$tp_x = e^{-\lambda t}$$

Hence, we have a unit whole-life insurance payable immediately upon death of $(x)$, where mortality is modeled to be exponential with parameter $\lambda$. 
Equivalence Principle

We obtain

\[ \tilde{P}_x = \frac{\mathbb{E} \left[ v^T x \right]}{\mathbb{E} \left[ \bar{a}_{Tx} \right]} = \frac{\bar{A}_x}{\bar{a}_x} = \delta \frac{\bar{A}_x}{1 - A_x} \]

\[ = \delta \frac{\int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} dt}{1 - \int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} dt} \]

\[ = \delta \frac{\frac{\lambda}{\lambda + \delta}}{1 - \frac{\lambda}{\lambda + \delta}} = \lambda \]

(136)
We obtain

\[ \bar{P}_x = \frac{\mathbb{E} \left[ v^{T_x} \right]}{\mathbb{E} \left[ \bar{a}_T \right]} = \frac{\bar{A}_x}{\bar{a}_x} = \delta \frac{\bar{A}_x}{1 - \bar{A}_x} \]

\[ = \delta \frac{\int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} dt}{1 - \int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} dt} \]

\[ = \delta \frac{\lambda}{1 - \frac{\lambda}{\lambda + \delta}} = \lambda \]

(HW: repeat the above calculation if \( S_0(x) = \frac{\omega - x}{\omega} \) for a finite lifetime model with maximal age \( \omega \).)
If we repeat the previous example, but now for the case of a unit whole-life insurance contract with level annual premium payment and benefit paid at the end of the death year, then

\[
Z = \nu^{K_x+1}
\]

\[
Y = \ddot{a}_{K_x+1}\]

\[
tp_x = e^{-\lambda t}
\]

(137)
It follows that

\[ P_x = d \frac{A_x}{1 - A_x} = d \frac{1}{1} - \sum_{k=0}^{\infty} e^{-\delta(k+1)} \cdot (k p_x - k+1 p_x) \]

\[ = d \left(1 - e^{-\lambda}\right) \cdot \sum_{k=0}^{\infty} e^{-\delta(k+1)} e^{-\lambda k} \]

\[ = d \frac{(1 - e^{-\lambda}) \cdot \sum_{k=0}^{\infty} e^{-\delta(k+1)} e^{-\lambda k}}{1 - (1 - e^{-\lambda}) \cdot \sum_{k=0}^{\infty} e^{-\delta(k+1)} e^{-\lambda k}} \]

\[ = d \frac{(1 - e^{-\lambda}) \cdot e^{-\delta} \cdot \frac{1}{1-e^{-\delta + \lambda}}}{1 - (1 - e^{-\lambda}) \cdot e^{-\delta} \cdot \frac{1}{1-e^{-\delta + \lambda}}} \]

\[ = (1 - e^{-\lambda}) \cdot e^{-\delta} \]
"Deterministic" Insurance

Consider an endowment insurance with sum insured 100000 issued to an age \((x)\) where 20 premiums are paid in return for the benefit 100000 paid at the end of year 20. Assume \(v = \frac{1}{1.05}\). Then

\[
Z = 100000v^{20}
\]

\[
Y = \ddot{a}_{20} = \frac{1 - v^{20}}{1 - v}
\]
"Deterministic" Insurance

Consider an endowment insurance with sum insured 100000 issued to an age ($x$) where 20 premiums are paid in return for the benefit 100000 paid at the end of year 20. Assume $\nu = \frac{1}{1.05}$. Then

$$Z = 100000\nu^{20}$$

$$Y = \ddot{a}_{20} = \frac{1 - \nu^{20}}{1 - \nu}$$

$$\Rightarrow P_d = \frac{100000\nu^{20}}{\left(\frac{1-\nu^{20}}{1-\nu}\right)}$$

$$= (1 - \nu) \times 100000 \times \frac{\nu^{20}}{1 - \nu^{20}}$$

$$\approx 2880.25.$$
Example 6.5

Now, consider an endowment insurance with sum insured 100000 issued to a select life aged [45] with term 20 years under which the death benefit is payable at the end of the year of death. Using the Standard Select Survival Model with interest at 5% per year, calculate the total amount of net premium payable in a year if premiums are payable annually.
Example 6.5

By the EPP, the fact that \( d = 1 - \frac{1}{1.05} \), and tables 6.1 and 3.7, we have

\[
100000 \cdot A_{[45]:20} = P \cdot \ddot{a}_{[45]:20}
\]

\[
\Rightarrow P = 100000 \cdot \frac{A_{[45]:20}}{\ddot{a}_{[45]:20}} = \frac{100000 \cdot \left(1 - d \ddot{a}_{[45]:20}\right)}{\ddot{a}_{[45]:20}} = \frac{100000 \cdot \left(1 - d \ddot{a}_{[45]:20}\right)}{\ddot{a}_{[45]:20}} = 100000 \cdot \left(1 - d \left(\ddot{a}_{45} - \frac{l_{65}}{l_{45}} v^{20} \ddot{a}_{65}\right)\right)
\]

\[
= 100000 \cdot \frac{0.383766}{12.94092} = 2965.52
\]

Is it reasonable that \( P = 2965.52 > 2880.25 = P_d \)?
Starting up an insurance company requires start-up capital like most other companies. Agents are charged with drumming up new business in the form of finding and issuing new life insurance contracts. This helps to diversify risk in the case of a large loss on one contract (more on this later.)

However, new contracts can incur larger losses up front in the first few years even without a benefit payout. This is due to initial commission payments to agents as well as contract preparation costs. Periodic maintenance costs can also factor into the premium calculation.
Consider offering an $n$–year endowment policy to an age $(x)$ in the aggregate population where the benefit $B$ is paid at the end of the year of death or on maturity. There are periodic renewal expenses of $r$ per policy.
Consider offering an \( n \)–year endowment policy to an age \((x)\) in the aggregate population where the benefit \( B \) is paid at the end of the year of death or on maturity. There are periodic renewal expenses of \( r \) per policy.

Then the premium \( P \) is calculated via the EPP as

\[
P \bar{a}_{x:n} = B \cdot A_{x:n} + r \bar{a}_{x:n} \tag{141}
\]

\[
\Rightarrow P = B \cdot \frac{A_{x:n}}{\bar{a}_{x:n}} + r
\]

and we see that periodic expenses are simply passed on to the consumer!
Consider offering an $n$–year endowment policy to an age $(x)$ in the aggregate population where the benefit $B$ is paid at the end of the year of death or on maturity. There are periodic *renewal expenses* of $r$ per policy and an initial preparation expense of $z$ per contract.
Consider offering an \( n \)--year endowment policy to an age \((x)\) in the aggregate population where the benefit \(B\) is paid at the end of the year of death or on maturity. There are periodic renewal expenses of \(r\) per policy and an initial preparation expense of \(z\) per contract.

Then the premium \(P\) is calculated via

\[
P = B \cdot \frac{A_{x:\overline{n}}}{\ddot{a}_{x:\overline{n}}} + r + \frac{z}{\ddot{a}_{x:\overline{n}}}
\]

(142)

and so the initial preparation expense is amortized over the lifetime of the contract.
An insurer issues a 25-year annual premium endowment insurance with sum insured 100000 to a select life aged [30]. The insurer incurs initial expenses of 2000 plus 50% of the first premium and renewable expenses of 2.5% of each subsequent premium. The death benefit is payable immediately upon death. What is the annual premium $P$?
We can see that

\[
L_0^g = 100000 v^{\min\{T_{[30],25}\}} + 2000 + 0.475P
\]
\[+ 0.025P \ddot{a}^{\min\{K_{[30]}+1,25\}} - P \ddot{a}^{\min\{K_{[30]}+1,25\}}
\]

\[\Rightarrow P = \frac{100000 \cdot E\left[v^{\min\{T_{[30],25}\}}\right] + 2000}{0.975 \cdot E\left[\ddot{a}^{\min\{K_{[30]}+1,25\}}\right] - 0.475}
\]

\[= \frac{100000 \cdot \bar{A}_{[30]:25} + 2000}{0.975 \cdot \bar{\ddot{a}}_{[30]:25} - 0.475}
\]

\[= \frac{100000 \cdot (0.298732) + 2000}{0.975 \cdot (14.73113) - 0.475}
\]

\[= 2295.04.
\]
Some more worked examples

Consider the following net loss random variables:

\[ L_0^n = v^T_x - P\bar{a}_{\min\{T_x, t\}} \]
Consider the following net loss random variables:

\[ L_0^n = v T_x - P \bar{a}_{\min\{T_x, t\}} \]

\[ L_0^n = v \min\{T_x, n\} - P \bar{a}_{\min\{T_x, t\}} \]
Some more worked examples

Consider the following net loss random variables:

\[ L^n_0 = \nu^T_x - P\bar{a}_{\min\{T_x,t\}} \]

\[ L^n_0 = \nu^{\min\{T_x,n\}} - P\bar{a}_{\min\{T_x,t\}} \]  

\[ L^n_0 = \nu^{K_x+1} - P\bar{a}_{\min\{K_x+1,t\}} \]  

(144)

What are the fair premiums under the EPP?
Some more worked examples

Consider the following net loss random variables:

\[
L^n_0 = \nu T_x - P \bar{a}_{\min\{T_x, t\}}
\]
\[
L^n_0 = \nu \min\{T_x, n\} - P \bar{a}_{\min\{T_x, t\}}
\]
\[
L^n_0 = \nu K_x + 1 - P \bar{a}_{\min\{K_x + 1, t\}}
\]

(144)

What are the fair premiums under the EPP?

\[
P = \frac{\bar{A}_x}{\bar{a}_{x: \bar{t}}}
\]
Some more worked examples

Consider the following net loss random variables:

\[ L^n_0 = vT_x - P\bar{\alpha}_{\min\{T_x, t\}} \]
\[ L^n_0 = v\min\{T_x, n\} - P\bar{\alpha}_{\min\{T_x, t\}} \]
\[ L^n_0 = v^{K_x + 1} - P\bar{\alpha}_{\min\{K_x + 1, t\}} \]  

(144)

What are the fair premiums under the EPP?

\[ P = \frac{\bar{A}_x}{\bar{a}_x:t} \]
\[ P = \frac{\bar{A}_x:n}{\bar{a}_x:t} \]
Some more worked examples

Consider the following net loss random variables:

\[
L^n_0 = \nu^{T_x} - P\bar{a}_{\min\{T_x,t\}} \\
L^n_0 = \nu^{\min\{T_x,n\}} - P\bar{a}_{\min\{T_x,t\}} \\
L^n_0 = \nu^{K_x+1} - P\bar{a}_{\min\{K_x+1,t\}}
\] (144)

What are the fair premiums under the EPP?

\[
P = \frac{\bar{A}_x}{\bar{a}_{x:t}} \\
P = \bar{A}_x^{n} \\
P = \bar{a}_{x:t}^{n} \\
P = \frac{\bar{A}_x}{\bar{a}_{x:t}}
\] (145)
Consider the case where a $n$–year deferred annual whole-life annuity due of 1 on a life $(x)$ where if the death occurs during the deferral period, the **single benefit premium** is refunded without interest at the end of the year of death. What is this single benefit premium $P$?
The net loss random variable is

\[ L_0^n = P v^{K_x+1} \cdot 1_{\{K_x+1 \leq n\}} + v^n 1_{\{K_x+1 > n\}} \cdot \ddot{a}_{K_x+1-n} - P \tag{146} \]

and by the EPP we have

\[ 0 = P A^{1}_{x:n} + n| \ddot{a}_x - P \tag{147} \]
The net loss random variable is

\[ L^n_0 = P v^{k_x+1} \cdot 1_{\{k_x+1 \leq n\}} + v^n 1_{\{k_x+1 > n\}} \cdot \ddot{a}_{k_x+1-n} - P \]  \hspace{1cm} (146)

and by the EPP we have

\[ 0 = PA^1_{x:n} + n | \ddot{a}_x - P \]  \hspace{1cm} (147)

This implies that

\[ P = \frac{n | \ddot{a}_x}{1 - A^1_{x:n}} \]

\[ = \frac{A_{x:n} - A^1_{x:n}}{1 - A^1_{x:n}} \cdot \ddot{a}_{x+n} \]  \hspace{1cm} (148)
Consider a 1–year term insurance contract issued to a select life \([x]\), with sum insured \(S = 1000\), interest rate \(i = 0.05\), and mortality \(q[x] = \mathbb{P}[T[x] \leq 1] = 0.01\).

It follows that \(L_0\), the future loss random variable calculated at the time of issuance, is

\[
L_0 = 1000v^11\{T[x] \leq 1\} - P
\]
Consider a 1-year term insurance contract issued to a select life $[x]$, with sum insured $S = 1000$, interest rate $i = 0.05$, and mortality $q_{[x]} = P[T_{[x]} \leq 1] = 0.01$

It follows that $L_0$, the future loss random variable calculated at the time of issuance, is

$$ L_0 = 1000v^{T_{[x]} \leq 1} - P $$

$$ \Rightarrow P = E[1000v^{T_{[x]} \leq 1}] = 1000v \cdot P[T_{[x]} \leq 1] $$

$$ = \frac{(1000)(0.01)}{1.05} = 9.52 $$
Consider that the company has issued a lot of these contracts, say $N \gg 1$, to independent select lives $[x]$. Let $D_{[x]}$ be the random variable representing the number of deaths in a year of this population, and assume

$$D_{[x]} \sim Bin(N, q_{[x]})$$

(150)
In general, we have the event that the insurer turns a profit on this group of policies is \( \{ \text{Profit} \} = \{ D_x \leq N \cdot q_x \} \), and so as \( N \to \infty \),

\[
\mathbb{P}[\text{Profit}] = \mathbb{P}[D_x \leq N \cdot q_x] = \mathbb{P}[D_x \leq \mathbb{E}[D_x]] = \mathbb{P}\left[\frac{D_x - \mathbb{E}[D_x]}{\sqrt{\text{Var}[D_x]}} \leq 0\right] 
\to \Phi(0) = \frac{1}{2} \text{ by the CLT.}
\]

**HW:** Compute

\[
\mathbb{E}\left[\frac{\text{Profit} | D_x \leq N \cdot q_x}{N \cdot P}\right]
\]

(152)
Consider now a whole-life contract issued to $[x]$ with sum insured $S$ and annual premium $P$. Then

$$\mathbb{P}[\text{Profit}] = \mathbb{P}[L_0 < 0] = \mathbb{P}[Sv^{K[x]+1} - P\bar{a}_{K[x]+1} < 0]$$

$$= \mathbb{P}[Sv^{K[x]+1} < P \cdot \frac{1 - v^{K[x]+1}}{d}]$$

$$= \mathbb{P}\left[v^{K[x]+1} < \frac{P}{P + d \cdot S}\right]$$

$$= \mathbb{P}\left[K[x] + 1 > \frac{1}{\delta} \ln \left(\frac{P + d \cdot S}{P}\right)\right]$$

$$= \mathbb{P}\left[K[x] \geq \left\lfloor \frac{1}{\delta} \ln \left(\frac{P + d \cdot S}{P}\right) \right\rfloor\right]$$

$$= \left\lfloor \frac{1}{\delta} \ln \left(\frac{P}{P + d \cdot S}\right) \right\rfloor \rho[x]$$

(153)
Define $L_0(k) = PV[\text{Loss} \mid K[x] = k]$. For a contract with a term $n$, it follows that

$$
E[L_0] = E[PV[\text{Loss}]] = E \left[ \sum_{k=0}^{n-1} PV[\text{Loss} \mid K[x] = k] \cdot 1\{K[x]=k\} \right] 
$$

$$
+ E \left[ PV[\text{Loss} \mid K[x] \geq n] \cdot 1\{K[x] \geq n\} \right] 
$$

$$
= \sum_{k=0}^{n-1} (PV[\text{Loss} \mid K[x] = k] \cdot P[K[x] = k]) + L_0(n) \cdot P[K[x] \geq n] 
$$

$$
= \sum_{k=0}^{n-1} L_0(k) \cdot k_0 q[x] + L_0(n) n p[x] 
$$

Q: Can we use this for the case $n = \infty$?
Example 6.9

A life insurer is about to issue a 25–year endowment insurance with a basic sum insured $S = 250000$ to a select life aged exactly [30]. Premiums are payable annually throughout the term of the policy. Initial expenses are 1200 plus 40% of the first premium and renewal expenses are 1% of the second and subsequent premiums. The insurer allows for a compound reversionary bonus of 2.5% of the basic sum insured, vesting on each policy anniversary (including the last.) The death benefit is payable at the end of the year of death. Assume the Standard Select Survival Model with interest rate 5% per year.
Example 6.9

In this case, we have the reversionary bonus exponentially grow the sum insured:

\[
L_0 = B_0(K_{[x]}) + E_0 - P_0(K_{[x]})
\]

\[
B_0(k) = 250000 \cdot 1.025^k v^{k+1} \text{ for } k \in \{0, 1, 2, \ldots, 24\}
\]

\[
B_0(25) = 250000 \cdot 1.025^{25} v^{25}
\]

\[
E_0 = 1200 + 0.39P
\]

\[
P_0(k) = 0.99P \bar{\alpha}_{\min \{k+1,25\}}
\]
Example 6.9

For $1 + j = \frac{1+i}{1.025}$, we have $j = 0.02439$ and so

$$\mathbb{E}[L_0] = \mathbb{E}[B_0(K[x])] + E_0 - \mathbb{E}[P_0(K[x])]$$

$$= \sum_{k=0}^{24} B_0(k)k|q_{[30]} + B_0(25)25p_{[30]} + E_0 - 0.99P\ddot{a}_{[30]:25}$$

$$= \sum_{k=0}^{24} \frac{250000 \cdot (1.025)^k}{(1.05)^{k+1}} k|q_{[30]} + B_0(25)25p_{[30]}$$

$$+ E_0 - 0.99P\ddot{a}_{[30]:25}$$

$$= \frac{250000}{1.025} A_{[30:25]}^1 + B_0(25) \cdot 25p_{[30]} + 1200 + 0.39P - 14.5838P$$

(156)

Under the EPP, we have $P = 9764.44$. 
Example 6.9

For \(1 + j = \frac{1+i}{1.025}\), we have \(j = 0.02439\) and so

\[
\mathbb{E}[L_0] = \mathbb{E}[B_0(K_{[x]})] + E_0 - \mathbb{E}[P_0(K_{[x]})]
\]

\[
= \sum_{k=0}^{24} B_0(k)_{k|q_{[30]}} + B_0(25)_{25}p_{[30]} + E_0 - 0.99P \dddot{a}_{[30]:25}
\]

\[
= \sum_{k=0}^{24} \frac{250000 \cdot (1.025)^k}{(1.05)^{k+1}} k|q_{[30]} + B_0(25)_{25}p_{[30]}
\]

\[
+ E_0 - 0.99P \dddot{a}_{[30]:25}
\]

\[
= \frac{250000}{1.025} A_{[30:25]j} + B_0(25) \cdot 25p_{[30]} + 1200 + 0.39P - 14.5838P
\]

Under the EPP, we have \(P = 9764.44\).

**HW:** Replicate Table 6.3 in the text by using a spreadsheet program. Compare this example with Example 6.10.
Assume once again that a company is about to issue insurance to \( N \) independent lives \([x]\), each with loss \( L_{0,i} \) for \( i \in \{1, 2, \ldots, N\} \). In this case

\[
L_0 = \sum_{i=1}^{N} L_{0,i}
\]

\[
\mathbb{E}[L_0] = \mathbb{E} \left[ \sum_{i=1}^{N} L_{0,i} \right] = \sum_{i=1}^{N} \mathbb{E} [L_{0,i}] = N \cdot \mathbb{E} [L_{0,1}] \tag{157}
\]

\[
\text{Var}[L_0] = N \cdot \text{Var}[L_{0,1}]
\]
If we require \( P \) such that \( P[L_0 < 0] = \alpha \), we can use CLT once again to show

\[
\alpha = P[L_0 < 0] = P\left[ \frac{L_0 - E[L_0]}{\sqrt{Var[L_0]}} < -\frac{E[L_0]}{\sqrt{Var[L_0]}} \right] \\
\rightarrow \Phi\left( -\frac{E[L_0]}{\sqrt{Var[L_0]}} \right) \quad \text{as} \quad N \rightarrow \infty
\]

For an individual present value of loss, stated wlog as \( L_{0,1} \), we recover the EPP as \( N \rightarrow \infty \)

\[
E[L_{0,1}] \approx -\frac{\Phi^{-1}(\alpha) \sqrt{Var[L_{0,1}]} \sqrt{N}}{\sqrt{N}} \rightarrow 0
\]
Example 6.11

An insurer issues whole life insurance policies to select lives aged [30]. The sum insured $S = 100000$ is paid at the end of the month of death and level monthly premiums are payable throughout the term of the policy. Initial expenses, incurred at the issue of the policy, are 15% of the total of the first year’s premiums. Renewal expenses are 4% of every premium, including those in the first year. Assume the SSSM with interest at 5% per year.

- Calculate the monthly premium $P$ using the EPP
- Calculate the monthly premium $P$ using the PPPP such that $\alpha = 0.95$ and $N = 10000$. 
Example 6.11

For the EPP calculation, we have

\[
E[PV(\text{Premiums})] = 12P\ddot{a}^{(12)}_{[30]} = 227.065P
\]

\[
E[PV(\text{Benefits})] = 100000A^{(12)}_{[30]} = 7866.18
\]

\[
E[PV(\text{Expenses})] = (0.15)(12P) + (0.04)(12P\ddot{a}^{(12)}_{[30]})
\]

\[
= 10.8826P
\]
Example 6.11

For the EPP calculation, we have

\[ \mathbb{E}[PV(\text{Premiums})] = 12P\ddot{a}_{30}^{(12)} = 227.065P \]

\[ \mathbb{E}[PV(\text{Benefits})] = 100000A_{30}^{(12)} = 7866.18 \]

\[ \mathbb{E}[PV(\text{Expenses})] = (0.15)(12P) + (0.04)(12P\ddot{a}_{30}^{(12)}) \]

\[ = 10.8826P \]

\[ \mathbb{E}[PV(\text{Premiums})] = \mathbb{E}[PV(\text{Benefits})] + \mathbb{E}[PV(\text{Expenses})] \]

\[ \Rightarrow P = 36.39 \]
Example 6.11

For all $i \in \{1, 2, \ldots, N\}$, we have the i.i.d. PV(Loss) random variables

$$L_{0,i} = 100000v^{K_{[30]}^{(12)} + \frac{1}{12}} + (0.15)(12P)$$

$$- (0.96) \left( 12P \ddot{a}_{\ddot{a}}^{(12)} \frac{K_{[30]}^{(12)} + \frac{1}{12}}{K_{[30]}^{(12)} + \frac{1}{12}} \right)$$

$$\mathbb{E}[L_{0,i}] = 100000A_{[30]}^{(12)} + (0.15)(12P) - (0.96)(12P \ddot{a}_{\ddot{a}}^{(12)})$$

$$= 7866.18 - 216.18P$$
Example 6.11

To find the variance, we rewrite

\[
L_{0,i} = \left(100000 + \frac{(0.96)(12P)}{d^{(12)}}\right) K^{(12)}_{[30]} + \frac{1}{12} + (0.15)(12P) - \frac{(0.96)(12P)}{d^{(12)}}
\]

\[
Var[L_{0,i}] = \left(100000 + \frac{(0.96)(12P)}{d^{(12)}}\right)^2 \cdot \left(2 A^{(12)}_{[30]} - \left(A^{(12)}_{[30]}\right)^2\right)
\]

\[
= (100000 + 236.59P)^2 \cdot (0.0053515)
\]
Collecting our results, we now have

\[ 0.95 = \alpha = P[L_0 < 0] \]

\[ \approx \Phi \left( -\frac{\mathbb{E}[L_0]}{\sqrt{\text{Var}[L_0]}} \right) \]

\[ = \Phi \left( -\sqrt{N} \cdot \frac{\mathbb{E}[L_{0,1}]}{\sqrt{\text{Var}[L_{0,1}]]} \right) \]

\[ = \Phi \left( \sqrt{10000} \cdot \frac{216.18P - 7866.18}{(100000 + 236.59P) \cdot \sqrt{0.0053515}} \right) \]

(163)
Example 6.11

It follows that

\[ 1.645 = \Phi^{-1}(0.95) \]

\[ \approx \sqrt{10000} \cdot \frac{216.18P - 7866.18}{(100000 + 236.59P) \cdot \sqrt{0.0053515}} \]  

\[ \Rightarrow P = 36.99 \]  

For general \( N \), we have

\[ \frac{216.18P - 7866.18}{(100000 + 236.59P) \cdot \sqrt{0.0053515}} = \frac{1.645}{\sqrt{N}} \]  

and as \( N \to \infty \), we have \( P \to 36.39 \), recovering the EPP premium as expected.
Imagine that a fully continuous whole life insurance is offered to \( N \) individuals aged \([x]\) with \( T_{[x]}^{(i)} \sim \exp(\lambda)\) for all \( i \in \{1, \ldots, N\}\). For each insured, the i.i.d. loss random variables are

\[
L_{0,i} = S v T_{[x]}^{(i)} - P \bar{a}_{T_{[x]}^{(i)}} = \frac{\delta S + P}{\delta} e^{-\delta T_{[x]}^{(i)}} - \frac{P}{\delta}
\]  

(166)

and so for \( L_0 = \sum_{i=1}^{N} L_{0,i} \), the PPPPP seeks to determine \( P \) such that

\[
P \left[ \sum_{i=1}^{N} \left( \frac{\delta S + P}{\delta} e^{-\delta T_{[x]}^{(i)}} - \frac{P}{\delta} \right) < 0 \right] = \alpha
\]  

(167)
Independent Exponential RV’s

In this case, we can rewrite this as

\[
\mathbb{P}\left[ \frac{1}{N} \sum_{i=1}^{N} e^{-\delta T^{(i)}_x} < \frac{P}{P + \delta S} \right] = \alpha
\]

(168)

In this case, for each \( i \) if we define \( Y_i := e^{-\delta T^{(i)}_x} \), then we know that

\[
\mathbb{P} [ Y_i > y ] = \mathbb{P} \left[ e^{-\delta T^{(i)}_x} > y \right] = 1 - y^{\frac{\lambda}{\delta}}.
\]
Independent Exponential RV’s

In this case, we can rewrite this as

\[
P \left[ \frac{1}{N} \sum_{i=1}^{N} e^{-\delta T[x]} < \frac{P}{P + \delta S} \right] = \alpha
\]  

(168)

In this case, for each \( i \) if we define \( Y_i := e^{-\delta T[x]} \), then we know that

\[
P [Y_i > y] = P \left[ e^{-\delta T[x]} > y \right] = 1 - y^\frac{\lambda}{\delta}.
\]

HW Using convolution techniques, find the above probability

\[
P \left[ \frac{1}{N} \sum_{i=1}^{N} Y_i < \frac{P}{P + \delta S} \right] = \alpha.
\]  

(169)

Are there any ergodic theory results that we can use?
Another principle is to find $P$ such that for any $\alpha \in (0, 1)$, the probability that any contract suffers a loss of $\beta < S \alpha \frac{\delta}{\lambda N} < S$ is set to $\alpha$ for a set of i.i.d. exponentially distributed times $T^{(i)}_{[x]}$:

$$\mathbb{P}\left[\max_{i=1..N} \left\{ \frac{\delta S + P}{\delta} e^{-\delta T^{(i)}_{[x]}} - \frac{P}{\delta} \right\} < \beta \right] = \alpha \quad (170)$$

We rewrite this as

$$\mathbb{P}\left[\min_{i=1..N} \left\{ T^{(i)}_{[x]} \right\} > -\frac{1}{\delta} \ln \left( \frac{P + \delta \beta}{P + \delta} \right) \right] = \alpha \quad (171)$$
Capped Maximal Loss

Another principle is to find $P$ such that for any $\alpha \in (0, 1)$, the probability that any contract suffers a loss of $\beta < S\alpha \frac{\delta}{\lambda N} < S$ is set to $\alpha$ for a set of i.i.d. exponentially distributed times $T^{(i)}_{[x]}$:

$$\mathbb{P} \left[ \max_{i=1..N} \left\{ \frac{\delta S + P}{\delta} e^{-\delta T^{(i)}_{[x]}} - \frac{P}{\delta} \right\} < \beta \right] = \alpha$$

(170)

We rewrite this as

$$\alpha = \mathbb{P} \left[ \min_{i=1..N} \left\{ T^{(i)}_{[x]} \right\} > -\frac{1}{\delta} \ln \left( \frac{P + \delta \beta}{P + \delta} \right) \right]$$

$$= \left( e^{-\lambda \left[ -\frac{1}{\delta} \ln \left( \frac{P + \delta \beta}{P + \delta} \right) \right]} \right)^N = \left( \frac{P + \delta \beta}{P + \delta S} \right)^\frac{\lambda N}{\delta}$$
Capped Maximal Loss

Another principle is to find $P$ such that for any $\alpha \in (0, 1)$, the probability that any contract suffers a loss of $\beta < S\alpha \frac{\delta}{\lambda N} < S$ is set to $\alpha$ for a set of i.i.d. exponentially distributed times $T^{(i)}_{[x]}$:

$$
\mathbb{P} \left[ \max_{i=1..N} \left\{ \frac{\delta S + P}{\delta} e^{-\delta T^{(i)}_{[x]}} - \frac{P}{\delta} \right\} < \beta \right] = \alpha \quad (170)
$$

We rewrite this as

$$
\alpha = \mathbb{P} \left[ \min_{i=1..N} \left\{ T^{(i)}_{[x]} \right\} > -\frac{1}{\delta} \ln \left( \frac{P + \delta \beta}{P + \delta} \right) \right] \\
= \left( e^{-\lambda \left[ -\frac{1}{\delta} \ln \left( \frac{P + \delta \beta}{P + \delta} \right) \right]} \right)^N = \left( \frac{P + \delta \beta}{P + \delta S} \right)^{\frac{\lambda N}{\delta}} \quad (171)
$$

$$
P = \delta \cdot \frac{S\alpha \frac{\delta}{\lambda N} - \beta}{1 - \alpha \frac{\delta}{\lambda N}} \rightarrow \infty \text{ as } N \rightarrow \infty
$$

For this risk measure, is there a number of policy holders $N$ that is too high?
Comments on PPPP

Notice that the PPPP only guarantees that the probability of a loss is $1 - \alpha$.

- **It says nothing about the size of what that loss could be if it arises.**

- This is a big problem if the loss is extremely large and bankrupts the insurer. It may seem very unlikely, but recent economic events have shown otherwise.

- **Further improvements to this model can be seen in the ERM for Strategic Management (Status Report) by Gary Venter, posted on the SOA.org website**

- **Also, there is a close link, perhaps to be explored in a project, with VAR in the financial world. Click [here](#) for an informative article in the *NY Times*™ for an article on VAR and the recent financial crisis.**
HW: 6.1, 6.2, 6.5, 6.7, 6.8, 6.12, 6.14, 6.15
When entering into a contract, the financial obligations of all parties should be specified at the time the agreement is signed. This includes disclosure of health status, age, and premium payments expected to fund benefits and expenses associated with the contract.

The **Policy Value** $tV$ is the expected value of the future loss random variable $L_t$ at time $t$:

$$tV = \mathbb{E}[L_t] = \mathbb{E}[ \text{Loss} \mid T_x > t]$$ (172)
Definition

- The **gross premium policy value** for a policy in force at duration \( t \geq 0 \) years after it was purchased is the expected value at that time of the gross future loss random variable on a specified basis. The premiums used in the calculation are the actual premiums payable under the contract.

- The **net premium policy value** for a policy in force at duration \( t \geq 0 \) years after it was purchased is the expected value at that time of the net future loss random variable on a specified basis (which makes no allowance for expenses.) The premiums used in the calculation are the net premiums calculated on the policy value basis using the equivalence principle, not the actual premiums payable.

It is important to note that usual practice dictates that when calculating \( tV \), premiums and premium-related expenses due at \( t \) are regarded as future payments and any insurance benefits and related expenses as past payments.
Define

- $P_t$ as the premium payable at time $t$
- $e_t$ as the premium-related expense payable at time $t$
- $S_{t+1}$ as the sum insured payable at time $t + 1$
- $E_{t+1}$ as the expense of paying the sum insured at time $t + 1$
- $t+1V$ as the gross premium policy value for a policy in force at time $t + 1$
- $L_t$ as the gross future loss random variable at time $t$
- $i_t$ as the interest rate from time $t$ to time $t + 1$. 
Then, using recursion, we obtain

$$ t V = e_t - P_t + q_{[x]+t} \cdot \frac{S_{t+1} + E_{t+1}}{1 + i_t} + p_{[x]+t} \cdot \frac{t+1 V}{1 + i_t}. \quad (173) $$

Notice that if there is a fixed term to the contract, such as an endowment or term insurance, then we have the boundary condition

$$ _n V = 0 \quad (174) $$

Also, if the premium is calculated using the EPP and the policy basis is the same as the premium basis, then

$$ _0 V = E[L_0] = 0 \quad (175) $$
For an endowment insurance contract with sum insured $S$, however, we have the pair of boundary conditions

$$n^{-} V = \lim_{\epsilon \to 0^+} n^{-\epsilon} V = S$$

$$n V = 0$$

In calculating $n^{-1} V$, we actually use $n^{-} V$ instead of $n V$. See next example!
Example 7.7

Consider a zero-expense, 20 year endowment policy purchased by a life aged 50. Level premiums of 23500 per year are payable annually throughout the term of the policy. A sum insured of 700000 is payable at the end of the term if the life survives to age 70. On death before age 70, a sum insured is payable at the end of the year of death equal to the policy value at the start of the year in which the policyholder dies. Assuming the SSSM with interest at 3.5% per year, calculate $V_{15}$, the policy value in force at the start of the 16th year.
Example 7.7

It follows that

\[ S_{t+1} = t V \]
\[ e_t = 0 = E_t \]
\[ S = 700000 \]
\[ P_t = 23500 \]

\[ t V = -23500 + q_{[50]+t} \frac{t V}{1.035} + p_{[50]+t} \frac{t+1 V}{1.035} \] (177)
Combining with our boundary value, we obtain the difference equation

\[ tV = \frac{p_{[50]+t} \cdot t+1 V - 24322.50}{p_{[50]+t} + 0.035} \]

\[ 20^- V = 700000. \]
Combining with our boundary value, we obtain the difference equation

$$tV = \frac{p_{[50]+t} \cdot t+1 V - 24322.50}{p_{[50]+t} + 0.035}$$

$$20^- V = 700000.$$  

Our initial iteration actually uses $$20^- V$$ to obtain

$$19 V = \frac{p_{69} \cdot (20^- V) - 24322.50}{p_{69} + 0.035} = 652401$$
Example 7.7

Combining with our boundary value, we obtain the difference equation

\[ tV = \frac{p_{[50]+t} \cdot t+1 V - 24322.50}{p_{[50]+t} + 0.035} \]

\[ 20^- V = 700000. \]

Our initial iteration actually uses \( 20^- V \) to obtain

\[ 19V = \frac{p_{69} \cdot (20^- V) - 24322.50}{p_{69} + 0.035} = 652401 \]

(179)

Use tables or spreadsheet to calculate SSSM values and obtain

\[ 15V = 478063. \]
Consider a 20–year endowment policy purchased by a life aged 50. Level premiums are payable annually throughout the term of the policy and the sum insured, \( S = 500000 \), is payable at the end of the year of death or at the end of the term, whichever is sooner. The basis used by the insurance company for all calculations is under the SSSM with 5% per year interest and no allowance for expenses. Calculate \( P \) under the EPPP and the corresponding policy values.
Substituting the information contained in the problem formation, we obtain

\[ tV = -P + q^{[50]+t} \cdot \frac{500000}{1.05} + p^{[50]+t} \cdot \frac{t+1V}{1.05} \]

\[ = p^{[50]+t} \cdot \frac{t+1V}{1.05} + \frac{500000}{1.05} - P \]

(180)

\[ 0V = 0 = \mathbb{E}[L_0] = 500000A^{[50]:20} - P\ddot{a}^{[50]:20} \]

\[ 20-V = 500000 \]

Solving for \( P \), we obtain \( P = 15114.33 \). Iteration of the resulting difference equation delivers the remaining policy values.
A man aged 50 purchases a deferred annuity policy. The annuity will be paid annually for life, with the first payment on his 60th birthday. Each annuity payment will be 10000. Level premiums of 11900 are payable annually for at most 10 years. On death before age 60, all premiums paid will be returned, without interest, at the end of the year of death. The insurer uses the following basis for calculation of policy values:

- SSSM with 5% interest per year
- Expenses of 10% of the first premium, 5% of subsequent premiums, 25 each time an annuity payment is paid, and 100 when a death claim is paid.

Calculate $tV$ for $t \in \{0, 1, \ldots, 9\}$
Our initial policy value is

\[ 0 V = P \cdot (IA)_{[50]:10}^1 + 100A_{[50]:10}^1 + 10025v^{10} \cdot 10 \cdot p_{[50]} \ddot{a}_{60} \]

\[ - \left( 0.95\ddot{a}_{[50]:10} - 0.05 \right) P \]

\[ = 485 > 0 \]  

(181)

This of course can now be used to *forward* iterate to find \( \{t V\}_{t=1}^9 \). Since \( 0 V = 485 > 0 \), the premiums charged correspond to a valuation basis that is more conservative than the premium basis.
Example 7.4

Our initial policy value is

\[ 0 V = P \cdot (IA)_{[50]:10}^1 + 100A_{[50]:10}^1 + 10025v^{10} p_{[50]} \ddot{a}_{60} \]
\[ - \left( 0.95\ddot{a}_{[50]:10} - 0.05 \right) p \]
\[ = 485 > 0 \]  

(181)

This of course can now be used to forward iterate to find \( \{ t V \}_{t=1}^9 \). Since \( 0 V = 485 > 0 \), the premiums charged correspond to a valuation basis that is more conservative than the premium basis.

In general, we have for \( t \in \{1, 2, \ldots, 9\} \),

\[ t V = P \cdot (IA)_{[50]+t:10-t}^1 + (tP + 100)A_{[50]+t:10-t}^1 \]
\[ + 10025v^{10-t} p_{[50]+t} \ddot{a}_{60} - 0.95P \ddot{a}_{[50]+t:10-t} \]  

(182)
Example 7.4

For $1 \leq t \leq 9$, we can see our recursion equation is

$$t V = -0.95P + q_{50} t \cdot \frac{(t + 1) \cdot P + 100}{1.05} + p_{50} t \cdot \frac{t+1 V}{1.05} \tag{183}$$

For $t \geq 10$, we have

$$t^- V = 10025 \ddot{a}_{50} t$$
$$t^+ V = 10025 a_{50} t = t^- V - 10025 \tag{184}$$

Which do we use to find $9 V$, $10^- V$ or $10^+ V$? Recall that from (182), we have

$$9 V = P \cdot (IA)^{1}_{59:1} + (9P + 100) A^{1}_{59:1} + 10025 p_{59} v a_{60} - 0.95 P \ddot{a}_{59:1}$$

$$= Pvq_{59} + (9P + 100) v q_{59} + 10025 p_{59} v a_{60} - 0.95 P$$

$$= -0.95 P + q_{59} \frac{10P + 100}{1.05} + p_{59} \frac{10025 \ddot{a}_{60}}{1.05} \tag{185}$$
The previous example shows that sometimes we need to calculate the initial value, given the information contained in the problem statement, to iterate forward, especially if there is no term $n$ and corresponding boundary condition $nV$. Also, no annuity payments have occurred yet and this reflects in the expenses.

It is likely that $DSAR := S_{t+1} + E_{t+1} - t+1V \neq 0$. The Death Strain At Risk, or DSAR, is the extra amount needed to increase the policy value to the death benefit at time $t+1$. This is a capital based risk measure, as it is a direct measure of what the insurer may be at risk of needing to close out a contract if a benefit must be paid. If the DSAR is large enough, management may want to purchase reinsurance in case a large DSAR (even with low probability) occurs.
Example 7.3

A woman aged 60 purchases a 20 year endowment insurance with a sum insured $S = 100000$ payable at the end of the year of death or on survival to age 80, whichever occurs first. **An annual premium of 5200 is payable for at most 10 years.** The insurer uses the following basis for calculation of policy values:

- SSSM with 5% interest per year
- Expenses of 10% of the first premium, 5% of subsequent premiums, and 200 on payment of the sum insured.

Let’s set up an algorithm to calculate the policy values $\{0 V, 1 V, ..., 9 V\}$. 
Example 7.3

Our initial recursion equation is

\[ 0V = -0.9P + q_{[60]} \cdot \frac{100200}{1.05} + p_{[60]} \cdot \frac{V}{1.05}. \]  

(186)

For \( 1 \leq t \leq 9 \), we have

\[ tV = -0.95P + q_{[60]+t} \cdot \frac{100200}{1.05} + p_{[60]+t} \cdot \frac{t+1V}{1.05}. \]  

(187)

For \( t = 10 \), we have

\[ 10V = \mathbb{E}[L_{10}] = \mathbb{E}\left[100200 \min\{K_{70}+1,10\}\right] \]
\[ = 100200A_{70:10} = 63703. \]  

(188)

HW Compute \( 9V \) and \( 12V \) explicitly.
A whole life insurance contract \((wl)\) is issued to an age 40, with death benefit 1000, has \(P_{40}\) defined as the premium for a whole life insurance issued to age \(40\) with benefit 1, under the equivalence pricing principle.

Now, assume that in fact a modified whole life \((mwl)\) contract issued to \(40\), where now the death benefit is 1000 for the first 20 years, 5000 for the next 5 years, and 1000 thereafter. The annual net premium is \(1000P_{40}\) for the first 20 years, \(5000P_{40}\) for the next 5 years, and a value \(\pi\) thereafter.

Assume

\[
(A_{40}, A_{60}, A_{60:5|}) = (0.16132, 0.36913, 0.06674)
\]
\[
(q_{60}, v^{20}_{20}p_{40}, v^{5}_{5}p_{60}) = (0.01376, 0.27414, 0.68756)
\]
\[
(\ddot{a}_{40}, \ddot{a}_{40:20|}, \ddot{a}_{60}, \ddot{a}_{60:5|}, \ddot{a}_{65}) = (14.8166, 11.7612, 11.1454, 4.3407, 9.8969)
\]

Given the information above, how can we calculate \(20V^{mwl}\) and \(21V^{mwl}\)?
Going forward, i.e. valuing the cash flows prospectively, and substituting the data found in (189), leads to

\[ 0 = 0 V^{wl} = 1000 \times A_{40} - 1000 P_{40} \ddot{a}_{40} \]

\[ \Rightarrow 1000 P_{40} = 1000 \times \frac{A_{40}}{\ddot{a}_{40}} = 1000 \times \frac{0.16132}{14.8166} = 10.89. \]  

(190)

Similarly,

\[ 0 = 0 V^{mwl} = \left( 1000 A_{40} + 4000 \nu^{20}_{20} p_{40} A_{60:51}^{1} \right) \]

\[ - \left( 1000 P_{40} \ddot{a}_{40:20} + 5000 P_{40} \nu^{20}_{20} p_{40} \ddot{a}_{60:51} + \pi \nu^{20}_{20} p_{40} \nu^{5}_{5} p_{60} \ddot{a}_{65} \right) \]

\[ \Rightarrow \pi = 22.32. \]

(191)
Continuing our prospective analysis, we calculate

\[ 20 \, V^{\text{mwl}} = 4000 A_{60:5}^1 + 1000 A_{60} - 5000 P_{40} \ddot{a}_{60:5} - \pi v^{\dddot{5}} p_{60} \ddot{a}_{65} \]

\[ = 247.86 \approx 248 \]

\[ \Rightarrow 21 \, V^{\text{mwl}} = \frac{(20 \, V + 5000 P_{40}) (1.06) - 5000 q_{60}}{p_{60}} \]

\[ = \frac{(248 + (5 \times 10.89)) (1.06) - (5000 \times 0.01376)}{1 - 0.01376} \]

\[ = 255.31 \approx 255. \]
However, notice that \( 0 = 0 V^\text{mwl} \) and that, like a regular whole life contract, for \( t \in \{0, 1, ..., 19\} \)

\[
t V^\text{mwl} = -1000 P_{40} + q_{40+t} \frac{1000}{1 + r_t} + p_{40+t} \frac{t+1 V^\text{mwl}}{1 + r_t}.
\] (193)

Iterating forward, we see that

\[
20 V^\text{mwl} = 20 V^\text{wl} = 1000 A_{60} - 1000 P_{40} \ddot{a}_{60}
\]

\[
= 1000 \times \left( A_{60} - \frac{A_{40}}{\ddot{a}_{40}} \ddot{a}_{60} \right)
\]

\[
= 1000 \times \left( 1 - d \ddot{a}_{60} - \frac{1 - d \ddot{a}_{40}}{\ddot{a}_{40}} \ddot{a}_{60} \right)
\]

\[
= 1000 \times \left( \frac{\ddot{a}_{40} - \ddot{a}_{60}}{\ddot{a}_{40}} \right) = 1000 \times \left( \frac{14.8166 - 11.1454}{14.8166} \right) \approx 248.
\] (194)
An insurer issues a large number of policies identical to the policy in Example 7.3 to women aged 60. Five years after they were issued, a total of 100 of these policies were still in force. In the following year, one person died ($d_5 = 1$) and

- expenses of 6% of each premium paid were incurred - i.e.  
  \[ e_5^{actual} = 0.06P_5 \]
- interest was earned at 6.5% on all assets - i.e.  
  \[ i_5^{actual} = 0.065 \]
- expenses of  
  \[ E_6^{actual} = 250 \] were incurred on the payment of the sum insured for the policyholder who died.

Calculate a.) the profit or loss on the group of policies for this year and b.) determine how much of this profit or loss is attributable to profit or loss from mortality, from interest, and from expenses.
To solve, we compute the difference between the growth of assets from $t = 5$ to $t = 6$ and subtract the total asset value at $t = 6$:

\[
Profit = N \cdot (5V + P_5 - 0.06P_5) \cdot (1 + i_5)^1 \\
- (d_5 \cdot (S + E_6) + (N - d_5) \cdot 6V) \\
= 100 \cdot (5V + (0.94)(5200)) \cdot (1.065)^1 \\
- (1 \cdot (S + E_6) + 99 \cdot 6V) \\
= 106.5 \cdot (5V + 4888) - (100250 + 99 \cdot 6V)
\]
Furthermore,

\[ 5 V = \mathbb{E}[L_5] = 100200A_{65:15} - 0.95 \cdot 5200\bar{a}_{65:5} \]
\[ = 29068 \]

\[ 6 V = \mathbb{E}[L_6] = 100200A_{66:14} - 0.95 \cdot 5200\bar{a}_{66:4} \]
\[ = 35324 \]

∴ Profit = 1065 \cdot (29068 + 35324) - (100250 + (99)(35324)) = 18919

HW: Read the solution for part b in the textbook.
Furthermore,

\[ 5V = \mathbb{E}[L_5] = 100200A_{65:15} - 0.95 \cdot 5200\bar{a}_{65.5} \]
\[ = 29068 \]

\[ 6V = \mathbb{E}[L_6] = 100200A_{66:14} - 0.95 \cdot 5200\bar{a}_{66.4} \]
\[ = 35324 \]

\[ \therefore \text{Profit} = 106.5 \cdot (29068 + 4888) - (100250 + (99)(35324)) \]
\[ = 18919 \]

HW: Read the solution for part b.) in the textbook.
Define $A S_t$ as the share of the insurer’s assets attributable to each policy in force at time $t$. Consider now a policy identical to the policy studied in Example 7.4 and suppose that this policy has now been in force for five years. Suppose that over the past five years, the insurer’s experience in respect of similar policies has realized annual interest on investments as $(i_1, i_2, i_3, i_4, i_5) = (0.048, 0.056, 0.052, 0.049, 0.047)$. 
Furthermore,

- Expenses at the start of the year in which a policy was issued were 15% of the premium
- Expenses at the start of the year after the year in which a policy was issued were 6% of the premium
- The expense of paying a death claim was, on average, 120
- The mortality rate $q_{[50]+t} \approx 0.0015$ for $t \in \{0, 1, 2, 3, 4\}$

Calculate $AS_t$ using the convention that $AS_t$ does not include the premium and related expense due at time $t$. (This of course means that $AS_0 = 0$, as no premiums paid in and no benefits paid out yet.)
We calculate $AS_1$ here and refer to Table 7.1 for the complete set of calculations.

- At time 0, insurer receives premiums minus expenses of $0.85 \cdot 11900N = 10115N$.
- At time 1, this accumulates to $10115N \cdot (1 + i_1) = 10601N$.
- A total of 0.0015$N$ policy holders die in the first year and death claims are $0.0015N \cdot (11900 + 120) = 18N$.
- Therefore, the value of the fund at the end of the first year is $10601N - 18N = 10582N$. 
It follows that

\[ AS_1 = \frac{\text{Fund Value at time } 1}{\text{Number of Policies in Force at time } 1} \]

\[ = \frac{10582N}{0.9985N} = 10598 \]

Now, read Section 7.4 on computing Valuation between premium dates.
Recall that

\[ tV = e_t - P_t + q_{x+t} \cdot \frac{S_{t+1} + E_{t+1}}{1 + i_t} + p_{x+t} \cdot \frac{t+1V}{1 + i_t}. \]  

(198)

Now, consider that \( t \) is real-valued and define

- \( P_t \) as the annual rate of premium payable at time \( t \)
- \( e_t \) as the annual rate of premium-related expense payable at time \( t \)
- \( S_t \) as the sum insured payable at time \( t \) if the policy holder dies exactly at \( t \)
- \( E_t \) as the expense of paying the sum insured at time \( t \)
- \( \mu_{x+t} \) as the force of mortality at age \([x] + t\)
- \( \delta_t \) as the force of interest assumed at time \( t \)
- \( tV \) as the policy value at time \( t \).
Now, as the force of interest varies, we have

\[ v(t) = e^{-\int_0^t \delta_u du} \]
\[ \frac{v(t)}{v(s)} = e^{-\int_s^t \delta_u du} \]

and so

\[ tV = \int_0^\infty \frac{v(t+s)}{v(t)} \cdot \left( [S_{t+s} + E_{t+s}] \cdot sp[x]+t\mu[x]+t+s \right) ds \]
\[ - \int_0^\infty \frac{v(t+s)}{v(t)} \cdot \left( [P_{t+s} - e_{t+s}] \cdot sp[x]+t \right) ds \]

Q: What happens if there is a finite term to contract?
By the product rule and the identities

\[ r - t p[x] + t = \frac{r p[x]}{t p[x]} \]

\[ \frac{d}{dt} (t p[x]) = -t p[x] \mu[x] + t \]

\[ \nu'(t) = -\delta_t \nu(t) \]

we obtain the ODE

\[ \frac{d}{dt} (t V) = \delta_t \cdot t V + P_t - e_t - (S_t + E_t - t V) \mu[x] + t \]
Boundary Conditions:

For $S$ sum insured, we have

\[
\lim_{t \to n^-} t V = S \quad \text{for an endowment policy with term } n \text{ years.}
\]

\[
\lim_{t \to n^-} t V = 0 \quad \text{for a term policy with term } n \text{ years.}
\]  \hspace{1cm} (203)

\[
\lim_{t \to \omega^-} t V = S \quad \text{for a whole life policy with upper limit } \omega \text{ years.}
\]

Forward Euler:

\[
t_{t+h} V = t V + h \cdot \left( \delta_t \cdot t V + P_t - e_t - (S_t + E_t - t V) \mu_{[x]+t} \right)
\]  \hspace{1cm} (204)
As an example, consider the case where for $S$ sum insured, we have

\[
S_{t+s} = S
\]
\[
e_{t+s} = 0 = E_{t+s}
\]
\[
\delta_t = \delta
\]
\[
\mu[x]+t+s = \lambda
\]
\[
P_{t+s} = Pe^{-\gamma(t+s)}
\]

Then it follows that

\[
tV = \int_0^\infty Se^{-\delta s} \cdot \lambda e^{-\lambda s} ds - \int_0^\infty e^{-\delta s} \cdot Pe^{-\gamma(t+s)} e^{-\lambda s} ds
\]

\[
= \frac{S\lambda}{\lambda + \delta} - \frac{Pe^{-\gamma t}}{\lambda + \delta + \gamma}
\]
Policy Alterations

In many cases, policyholders may wish to change the terms of their contract if it is still in effect. For example:

- They may wish to stop making premiums, or to change the terms of their benefit payout.
- They may wish to cash out their position, or simply wish to shorten the time remaining until payout.
Policy Alterations

In many cases, policyholders may wish to change the terms of their contract if it is still in effect. For example:

- They may wish to stop making premiums, or to change the terms of their benefit payout.
- They may wish to cash out their position, or simply wish to shorten the time remaining until payout.

One may argue that the insurer is under no obligation to make such changes if they are not written expressly into the initial contract. For example:

- The policyholder (but not insurer) may know something about their health status that would make it better for them to cash out now.
- By having to liquidate assets that cover the policy, the insurer may have to take a loss to be able to settle the alteration, and this could affect other policyholders adversely.
Because of these concerns, the lender may agree to alter the terms of the contract, but only paying a **Surrender (Cash) Value** $C_t$ of a fraction of $tV$ or $AS_t$. 

In allowing the policy to lapse, the policy holder is cashing out a policy and using the proceeds to enter into a new contract. If the period between lapsing and entering into a new contract is too short, then the insurer may suffer from not earning enough income to cover the new business strain of writing the first contract. Hence, some countries including the US have non-forfeiture laws that allow for zero cash values for early surrenders.
Because of these concerns, the lender may agree to alter the terms of the contract, but only paying a **Surrender (Cash) Value** $C_t$ of a fraction of $V_t$ or $AS_t$.

$$C_t = \mathbb{E} [PV_t(\text{future benefits + expenses, altered contract})] - \mathbb{E} [PV_t(\text{future premiums, altered contract})]$$

In allowing the policy to **lapse**, the policy holder is cashing out a policy and using the proceeds to enter into a new contract. If the period between lapsing and entering into a new contract is too short, then the insurer may suffer from not earning enough income to cover the new business strain of writing the first contract. Hence, some countries including the US have **non-forfeiture** laws that allow for zero cash values for early surrenders.
Consider the policy discussed in Examples 7.4 and 7.9. Given the experience of the insurer detailed in Example 7.9, at the start of the $6^{th}$ year but before paying the premium due, the policyholder requests that the policy be altered in one of the following ways:
1. The policy is surrendered immediately.

2. No more premiums are paid, and a reduced annuity is payable from age 60. In this case, all premiums paid are refunded at the end of the year of death if the policyholder dies before age 60.

3. Premiums continue to be paid, but the benefit is altered from an annuity to a lump sum (pure endowment) payable on reaching age 60. Expenses and benefits on death before age 60 follow the original policy terms. There is an expense of 100 associated with paying the sum insured at the new maturity date.
Calculate the surrender value, the reduced annuity, and sum insured assuming the insurer uses

- 90% of the asset share less a charge of 200 or
- 90% of the policy value less a charge of 200

together with the assumptions in the policy value basis when calculating revised benefits and premiums. Use the associated values

\[ 5V = 65470 \]
\[ AS_5 = 63509 \]  

(208)
Policy Alterations: Example 7.13

(1) \[
C_{5}^{\text{asset share}} = 0.9 \cdot AS_{5} - 200 = 56958
\]
\[
C_{5}^{\text{policy value}} = 0.9 \cdot 5V - 200 = 58723
\]
Policy Alterations: Example 7.13

1

\[ C_{5}^{\text{assetshare}} = 0.9 \cdot AS_{5} - 200 = 56958 \]  
\[ C_{5}^{\text{policyvalue}} = 0.9 \cdot 5 V - 200 = 58723 \]  

(209)

2

\[ C_{5} = 5 \cdot 11900A_{55:5}^{1} + 100A_{55:5}^{1} + (X + 25) \cdot v_{55}^{p_{55}} \cdot \ddot{a}_{60} \]  
\[ X^{\text{assetshare}} = 4859 \]  
\[ X^{\text{policyvalue}} = 5012 \]  

(210)
Policy Alterations: Example 7.13

\[ C_5^{\text{assetshare}} = 0.9 \cdot AS_5 - 200 = 56958 \]  
\[ C_5^{\text{policyvalue}} = 0.9 \cdot 5V - 200 = 58723 \]  
\[ C_5 = 5 \cdot 11900A^{1}_{55:5} + 100A^{1}_{55:5} + (X + 25) \cdot v^5 p_{55} \cdot \ddot{a}_{60} \]  
\[ X^{\text{assetshare}} = 4859 \]  
\[ X^{\text{policyvalue}} = 5012 \]  
\[ C_5 + 0.95 \cdot 11900\ddot{a}_{55:5} = 11900 \left( (IA)^{1}_{55:5} + 5A^{1}_{55:5} \right) + 100A^{1}_{55:5} + v^5 p_{55} (S + 100) \]  
\[ S^{\text{assetshare}} = 138314 \]  
\[ S^{\text{policyvalue}} = 140594 \]
Over- or underestimated interest rates are only one risk factor for actuarial reserving. Another very real factor is known as **longevity risk**, which is due to the possibility that a pensioner may live longer than expected. Hedging against such a possibility is extremely important. Please consult the paper by Tsai, Tzeng, and Wang on *Hedging Longevity Risk When Interest Rates Are Uncertain*. 
Related Project Topics

- For those of you interested in more sophisticated, cutting edge coding methods for reserving, code.google.com has a site dedicated to ChainLadder (google code name chainladder) that contains an R package providing methods which are typically used in insurance claims reserving. Links to slides explaining the method are also on the site.

- *An Introduction to R: Examples for Actuaries* by **Nigel de Silva** is a very nice primer on using R.
Homework Questions

HW: 7.1, 7.2, 7.4, 7.5, 7.8, 7.12, 7.14, 7.15
Recall that for the survival time $T_x$ of an individual $(x)$, we have

$$S_x(t) = 1 - F_x(t) = 1 - \mathbb{P}[T_x \leq t]$$  \hspace{1cm} (212)

We now extend the model to include multiple states, but first we define the random variable $Y(t) \in \{0, 1\}$ as the state of the individual $(x)$. If $(x)$ is alive at time $x + t$, then $Y(t) = 0$. Otherwise, $Y(t) = 1$. Hence, we can define

$$T_x = \max \{ t \mid Y(t) = 0 \}$$  \hspace{1cm} (213)

and the model flow $0 \rightarrow 1$. 
Accidental Death Model

We can also define

\[
Y(t) = \begin{cases} 
0 & \text{if } (x) \text{ is alive at time } x + t \\
1 & \text{if } (x) \text{ is dead at time } x + t \text{ of accidental cause} \\
2 & \text{if } (x) \text{ is dead at time } x + t \text{ of other cause}
\end{cases}
\]

Figure: ADM Flow Chart

There is a sum insured upon leaving state 0, but that sum is dependent on entering state 1 or 2.
However, we can go even further and define

\[ Y(t) = \begin{cases} 
0 & \text{if } (x) \text{ is alive at time } x + t \\
1 & \text{if } (x) \text{ is disabled at time } x + t \\
2 & \text{if } (x) \text{ is dead at time } x + t 
\end{cases} \]

There is a lump sum paid upon entering state 1, an annuity paid while in state 1, and a lump sum paid upon entering state 2.
Disability Income Insurance Model

However, we can go even further and define

\[ Y(t) = \begin{cases} 
0 & \text{if } (x) \text{ is alive and healthy at time } x + t \\
1 & \text{if } (x) \text{ is alive and sick at time } x + t \\
2 & \text{if } (x) \text{ is dead at time } x + t 
\end{cases} \]

Figure: DIIM Flow Chart

Premium is paid while in state 0, is a lump sum paid upon entering state 1, an annuity paid while in state 1, and a lump sum paid upon entering state 2.
Define

- **Joint Life Annuity** - annuity that pays until the first death among a group of lives
- **Last Survivor Annuity** - annuity that pays until the last death among a group of lives
- A common feature is payment rate decreases upon each death
- **Reversionary Annuity** - life annuity that starts payment upon death of a specified life, as long as another member of group is alive
- **Joint Life Insurance** - life annuity that starts payment upon first death of a member of group
- Usually, group consists of two members, a *husband* and a *wife*
Joint Model

For example, consider a policy issued to a group \((H, W)\) of age \((x, y)\). Then,

\[
Y(t) = \begin{cases} 
0 & \text{if } H \text{ is alive at } x + t \text{ and } W \text{ is alive at } y + t \\
1 & \text{if } H \text{ is alive at } x + t \text{ and } W \text{ is dead at } y + t \\
2 & \text{if } H \text{ is dead at } x + t \text{ and } W \text{ is alive at } y + t \\
3 & \text{if } H \text{ is dead at } x + t \text{ and } W \text{ is dead at } y + t 
\end{cases}
\]

Figure: Joint Model Flow Chart
Assuming that the group (which can consist of 1, 2, or more individuals) can be found in any one of the $n + 1$ states \( \{0, 1, 2, \ldots, n - 1, n\} \), we define the event

\[
\{ Y(t) = i \} \quad (214)
\]

to mean the group is in state $i$ at time $t$.

It follows that \( \{ Y(t) \}_{t \geq 0} \) is a discrete valued stochastic process.
Assumptions

We make the following assumptions and definitions about transitions between states and their associated probabilities:

\[ \mathbb{P}[Y(x + t) = j \mid Y(x) = i] := t p_{ij} \] (Markovity)
Assumptions

We make the following assumptions and definitions about transitions between states and their associated probabilities:

\[ P[Y(x + t) = j \mid Y(x) = i] := t p_{ij}^{X} \text{ (Markovity)} \]

\[ P[Y(x + s) = i \text{ for all } s \in [0, t] \mid Y(x) = i] := t p_{ii}^{X} \]
We make the following assumptions and definitions about transitions between states and their associated probabilities:

\[
\mathbb{P}[Y(x + t) = j \mid Y(x) = i] := t \rho_{ij}^x \quad \text{(Markovity)}
\]

\[
\mathbb{P}[Y(x + s) = i \text{ for all } s \in [0, t] \mid Y(x) = i] := t \rho_{ii}^x
\]

\[
\lim_{h \to 0} \frac{\mathbb{P}[2 \text{ or more transitions in interval of length } h]}{h} = 0
\]
Assumptions

We make the following assumptions and definitions about transitions between states and their associated probabilities:

\[
\mathbb{P}[Y(x + t) = j \mid Y(x) = i] := t p_{ij}^x \quad \text{(Markovity)}
\]

\[
\mathbb{P}[Y(x + s) = i \text{ for all } s \in [0, t] \mid Y(x) = i] := t \overline{p}_{ii}^x
\]

\[
\lim_{h \to 0} \mathbb{P}[\text{2 or more transitions in interval of length } h] = 0
\]

\[
\lim_{h \to 0^+} \frac{h p_{ij}^x}{h} := \mu_{ij}^x
\]
Assumptions

We make the following assumptions and definitions about transitions between states and their associated probabilities:

\[
\mathbb{P}[Y(x + t) = j \mid Y(x) = i] := t p_{x}^{ij} \quad \text{(Markovity)}
\]

\[
\mathbb{P}[Y(x + s) = i \text{ for all } s \in [0, t] \mid Y(x) = i] := t p_{x}^{ji}
\]

\[
\lim_{h \to 0} \frac{\mathbb{P}[2 \text{ or more transitions in interval of length } h]}{h} = 0
\]

\[
\lim_{h \to 0^+} \frac{h p_{x}^{ij}}{h} := \mu_{x}^{ij}
\]

\[
\frac{d}{dt} (t p_{x}^{ij}) \text{ exists for all } t \geq 0
\]
Note:

\[ t p_{x}^{00} = t p_{x} \]
\[ t p_{x}^{01} = t q_{x} \]
\[ t p_{x}^{10} = 0 \]
\[ 0 p_{x}^{ij} = 1 \{ i=j \} \]
\[ \mu_{x}^{01} = \mu_{x} \]
Note:

\[ t p_x^{00} = t p_x \]
\[ t p_x^{01} = t q_x \]
\[ t p_x^{10} = 0 \]
\[ 0 p_x^{ij} = 1 \{i=j\} \]
\[ \mu_x^{01} = \mu_x \]
\[ h p_x^{ij} = h \cdot \mu_x^{ij} + o(h) \]
\[ t p_x^{ii} \leq t p_x^{ij} \]

(216)
Note:

\[
\begin{align*}
    t p_{x}^{00} &= t p_{x} \\
    t p_{x}^{01} &= t q_{x} \\
    t p_{x}^{10} &= 0 \\
    0 p_{x}^{ij} &= 1_{\{i=j\}} \\
    \mu_{x}^{01} &= \mu_{x} \\
    h p_{x}^{ij} &= h \cdot \mu_{x}^{ij} + o(h) \\
    t p_{x}^{ii} &\leq t p_{x}^{ii}
\end{align*}
\]  

(216)

As an example, we can show that for the permanent disability model

\[
    u p_{x}^{01} = \int_{0}^{u} t p_{x}^{00} \cdot \mu_{x+t}^{01} \cdot u - t p_{x+t}^{11} dt.
\]  

(217)
Theorem

\[
\begin{align*}
    h p_x^{ii} &= 1 - h \cdot \sum_{j=0, j \neq i}^{n} \mu_{x}^{ij} + o(h) \\
    t p_x^{ii} &= \exp \left( - \int_{0}^{t} \sum_{j=0, j \neq i}^{n} \mu_{x+s}^{ij} ds \right)
\end{align*}
\]
Proof.

\[ P[(x, i) \rightarrow (x + h, i)] = 1 - P[(x, i) \not\rightarrow (x + h, i)] \]

\[ = 1 - h \cdot \sum_{j=0, j\neq i}^{n} \mu_{x}^{ij} + o(h). \]

\[ \therefore t + h p_{x}^{ii} = t p_{x}^{ii} \cdot h p_{x+t}^{ii} = t p_{x}^{ii} \cdot \left(1 - h \cdot \sum_{j=0, j\neq i}^{n} \mu_{x+t}^{ij} + o(h)\right) \]

\[ \Rightarrow \frac{d}{dt} \left( t p_{x}^{ii} \right) = \lim_{h \to 0} \frac{t + h p_{x}^{ii} - t p_{x}^{ii}}{h} = -t p_{x}^{ii} \cdot \sum_{j=0, j\neq i}^{n} \mu_{x+t}^{ij} \]

\[ \Rightarrow t p_{x}^{ii} = 0 p_{x}^{ii} \cdot e^{-\int_{0}^{t} \sum_{i \neq j} \mu_{x+s}^{ij} ds} = e^{-\int_{0}^{t} \sum_{i \neq j} \mu_{x+s}^{ij} ds}. \]

(219)
Using the vocabulary of probabilists, we define the Kolmogorov forward equations for the evolution of the densities of the birth death Markov process $Y$ as

\[
\frac{d}{dt} tp_{x}^{ij} = \sum_{k=0,k\neq j}^{n} tp_{x}^{ik} \mu_{x+t}^{kj} \mu_{x+t}^{jk} - tp_{x}^{ij} \sum_{k=0,k\neq j}^{n} \mu_{x+t}^{jk}
\]  

(220)
In matrix notation, for a fixed $x$, define the matrices $P(t)$, $Q(t)$ such that

$$[P(t)]_{i,j} = t p_{x}^{ij}$$

and the corresponding ODE system is

$$P'(t) = P(t) Q(t)$$

$P(0) = I =$ Identity Matrix.
In matrix notation, for a fixed \( x \), define the matrices \( P(t) \), \( Q(t) \) such that

\[
P(t)_{i,j} = t p_{x}^{ij}
\]

\[
Q(t) = \begin{pmatrix}
- \sum_{k=1}^{n} \mu_{x+t}^{0k} & \mu_{x+t}^{01} & \cdots & \mu_{x+t}^{0n} \\
\mu_{x+t} & - \sum_{k=0, k \neq 1}^{n} \mu_{x+t}^{1k} & \cdots & \mu_{x+t}^{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{x+t}^{n0} & \mu_{x+t}^{n1} & \cdots & - \sum_{k=0}^{n-1} \mu_{x+t}^{nk}
\end{pmatrix}
\]

and the corresponding ODE system is

\[
P'(t) = P(t)Q(t)
\]

\[
P(0) = I = \text{Identity Matrix.}
\]
Here, $Q$ is referred to as the **transition intensity matrix**. We can work with off diagonal entries as the diagonal entries are dependent on them. Also, a whole row of the matrix is filled by zeros if there is no transition out of the state corresponding to the row.
Consider the case where $Q$ is time-independent. Also, consider the diagonalization of $Q$ via

$$Q = U D U^{-1}$$

where $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$

are the eigenvalues of $Q$ and $U$ is the matrix composed of the corresponding eigenvectors.
Kolmogorov Forward Equations

Then \( P(t) = e^{tQ}P(0) \), where \( e^{tQ} = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k \).

If \( Q \) is diagonalizable, then

\[
e^{tQ} = U e^{tD} U^{-1}
\]

\[
e^{tD} = \\
\begin{pmatrix}
e^{t\lambda_1} & 0 & \cdots & 0 \\
0 & e^{t\lambda_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{t\lambda_n}
\end{pmatrix}
\] (224)

**Question:** What if \( Q \) is in fact time dependent?
For the regular alive-dead model with constant force of mortality $\mu$, the rate matrix is

$$Q = \begin{pmatrix} -\mu & \mu \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\mu & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

(225)

It follows that we retain

$$P(t) = e^{tQ} = \begin{pmatrix} e^{-\mu t} & 1 - e^{-\mu t} \\ 0 & 1 \end{pmatrix}$$

(226)

as in the previous sections.
Consider a fun toy problem that remains in the $2 \times 2$ matrix setting:

A zombie population has the rate matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad (227)$$

It follows that

$$P(t) = e^{tQ} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad (228)$$
Consider a fun toy problem that remains in the $2 \times 2$ matrix setting:

A zombie population has the rate matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

It follows that

$$P(t) = e^{tQ} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

**Question** As $t \to \infty$, can you say anything about the percentage of humans in the total population?
Zombies - the general case..

For a general zombie population constant rate matrix

\[ Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix} = \begin{pmatrix} 1 & -\frac{a}{b} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -(a + b) \end{pmatrix} \begin{pmatrix} -\frac{b}{a+b} & \frac{a}{b} \\ \frac{b}{a+b} & \frac{b}{a+b} \end{pmatrix} \] (229)
Zombies - the general case.

For a general zombie population constant rate matrix

\[
Q = \begin{pmatrix}
-a & a \\
-\frac{b}{a+b} & \frac{b}{a+b}
\end{pmatrix} = \begin{pmatrix}
1 & -\frac{a}{b} \\
1 & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
-(a+b) & -(a+b)
\end{pmatrix} \begin{pmatrix}
\frac{b}{a+b} & \frac{b}{a+b} \\
\frac{b}{a+b} & \frac{b}{a+b}
\end{pmatrix}
\] (229)

It follows that

\[
P(t) = \begin{pmatrix}
1 & -\frac{a}{b} \\
1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & e^{-(a+b)t}
\end{pmatrix} \begin{pmatrix}
\frac{b}{a+b} & \frac{b}{a+b} \\
\frac{b}{a+b} & \frac{b}{a+b}
\end{pmatrix}
\]
Zombies - the general case..

For a general zombie population constant rate matrix

\[
Q = \begin{pmatrix}
-a & a \\
b & -b \\
\end{pmatrix} = \begin{pmatrix}
1 & -\frac{a}{b} \\
1 & 1 \\
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & -(a + b) \\
\end{pmatrix} \begin{pmatrix}
\frac{b}{a+b} & \frac{a}{b} \\
\frac{b}{a+b} & \frac{a}{b+a} \\
\end{pmatrix}
\] (229)

It follows that

\[
P(t) = \begin{pmatrix}
1 & -\frac{a}{b} \\
1 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & e^{-(a+b)t} \\
\end{pmatrix} \begin{pmatrix}
\frac{b}{a+b} & \frac{a}{b} \\
\frac{b}{a+b} & \frac{a}{b+a} \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{b}{a+b} + \frac{a}{a+b}e^{-(a+b)t} & \frac{a}{a+b} - \frac{a}{a+b}e^{-(a+b)t} \\
\frac{b}{a+b} - \frac{a}{a+b}e^{-(a+b)t} & \frac{a}{a+b} + \frac{a}{a+b}e^{-(a+b)t} \\
\end{pmatrix}
\]
Zombies - the general case.

For a general zombie population constant rate matrix

\[
Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 1 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -(a + b) \end{pmatrix} \begin{pmatrix} \frac{b}{a+b} & \frac{a}{b(a+b)} \\ b & a \end{pmatrix} \quad (229)
\]

It follows that

\[
P(t) = \begin{pmatrix} 1 & -a \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-(a+b)t} \end{pmatrix} \begin{pmatrix} \frac{b}{a+b} & \frac{a}{b(a+b)} \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{b+ae^{-(a+b)t}}{a+b} & \frac{a-ae^{-(a+b)t}}{a+b} \\ \frac{b}{a+b} & \frac{a+be^{-(a+b)t}}{a+b} \end{pmatrix} \quad (230)
\]
Example 8.4

Suppose you are given the **transition intensity matrix** for the permanent disability model as follows:

\[
\begin{pmatrix}
\mu_{00} & \mu_{01} & \mu_{02} \\
\mu_{10} & \mu_{11} & \mu_{12} \\
\mu_{20} & \mu_{21} & \mu_{22}
\end{pmatrix} =
\begin{pmatrix}
-0.0508 & 0.0279 & 0.0229 \\
0.0000 & -0.0229 & 0.0229 \\
0.0000 & 0.0000 & 0.0000
\end{pmatrix}
\]

(231)

Then

\[
10 p_{60}^{00} = 10 p_{60}^{00} = e^{-\int_0^{10} (0.0279 + 0.0229) ds} = 0.60170
\]

\[
10 p_{60}^{01} = \int_0^{10} t p_{60}^{00} \mu_{60+t}^{01} 10 - t p_{60+t}^{11} dt
\]

\[
= \int_0^{10} e^{-\int_0^t (0.0279 + 0.0229) ds} \cdot 0.0279 \cdot e^{-\int_0^{10-t} (0.0229) ds} dt
\]

\[
= 0.19363
\]

(232)
A party of scientists arrives at a remote island. Unknown to them, a hungry tyrannosaur lives on the island. You model the future lifetimes of the scientists as a three-state model, where:

- State 0: no scientists have been eaten.
- State 1: exactly one scientist has been eaten.
- State 2: at least two scientists have been eaten.

You are given:

- (i) Until a scientist is eaten, they suspect nothing, so $\mu_{01}^t = 0.01 + 0.02 \cdot 2^t$.
- (ii) Until a scientist is eaten, they suspect nothing, so the tyrannosaur may come across two together and eat both, with $\mu_{02}^t = 0.5 \cdot \mu_{01}^t$.
- (iii) After the first death, scientists become much more careful, so $\mu_{12}^t = 0.01$.

Calculate the probability that no scientists are eaten in the first year.
This is essentially a Permanent Disability model, and so we can compute the transition probabilities accordingly:

\[ t p_{00}^0 = \exp \left( - \int_0^t \sum_{j=1}^{2} \mu_{s}^{0j} ds \right) \]

\[ = \exp \left( - \int_0^t 1.5(0.01 + 0.02 \cdot 2^t) ds \right) \]

\[ \Rightarrow \ 1 p_{00}^0 = 0.943. \]

**HW** Compute the other transition probabilities using both the integral equation *and* the matrix exponential method.
Example 8.5 (Forward Euler Method)

Suppose you are given the transition intensity matrix for the disability income insurance model as follows:

\[
\begin{pmatrix}
\mu_{00} & \mu_{01} & \mu_{02} \\
\mu_{10} & \mu_{11} & \mu_{12} \\
\mu_{20} & \mu_{21} & \mu_{22}
\end{pmatrix} = \begin{bmatrix}
-\mu_{x} & 0 \cdot 0 \cdot 0 & a_1 + b_1 e^{c_1 x} \\
0.1 \cdot (\mu_{x}) & -\mu_{x} & a_2 + b_2 e^{c_2 x} \\
0 & 0 & 0
\end{bmatrix}
\] (234)

for parameters

\[
\begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2
\end{pmatrix} = \begin{bmatrix}
4 \times 10^{-4} & 3.4674 \times 10^{-6} & 0.138155 \\
5 \times 10^{-4} & 7.5858 \times 10^{-6} & 0.087498
\end{bmatrix}
\] (235)
Then Forward Euler applied the Kolmogorov equations leads to

\[
\begin{align*}
t + h p_{60}^{00} &= t p_{60}^{00} - h \cdot t p_{60}^{00} \cdot (\mu_{60+t}^{01} + \mu_{60+t}^{02}) \\
&\quad + h \cdot t p_{60}^{01} \cdot \mu_{60+t}^{10} + o(h)\\
\end{align*}
\]

\[
\begin{align*}
t + h p_{60}^{01} &= t p_{60}^{01} - h \cdot t p_{60}^{01} \cdot (\mu_{60+t}^{12} + \mu_{60+t}^{10}) \\
&\quad + h \cdot t p_{60}^{00} \cdot \mu_{60+t}^{01} + o(h)
\end{align*}
\]

Ignoring the \( o(h) \) terms, we can iterate forward using, for example, \( h = \frac{1}{12} \).
In matrix-vector notation, we have

\[
\begin{pmatrix}
    t + hp_{00}^0 \\
    t + hp_{01}^0 \\
    t + hp_{10}^0 \\
    t + h p_{01}^0 \\
\end{pmatrix}
= \left[ I - hA(t) \right]
\begin{pmatrix}
    t p_{00}^0 \\
    t p_{01}^0 \\
    t p_{10}^0 \\
    t p_{01}^0 \\
\end{pmatrix}
\]
In matrix-vector notation, we have

\[
\begin{pmatrix}
(t+h)p_{60}^{00} \\
(t+h)p_{60}^{01}
\end{pmatrix} = \left[ I - hA(t) \right] \begin{pmatrix}
(t)p_{60}^{00} \\
(t)p_{60}^{01}
\end{pmatrix}
\]

\[
I = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
A(t) = \begin{pmatrix}
\mu_{60+t}^{01} + \mu_{60+t}^{02} & -\mu_{60+t}^{10} \\
-\mu_{60+t}^{01} & \mu_{60+t}^{12} + \mu_{60+t}^{10}
\end{pmatrix}
\]

(237)
Example 8.5

In matrix-vector notation, we have

\[
\begin{pmatrix}
t + h p_{60}^{00} \\
t + h p_{60}^{01}
\end{pmatrix} = \left[ I - h A(t) \right] \begin{pmatrix} t p_{60}^{00} \\ t p_{60}^{01} \end{pmatrix}
\]

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(237)

\[
A(t) = \begin{pmatrix} 
\mu_6^{01} + \mu_6^{02} + t & -\mu_6^{10} + t \\
-\mu_6^{01} + t & \mu_6^{12} + \mu_6^{10} + t
\end{pmatrix}
\]

Keep in mind that \( A \) is determined by the given transition intensity matrix.

**HW** Compute this vector over the interval \( t \in [0, 10] \) using a time step of \( h = \frac{1}{12} \). Use any numerical solver you like, but please have the values computed into a pair of columns.
Upon finding the right diagonalization, one can show that for a PDM with
the rate matrix

$$Q = \begin{pmatrix}
-(a + b) & a & b \\
0 & -c & c \\
0 & 0 & 0
\end{pmatrix}$$

(238)

where $a, b, c > 0$, it follows that for $a + b \neq c$, $P(t) =$

$$\begin{pmatrix}
    e^{-(a+b)t} & \frac{a}{a+b-c}(e^{-ct} - e^{-(a+b)t}) & 1 - \frac{a}{a+b-c}e^{-ct} + \frac{c-b}{a+b-c}e^{-(a+b)t} \\
    0 & e^{-ct} & 1 - e^{-ct} \\
    0 & 0 & 1
\end{pmatrix}$$

(239)
Consider a Modified Disability Model where observed transition intensities are \((\mu^0_{t}, \mu^{10}_{t}, \mu^{12}_{t}) = (0.02, 0.06, 0.10)\).

Using the Kolmogorov forward equations with step \(h = 0.5\), calculate the probability that a person currently disabled will be healthy at the end of one year.
Recall that the Forward Euler method can be written explicitly as

\[ f(t + h) \approx f(t) + h \cdot f'(t) \]  

(240)

and so for the Kolmogorov Forward ODE system,

\[ t + h p_x^{ij} = t p_x^{ij} + h \cdot \left( \sum_{k=0, k \neq j}^{n} t p_x^{ik} \mu_{x+t}^{kj} - t p_x^{ij} \sum_{k=0, k \neq j}^{n} \mu_{x+t}^{jk} \right). \]  

(241)
Again, wlog set $x = 0$ and fix $i, j$. For

$$f(t) := t^{ij}p_0$$

$$f(h) = f(0) + h \cdot f'(0)$$

we have the system of equations

$$hp_0^{10} = p_0^{10} + h \cdot \left( (p_0^{11} \mu_0^{10} + p_0^{12} \mu_0^{20}) - p_0^{10} \cdot (\mu_0^{01} + \mu_0^{02}) \right)$$

$$hp_0^{11} = p_0^{11} + h \cdot \left( (p_0^{10} \mu_0^{11} + p_0^{12} \mu_0^{21}) - p_0^{11} \cdot (\mu_0^{10} + \mu_0^{12}) \right)$$

$$hp_0^{12} = p_0^{12} + h \cdot \left( (p_0^{10} \mu_0^{02} + p_0^{11} \mu_0^{12}) - p_0^{12} \cdot (\mu_0^{20} + \mu_0^{21}) \right).$$
Using the above, and $h = 0.5$, we obtain for the first iterates:

\[ h p_{0}^{10} = h \cdot \mu_{0}^{10} = 0.03 \]
\[ h p_{0}^{12} = h \cdot \mu_{0}^{12} = 0.05 \]
\[ h p_{0}^{11} = 1 - h \cdot (\mu_{0}^{10} + \mu_{0}^{12}) = 0.92. \]  

(244)

Recursively, $f(2h) = f(h) + h \cdot f'(h)$ and so for $f(2h) = 2h p_{0}^{ij}$ we obtain

\[ 2h p_{0}^{10} = h p_{0}^{10} + h \cdot \left( (h p_{0}^{11} \mu_{0}^{10} + h p_{0}^{12} \mu_{0}^{20}) - h p_{0}^{10} \cdot (\mu_{0}^{01} + \mu_{0}^{02}) \right) \]
\[ = 0.03 + 0.5 \cdot \left( (0.92)(0.06) + 0 - (0.03)(0.02 + 0) \right) \]
\[ = 0.0573. \]  

(245)
Note that

\[
Q = \begin{pmatrix}
-0.02 & 0.02 & 0 \\
0.06 & -0.16 & 0.10 \\
0 & 0 & 0
\end{pmatrix}
= UDU^{-1}
\]
Note that

\[
Q = \begin{pmatrix} -0.02 & 0.02 & 0 \\ 0.06 & -0.16 & 0.10 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[= UDU^{-1} \]

where \( U = \begin{pmatrix} -0.133827 & 0.92683 & 0.57735 \\ 0.991005 & 0.375482 & 0.57735 \\ 0 & 0 & 0.57735 \end{pmatrix} \)

\( D = \begin{pmatrix} -0.168102 & 0 & 0 \\ 0 & -0.0118975 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

\( U^{-1} = \begin{pmatrix} -0.387597 & 0.956735 & -0.569138 \\ 1.02298 & 0.138145 & -1.16113 \\ 0 & 0 & 1.73205 \end{pmatrix} \)
It follows that

\[ P(t) = U e^{tD} U^{-1} \]
It follows that

\[ P(t) = U e^{tD} U^{-1} \]

where

\[
U = \begin{pmatrix}
-0.133827 & 0.92683 & 0.57735 \\
0.991005 & 0.375482 & 0.57735 \\
0 & 0 & 0.57735 \\
\end{pmatrix}
\]

\[
e^{tD} = \begin{pmatrix}
e^{-0.168102t} & 0 & 0 \\
0 & e^{-0.0118975t} & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
U^{-1} = \begin{pmatrix}
-0.387597 & 0.956735 & -0.569138 \\
1.02298 & 0.138145 & -1.16113 \\
0 & 0 & 1.73205 \\
\end{pmatrix}
\]

(247)

Compare these with your previously obtained Forward-Euler results.
Consider an annuity issued to a life (x) that pays at rate 1 per year continuously while the life is in state $j$. Then the EPV of this annuity at force of interest $\delta$ per year is

$$\bar{a}_{ij}^x = E \left[ \int_0^\infty e^{-\delta t} 1\{Y(t) = j \mid Y(0) = i\} \, dt \right]$$

$$= \int_0^\infty e^{-\delta t} E \left[ 1\{Y(t) = j \mid Y(0) = i\} \right] \, dt = \int_0^\infty e^{-\delta t} t p_{ij}^x \, dt$$

(248)

If the annuity is payable at the start of each year from the current time, based on the conditional event $\{Y(t) = j \mid Y(0) = i\}$, then the EPV is

$$\ddot{a}_{ij}^x = \sum_{k=0}^\infty v^k k p_{ij}^x$$

(249)
If a unit benefit is payable to a life \((x)\) on transition to state \(k\), given that it is currently in state \(i\), then the EPV of this benefit is

\[
\bar{A}_x^{ik} = \int_0^\infty \sum_{j \neq k} e^{-\delta t} t p_{ij}^{jk} \mu_{x+t} dt
\]  

(250)
An insurer issues a 10–year disability income insurance policy to a healthy life aged 60. Use the model and parameters from i.) Example 8.5 and ii.) Example 8.4. Assume an effective rate of 5% per year and no expenses. Calculate the premiums for the following designs

- (a) Premiums are payable continuously while in the healthy state. A benefit of 20000 per year is payable continuously while in the disabled state. A death benefit of 50000 is payable immediately upon death.
- (b) Premiums are payable monthly in advance conditional on the life being in the healthy state at the premium date. The sickness benefit of 20000 per year is payable monthly in arrear, if the life is in the sick state at the payment date. A death benefit of 50000 is payable immediately upon death.
Example 8.6

For case a.), we have via the EPP principle that

\[ 20000 \bar{a}_{60:10}^{01} + 50000 \bar{A}_{60:10}^{02} - P \bar{a}_{60:10}^{00} = 0 \quad (251) \]

and so

\[
P = \frac{20000 \int_0^{10} e^{-\delta t} t \bar{p}_{60}^{01} \, dt}{\int_0^{10} e^{-\delta t} t \bar{p}_{60}^{00} \, dt} + \frac{50000 \int_0^{10} e^{-\delta t} (t \bar{p}_{60}^{00} \mu_{60+t}^{02} + t \bar{p}_{60}^{01} \mu_{60+t}^{12}) \, dt}{\int_0^{10} e^{-\delta t} t \bar{p}_{60}^{00} \, dt}. \quad (252)\]

For a time step of \( h = \frac{1}{12} \) (monthly), we can use the forward-Euler results from the previous example to calculate \( P = 3254.65 \).
Example 8.6

For case \( b. \), we have via the EPP principle that

\[
20000\bar{a}_{60:10}^{(12)01} + 50000\bar{A}_{60:10}^{02} - P\bar{a}_{60:10}^{(12)00} = 0
\]  

(253)

and so

\[
P = \frac{20000 \sum_{k=0}^{119} \nu_{12}^k p_{60}^{01}}{\sum_{k=0}^{119} \nu_{12}^k p_{60}^{00}} \]

\[
+ \frac{50000 \int_{0}^{10} e^{-\delta t} \left( t p_{60}^{00} \mu_{60}^{02} + t p_{60}^{01} \mu_{60}^{12} \right) dt}{\sum_{k=0}^{119} \nu_{12}^k p_{60}^{00}} .
\]

(254)

Again, we can use the previously computed values of \( \{ \nu_{12}^k p_{60}^{0j} \}_{(j,k)=(0,0)}^{(1,119)} \) to calculate \( P = 3257.20 \).
Consider the special case for transition matrix

\[
Q(t) = \begin{pmatrix}
- \sum_{k=1}^{n} \mu_{0k}^0 & \mu_{01}^1 & \cdots & \mu_{0n}^0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

(255)

Here, there are multiple exits from state 0, but no further transitions.

Figure: Multiple Decrement Flow Chart
Multiple Decrement Models

\[ t p_{x}^{00} = t \overline{p}_{x}^{00} = \exp \left[ - \int_{0}^{t} \sum_{i=1}^{n} \mu_{x+s}^{0i} ds \right] \]

\[ t p_{x}^{0i} = \int_{0}^{t} s p_{x}^{00} \mu_{x+s}^{0i} ds \]

\[ 0 p_{x}^{ij} = 1 \{ i = j \} \]

**Note:**

- Premium is now different when compared to a policy that only allows transition 0 \(\rightarrow\) 1.
- This is because \(\mu_{x+t}^{00} = - \sum_{k=1}^{n} \mu_{x+t}^{0k} < -\mu_{x+t}^{01}\) and so \(t p_{x}^{00}\) changes accordingly.
- See Example 8.8, where for example an insurer may allow for lower premiums via lapse support.
Double Decrement Models

Consider the extra special case for transition matrix

\[
Q(t) = \begin{pmatrix}
-(a + b) & a & b \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  

(257)

Here, there are *two possible* exits from state 0, but no further transitions.
Double Decrement Models

Consider the extra special case for transition matrix

\[
Q(t) = \begin{pmatrix} -(a + b) & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{257}
\]

Here, there are two possible exits from state 0, but no further transitions.

\[
P(t) = \begin{pmatrix} e^{-(a+b)t} & \frac{a}{a+b}(1 - e^{-(a+b)t}) & \frac{b}{a+b}(1 - e^{-(a+b)t}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{258}
\]
Double Decrement Models

Consider the extra special case for transition matrix

\[
Q(t) = \begin{pmatrix}
-(a + b) & a & b \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  

(257)

Here, there are two possible exits from state 0, but no further transitions.

\[
P(t) = \begin{pmatrix}
e^{-(a+b)t} & \frac{a}{a+b}(1 - e^{-(a+b)t}) & \frac{b}{a+b}(1 - e^{-(a+b)t}) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(258)

HW For this double decrement model, compute \( \ddot{a}_x^{01}, \ddot{a}_x^{02} \) as well as \( \ddot{A}_x^{01} \) and \( \ddot{A}_x^{02} \). What can you say about these financial instruments as \( b \rightarrow 0 \)?
Policy Values

Define

- $\mu_{ij}^y$ as the transition intensity between states $i$ and $j$ at age $y$
- $\delta_t$ as the force of interest per year at time $t$
- $B_t^{(i)}$ as the benefit payment rate while the policyholder is in state $i$
- $S_t^{(ij)}$ as the lump sum payment instantaneously at time $t$ on transition from state $i$ to state $j$.

Assume the above are all members of $C^0[0, n]$. Then $\forall i \in \{0, 1, \cdots, n\}$, Thiele’s Differential Equation is

$$
\frac{d}{dt} t V^i = \delta_t t V^i - B_t^{(i)} - \sum_{j \neq i} \mu_{x+t}^{ij} \left( S_t^{(ij)} + t V^j - t V^i \right)
$$

(259)
Figure: Joint Model Transition Rates

Define the joint transition matrix via the flow chart above.
Joint Life - Last Survivor Benefits

Define

\[ t p_{xy} = t p_{xy}^{00} = \mathbb{P}[(x) \text{ and } (y) \text{ are both alive in } t \text{ years}] \]

\[ t q_{xy} = t p_{xy}^{01} + t p_{xy}^{02} + t p_{xy}^{03} = \mathbb{P}[(x) \text{ and } (y) \text{ are not both alive in } t \text{ years}] \]

\[ t p_{\overline{xy}} = t p_{xy}^{00} + t p_{xy}^{01} + t p_{xy}^{02} = \mathbb{P}[ \text{ at least one of } (x) \text{ and } (y) \text{ is alive in } t \text{ years}] \]

\[ t q_{\overline{xy}} = 1 - t p_{\overline{xy}} \]

\[ \mu_{x+t:y+t} = \mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} \]
Also define

\[ t q_{xy}^1 = \mathbb{P}[(x) \text{ dies before } (y) \text{ and within } t \text{ years}] \]

\[ = \int_0^t r p_{xy}^{00} \mu_{x+r}^{02} + r dr \]

\[ \neq t p_{xy}^{02} \]

\[ t q_{xy}^2 = \mathbb{P}[(x) \text{ dies after } (y) \text{ and within } t \text{ years}] \]

\[ = \int_0^t r p_{xy}^{01} \mu_{x+r}^{13} dr \]
For two lives, (80) and (90), with independent future lifetimes, you are given

\[ p_{80+k} = 0.9 - 0.1k \text{ for } k \in \{0, 1, 2\} \]
\[ p_{90+k} = 0.6 - 0.1k \text{ for } k \in \{0, 1, 2\} \]  \hspace{1cm} (262)

Calculate the probability that the last survivor will die in the third year.
For two lives, (80) and (90), with independent future lifetimes, you are given

\[
p_{80+k} = 0.9 - 0.1k \text{ for } k \in \{0, 1, 2\}
\]

\[
p_{90+k} = 0.6 - 0.1k \text{ for } k \in \{0, 1, 2\}
\]

(262)

Calculate the probability that the last survivor will die in the third year. By definition,

\[
tp_{xy} = tp^{00}_{xy} + tp^{01}_{xy} + tp^{02}_{xy}
\]

\[
= \mathbb{P}[ \text{ at least one of } (x) \text{ and } (y) \text{ is alive in } t \text{ years}]
\]

(263)

is the probability we seek to compute.
Using the assumption of independence of lives,

\[ 2p_{80:90} - 3p_{80:90} = \mathbb{P}[ \text{at least one of (80) and (90) is alive in 2 years}] - \mathbb{P}[ \text{at least one of (80) and (90) is alive in 3 years}] = \mathbb{P}[ \text{last survivor dies in the 3rd year}] = \left(2p_{80} + 2p_{90} - 2p_{80:90}\right) - \left(3p_{80} + 3p_{90} - 3p_{80:90}\right) = \left(p_{80}p_{81} + p_{90}p_{91} - p_{80}p_{81}p_{90}p_{91}\right) - \left(p_{80}p_{81}p_{82} + p_{90}p_{91}p_{92} - p_{80}p_{81}p_{82}p_{90}p_{91}p_{92}\right) = 0.24048. \]
Joint Life - Last Survivor Benefits

Insurance Notation

- \( \bar{a}_{xy} = \bar{a}_{xy}^{00} \), the \textbf{Joint Life Annuity} with continuous payment of 1 per year while both husband and wife are alive.

- \( \bar{A}_{xy} \), the \textbf{Joint Life Insurance} with a unit payment immediately upon first death.

- \( \bar{A}^1_{xy} \), the \textbf{Contingent Insurance}, a unit payment immediately upon death of the husband given that he dies before his wife.
Joint Life - Last Survivor Benefits

Insurance Notation

- $\bar{A}_{xy} = \bar{A}^{03}_{xy}$, the Last Survivor Insurance with unit payment immediately upon second death.
- $\bar{a}_{x\mid y} = \bar{a}^{02}_{xy}$ the Reversionary Annuity with a continuous payment at unit rate per year while wife is alive given that husband has died.
- $\bar{a}_{xy} = \bar{a}^{00}_{xy} + \bar{a}^{01}_{xy} + \bar{a}^{02}_{xy}$, the Last Survivor Annuity, a continuous payment at rate 1 per year while at least one person is alive.
Joint Life - Last Survivor Benefits

It can be shown that

\[
\bar{a}_{xy} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}
\]
\[
\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}
\]
\[
\bar{A}_{xy} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}
\]
\[
\bar{a}_{xy} = \frac{1 - \bar{A}_{xy}}{\delta}
\]

(265)

HW

• Prove this using explicit integral formulations.
• Read over Examples 8.10 and 8.11.
For a fully continuous whole life insurance issued on \((x)\) and \((y)\), you are given \(\forall t \geq 0:\)

- The death benefit of 100 is payable at the second death.
- Premiums are payable until the first death.
- The future lifetimes of \((x)\) and \((y)\) are dependent.
- \(t\rho_{xy} = \lambda e^{-at} + (1 - \lambda)e^{-bt}\) for some \(\lambda \in [0, 1]\).
- \(t\rho_x = e^{-at}\)
- \(t\rho_y = e^{-ct}\) for some \(c < a < b\).
- The force of interest is constant at \(\delta > 0\).

Calculate the annual benefit premium rate \(P\) for this insurance.
The equation of value here, assuming the EPP, is

\[ 0 = 100\bar{A}_{xy} - P\bar{a}_{xy} \]
The equation of value here, assuming the EPP, is

\[ 0 = 100 \bar{A}_{xy} - P \bar{a}_{xy} \]

\[ \Rightarrow P = 100 \frac{\bar{A}_{xy}}{\bar{a}_{xy}} \]

\[ = 100 \frac{\bar{A}_x + \bar{A}_y - \bar{A}_{xy}}{\bar{a}_{xy}} \]

\[ = 100 \frac{\bar{A}_x + \bar{A}_y - (1 - \delta \bar{a}_{xy})}{\bar{a}_{xy}} \]

(266)
However,

\[ \overline{a}_{xy} = \int_{0}^{\infty} e^{-\delta t} \left( \lambda e^{-at} + (1 - \lambda) e^{-bt} \right) dt \]

\[ = \frac{\lambda}{a + \delta} + \frac{1 - \lambda}{b + \delta} \]

\[ \overline{A}_x = \int_{0}^{\infty} e^{-\delta t} ae^{-at} dt \]

\[ = \frac{a}{a + \delta} \]

\[ \overline{A}_y = \int_{0}^{\infty} e^{-\delta t} ce^{-ct} dt \]

\[ = \frac{c}{c + \delta} \]

\[ \Rightarrow P = 100 \frac{\frac{a}{a+\delta} + \frac{c}{c+\delta} - 1 + \delta \left( \frac{\lambda}{a+\delta} + \frac{1-\lambda}{b+\delta} \right)} {\frac{\lambda}{a+\delta} + \frac{1-\lambda}{b+\delta}} \]

(267)
Example: Joint Life Benefits

For a special whole life insurance policy on \((x)\) and \((y)\) with dependent future lifetimes, you are given:

- A death benefit of 105,000 is paid at the end of the year of death if both \((x)\) and \((y)\) die within the same year. No death benefits are payable otherwise.
- \(p_{x+k} = 0.85\) for all \(k \in \{0, 1, 2, ..\}\)
- \(p_{y+k} = 0.8\) for all \(k \in \{0, 1, 2, ..\}\)
- \(p_{x+k:y+k} = 0.75\) for all \(k \in \{0, 1, 2, ..\}\)
- The yearly interest rate used is \(r = 0.05\).

Calculate the expected present value of the death benefit.
Example: Joint Life Benefits

First, notice that

\[ 1p_{x+k:y+k} = p_{x+k} + p_{y+k} - p_{x+k:y+k} = 0.9 \]
\[ 1q_{x+k:y+k} = 1 - 1p_{x+k:y+k} = 0.1 \]

\[ kp_{xy} = \prod_{j=0}^{k-1} p_{x+j:y+j} = 0.75^k. \]

It follows that the

\[ EPV = 105000 \sum_{k=0}^{\infty} \frac{1}{(1 + r)^{k+1} k p_{xy} q_{x+k:y+k}} \]
\[ = 105000 \sum_{k=0}^{\infty} \frac{1}{(1.05)^{k+1} (0.75^k)(0.1)} \]
\[ = \frac{10000}{1 - \frac{5}{7}} = 35000. \]
Example 8.12: The employees (0) of a large corporation can leave the corporation in three ways: they can retire (1), they can withdraw from the corporation (2), or they can die while they are still employees (3). Consider the model

\[
\mu_x^{03} \equiv \mu_x^{13} \equiv \mu_x^{23} = \mu_x
\]

\[
\mu_x^{02} = \begin{cases} 
\mu_x^{02}, & \text{if } x < 60 \\
0, & \text{if } x \geq 60
\end{cases}
\]

(270)

where retirement can only take place only on an employee’s 60th, 61st, 62nd, 63rd, 64th, or 65th birthday. Assume that 40\% of employees reaching exact age 60, 61, 62, 63 or 64 will retire at that age and that 100\% of all employees who reach age 65 retire immediately.
Transitions at Specific Ages

The corporation offers the following benefits to the employees:

- For those employees who die while still employed, a lump sum of 200000 is payable immediately upon death.
- For those employees who retire, a lump sum of 150000 is payable immediately upon death after retirement.

Theorem

Assuming a constant force of interest of \( \delta \) per year and the notation of \( \bar{A}_x \) and \( nE_x \) from single life computations based on a force of mortality \( \mu_x \), it follows that the EPV of the Death after retirement benefit of an employee currently aged 40 is

\[
150000 \cdot 20E_{40}e^{-20\mu_{02}} \left( 0.4 \cdot \left[ \sum_{k=0}^{4} 0.6^k \bar{A}_{60} \right] + 0.6^5 \cdot 5\bar{A}_{60} \right) \quad (271)
\]

Albert Cohen (MSU)
To begin our proof, we can compute

\[
\mathbb{E}_{(40)}[\text{PV(Benefit) } | \text{ retire at age 60}] = 150000e^{-20\delta} \bar{A}_{60} \tag{272}
\]

\[
20 - p_{40}^{00} = \exp \left[ - \int_{0}^{20} (\mu_{02} + \mu_{40+t}) \, dt \right]
\]

\[
= \exp \left[ - \int_{0}^{20} (\mu_{40+t}) \, dt \right] e^{-20\mu_{02}}
\]

\[
= 20p_{40}e^{-20\mu_{02}} \tag{273}
\]

\[
\mathbb{P}_{(40)}[\text{retire at age 60}] = 0.4 \cdot 20 - p_{40}^{00} = 0.4 \cdot 20p_{40}e^{-20\mu_{02}}
\]

\[
20 + p_{40}^{00} = 0.6 \cdot 20 - p_{40}^{00}
\]

\[
21 - p_{40}^{00} = 20 + p_{40}^{00} \cdot p_{60}
\]

\[
21 + p_{40}^{00} = 0.6 \cdot 21 - p_{40}^{00} = 0.6^2 \cdot 21p_{40}e^{-20\mu_{02}}
\]
Also,

\[ \mathbb{P}(40)[\text{retire at age 61}] = X_1 X_2 X_3 X_4 \]
\[ X_1 = \mathbb{P}(40)[\text{survive in employment to age } 60^-] = 20 - p_{40}^{00} \]
\[ X_2 = \mathbb{P}(40)[\text{will not retire at age 60}] = 0.6 \]
\[ X_3 = \mathbb{P}(40)[(60^+) \text{ will survive to age } 61^-] = 1 p_{60} \]
\[ X_4 = \mathbb{P}(40)[\text{will retire at age 61}] = 0.4 \]
\[ \mathbb{E}(40)[\text{PV(Benefit)} | \text{retire at age 61}] = 150000 e^{-21 \delta} \bar{A}_{61} \]
We can repeat this until

\[
P_{(40)}[\text{retire at age 65}] = 20 - p_{40}^{00} \cdot 0.6^5 \cdot 1p_{60} \cdot 1p_{61} \cdot 1p_{62} \cdot 1p_{63} \cdot 1p_{64}
\]

\[
= 25p_{40}e^{-20\mu_{02}} \cdot 0.6^5
\]

\[
E_{(40)}[PV(\text{Benefit}) | \text{retire at age 65}] = 150000e^{-25\delta A_{65}}
\]

We now have enough information to complete the proof.
Transitions at Specific Ages

Proof.

For benefit after retirement, we have

\[ \mathbb{E}(40)[\text{PV(Benefit)}] = \sum_{k=0}^{5} \mathbb{B}^k_{(40)} \mathbb{P}^k_{(40)} \]

\[ \mathbb{B}^k_{(40)} = \mathbb{E}(40)[\text{PV(Benefit)} \mid \text{retire at age } 60 + k] \]

\[ = 150000e^{-(20+k)\delta} \bar{A}_{60+k} \text{ for } k \in \{0, \ldots , 5\} \quad (276) \]

\[ \mathbb{P}^k_{(40)} := \mathbb{P}(40)[\text{retire at age } 60 + k] \]

\[ = 20+k p_{40} e^{-20\mu^0} \cdot 0.6^k \cdot 0.4 \text{ for } k \in \{0, \ldots , 4\} \]

\[ \mathbb{P}^5_{(40)} = 25 p_{40} e^{-20\mu^0} \cdot 0.6^5 \]

Substitution and arithmetic lead to the form in the theorem statement.
Transitions at Specific Ages

Also notice that via the Tower property for conditional expectations, we have a direct version of the proof:

\[
\mathbb{E}_{(40)}[PV(Benefit)] = \mathbb{E}_{(40)}\left[\mathbb{E}_{(60)}[PV(Benefit)]\right] \\
= 20 E_{40} e^{-20\mu^0} \cdot \sum_{k=0}^{5} B^k_{(60)} P^k_{(60)} 
\]

(277)

where

\[
B^k_{(60)} = \mathbb{E}_{(60)}[PV(Benefit) | \text{retire at age } 60 + k] \\
= 150000e^{-k\delta} \bar{A}_{60+k} \text{ for } k \in \{0, \cdots, 5\}
\]

\[
P^k_{(60)} := \mathbb{P}_{(60)}[\text{retire at age } 60 + k] \\
= kp_{60} \cdot 0.6^k \cdot 0.4 \text{ for } k \in \{0, \cdots, 4\}
\]

(278)

\[
P^5_{(60)} = 5p_{60} \cdot 0.6^5
\]
For example, consider a puppy at point 0 of a square lake. The state of the puppy is

\[
Y(k) = \begin{cases} 
0 & \text{if the puppy is at corner 0 time } k \\
1 & \text{if the puppy is at corner 1 time } k \\
2 & \text{if the puppy is at corner 2 time } k \\
3 & \text{if the puppy is at corner 3 time } k 
\end{cases}
\]

Figure: Joint Model Flow Chart
Now, consider the random variable

\[ T_{ij} = \text{the time it takes to get from } i \text{ to } j. \] (279)

and the expected values

\[ t_{ij} = \mathbb{E}[T_{ij}]. \] (280)

Consider a transition probability matrix \( P \) such that

\[
P = \begin{pmatrix}
p_{00} & p_{01} & p_{02} & p_{03} \\
p_{10} & p_{11} & p_{12} & p_{13} \\
p_{20} & p_{21} & p_{22} & p_{23} \\
p_{30} & p_{31} & p_{32} & p_{33}
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{pmatrix}. \] (281)

For this transition model, calculate \( t_{03} \).
Markov Transportation Models

For the above matrix, we can see that

\[
t_{03} = \frac{1}{2} \left( 1 + t_{13} \right) + \frac{1}{2} \left( 1 + t_{23} \right)
\]
\[
t_{13} = \frac{1}{2} \left( 1 + t_{03} \right) + \frac{1}{2} \left( 1 \right)
\]
\[
t_{23} = \frac{1}{2} \left( 1 + t_{03} \right) + \frac{1}{2} \left( 1 \right).
\]

It follows that

\[
t_{03} = \frac{1}{2} \left( 1 + \frac{1}{2} \left( 1 + t_{03} \right) + \frac{1}{2} \left( 1 \right) \right) + \frac{1}{2} \left( 1 + \frac{1}{2} \left( 1 + t_{03} \right) + \frac{1}{2} \left( 1 \right) \right)
\]
\[
= 2 + \frac{1}{2} t_{03}
\]
\[
\therefore t_{03} = 4.
\]
If the transition matrix is adapted to

\[
P = \begin{pmatrix}
p_{00} & p_{01} & p_{02} & p_{03} \\
p_{10} & p_{11} & p_{12} & p_{13} \\
p_{20} & p_{21} & p_{22} & p_{23} \\
p_{30} & p_{31} & p_{32} & p_{33}
\end{pmatrix} = \begin{pmatrix}
0 & p & 1 - p & 0 \\
1 - p & 0 & 0 & p \\
p & 0 & 0 & 1 - p \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(284)

then the expected time \( t_{03} \) is computed via

\[
t_{03} = p (1 + t_{13}) + (1 - p)(1 + t_{23})
\]

\[
t_{13} = (1 - p)(1 + t_{03}) + p (1)
\]

\[
t_{23} = p (1 + t_{03}) + (1 - p)(1).
\]

(285)
The matrix form of (285) is

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
1 & -p & -(1 - p) \\
-(1 - p) & 1 & 0 \\
-p & 0 & 1
\end{pmatrix}
\begin{pmatrix}
t_{03} \\
t_{13} \\
t_{23}
\end{pmatrix}
\]

(286)

and inversion via WolframAlpha leads to

\[
\begin{pmatrix}
t_{03} \\
t_{13} \\
t_{23}
\end{pmatrix}
= 
\begin{pmatrix}
1 & -p & -(1 - p) \\
-(1 - p) & 1 & 0 \\
-p & 0 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} = 
\begin{pmatrix}
\frac{2}{2p^2 - 2p + 1} \\
\frac{2p^2 - 4p + 3}{2p^2 - 2p + 1} \\
\frac{2p^2 + 1}{2p^2 - 2p + 1}
\end{pmatrix}.
\]

(287)

Note that \( t_{03} = 4 \) when \( p = \frac{1}{2} \).
Now consider an ant at corner 1 of a cube. The state of the ant is

$$\{Y(k) = j\} \text{ if the ant is at corner } j \in \{1, 2, \ldots, 8\} \text{ at time } k. \quad (288)$$

Consider the general transition probability matrix $P$

$$P = \begin{pmatrix}
p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\
p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} \\
p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p_{47} & p_{48} \\
p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p_{57} & p_{58} \\
p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} & p_{68} \\
p_{71} & p_{72} & p_{73} & p_{74} & p_{75} & p_{76} & p_{77} & p_{78} \\
p_{81} & p_{82} & p_{83} & p_{84} & p_{85} & p_{86} & p_{87} & p_{88}
\end{pmatrix}. \quad (289)$$
For an ant at corner 1 that would like to get to corner 8,

\[
P = \begin{pmatrix}
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(290)

Compute \( t_{18} = \mathbb{E}[T_{18}] \).
We can summarize the equations for the expected time to arrival as

\[
\begin{align*}
  t_{18} &= 1 + \frac{1}{3} (t_{28} + t_{48} + t_{68}) \\
  t_{28} &= 1 + \frac{1}{3} (t_{18} + t_{38} + t_{78}) \\
  t_{38} &= 1 + \frac{1}{3} (t_{28} + t_{48} + t_{88}) \\
  t_{48} &= 1 + \frac{1}{3} (t_{18} + t_{38} + t_{58}) \\
  t_{58} &= 1 + \frac{1}{3} (t_{48} + t_{68} + t_{88}) \\
  t_{68} &= 1 + \frac{1}{3} (t_{18} + t_{58} + t_{78}) \\
  t_{78} &= 1 + \frac{1}{3} (t_{28} + t_{68} + t_{88}) \\
  t_{88} &= 0.
\end{align*}
\]
Markov Chain Model of Employment

The US government has studied models of employment and has come up with the following observation:

- \( P[\text{Unemployed finds job by end of the year}] = p_f \in (0, 1) \)
- \( P[\text{Employed loses job by end of the year}] = p_l \in (0, 1) \)

Define \( W_k \) to be the probability a worker is employed at the beginning of year \( k \), \( N_k \) the probability she is not working at the beginning of year \( k \).

Then

\[
\begin{pmatrix}
W_{k+1} \\
N_{k+1}
\end{pmatrix} =
\begin{bmatrix}
1 - p_l & p_f \\
p_l & 1 - p_f
\end{bmatrix}
\begin{pmatrix}
W_k \\
N_k
\end{pmatrix}
\]  
(292)

Finally, assume that \( W_k + N_k = 1 \).
Markov Chain Model of Employment

The US government has studied models of employment and has come up with the following observation:

- \( \mathbb{P} \{ \text{Unemployed finds job by end of the year} \} = p_f \in (0, 1) \)
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\[
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W_{k+1} \\
N_{k+1}
\end{pmatrix} =
\begin{bmatrix}
1 - p_l & p_f \\
p_l & 1 - p_f
\end{bmatrix}
\begin{pmatrix}
W_k \\
N_k
\end{pmatrix}
\]

Finally, assume that \( W_k + N_k = 1 \).

**Question:** Does \( W_k \to W \) for some \( W \in (0, 1) \)?
It follows that the matrix can be diagonalized as

\[
\begin{pmatrix}
1 - p_I & p_f \\
p_I & 1 - p_f
\end{pmatrix} = \begin{pmatrix}
p_f/p_I & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1 - p_f - p_I
\end{pmatrix} \begin{pmatrix}
p_I/p_f + p_I & p_f/p_f + p_I \\
p_I/p_f + p_I & p_f/p_f + p_I
\end{pmatrix}
\]

(293)
Markov Chain Model of Employment

It follows that the matrix can be diagonalized as

\[
\begin{pmatrix}
1 - p_l & p_f \\
p_l & 1 - p_f
\end{pmatrix} = \begin{pmatrix}
p_f/
\hfill
p_l \\
\hfill
1
\end{pmatrix} \begin{pmatrix}
1 & -1 \\
0 & 1 - p_f - p_l
\end{pmatrix} \begin{pmatrix}
p_l/p_f + p_l \\
p_f/p_f + p_l
\end{pmatrix}
\]

Define

\[
\tilde{q}_k = \begin{pmatrix}
p_l/p_f + p_l \\
p_l/p_f + p_l
\end{pmatrix} \begin{pmatrix}
W_k \\
N_k
\end{pmatrix}
\]
Markov Chain Model of Employment

Hence,

$$
\vec{q}_{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - p_f - p_l \end{pmatrix} \vec{q}_k \Rightarrow \vec{q}_k = \begin{pmatrix} A \\ B(1 - p_f - p_l)^k \end{pmatrix}
$$

(295)
Markov Chain Model of Employment

Hence,

\[
\vec{q}_{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - p_f - p_l \end{pmatrix} \vec{q}_k \implies \vec{q}_k = \begin{pmatrix} A \\ B(1 - p_f - p_l)^k \end{pmatrix}
\]

(295)

Returning to our original notation,

\[
\begin{pmatrix} W_k \\ N_k \end{pmatrix} = \begin{pmatrix} \frac{p_f}{p_l} & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - p_f - p_l \end{pmatrix} \vec{q}_k
\]

\[
= \begin{pmatrix} \frac{p_f}{p_l} & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - p_f - p_l \end{pmatrix} \left( \begin{pmatrix} A \\ B(1 - p_f - p_l)^k \end{pmatrix} \right)
\]

(296)
Solving for our parameters $A, B$, we see that

$$
\begin{pmatrix}
W_0 \\
N_0
\end{pmatrix}
= 
\begin{pmatrix}
A \frac{p_f}{p_l} - B(1 - p_f - p_l) \\
A + B(1 - p_f - p_l)
\end{pmatrix}
$$

$$
\Rightarrow
\begin{pmatrix}
A \\
B
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{1 + \frac{p_f}{p_l}} \\
\frac{\frac{p_f}{p_l} N_0 - W_0}{1 - p_f - p_l}
\end{pmatrix}
$$
Markov Chain Model of Employment

Solving for our parameters $A, B$, we see that

\[
\begin{pmatrix}
W_0 \\
N_0
\end{pmatrix}
= \begin{pmatrix}
A^{\frac{p_f}{p_l}} - B(1 - p_f - p_l) \\
A + B(1 - p_f - p_l)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{1 + \frac{p_f}{p_l}} \\
\frac{pf}{pl} N_0 - W_0 \\
\frac{1 - p_f - p_l}{1 - p_f - p_l}
\end{pmatrix}
\]

\[
\Rightarrow \begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
\frac{p_f}{p_l} \\
\frac{p_f}{p_l} N_0 - W_0 \\
\frac{1 - p_f - p_l}{1 - p_f - p_l}
\end{pmatrix} = \begin{pmatrix}
\frac{p_f}{p_l + p_f} \\
\frac{p_f}{p_l + p_f}
\end{pmatrix}
\]

(297)
Examples to Read:

- 8.13, 8.14, 8.15

HW Exercises:

- 8.1, 8.2, 8.3, 8.4, 8.5, 8.8, 8.10, 8.11, 8.12, 8.13, 8.15, 8.16, 8, 18, 8.20
We consider two types of retirement plans.

- A **Defined Contribution** plan specifies how much an employer will contribute, as a percentage of salary, into a plan.

- A **Defined Benefit** plan specifies a level of benefit, most likely related to the employee’s salary near retirement. Here, contributions may need to be updated based on the investment returns to ensure that the benefit is met.
Also, defining the **Replacement Ratio**

\[
R := \frac{\text{pension income in the year after retirement}}{\text{salary in the year before retirement}}
\]  \hspace{1cm} (298)

the benefit under DB plans and target under DC plans may aim for

\[
R \in (0.5, 0.7).
\]  \hspace{1cm} (299)

This of course assumes the member survives the year following retirement.
We can also use a deterministic model to define the salary scale \( \{ s_y \}_{y \geq x_0} \) beginning at some suitable initial age \( x_0 \) where the value of \( s_{x_0} \) can be set arbitrarily.
We can also use a deterministic model to define the **salary scale** \( \{ s_y \}_{y \geq x_0} \) beginning at some suitable initial age \( x_0 \) where the value of \( s_{x_0} \) can be set arbitrarily.

The ratio usually given is

\[
\frac{s_y}{s_x} = \frac{\text{salary received in year } y \text{ to } y + 1}{\text{salary received in year } x \text{ to } x + 1}
\]

and, assuming that salaries are increased continuously, the **salary rate** at age \( x \) is defined to be \( s_{x - \frac{1}{2}} \).
Example 10.2

The final average salary for the pension benefit provided by a pension plan is defined as the average salary in the three years before retirement. Members’ salaries are increased each year, six months before the valuation date.

- A member aged exactly 35 at the valuation date received 75000 in salary in the year to the valuation date. Calculate his predicted final average salary assuming retirement at age 65.
- A member aged exactly 55 at the valuation date was paid salary at a rate of 100000 per year at that time. Calculate her predicted final average salary assuming retirement at age 65.

Assume a salary scale where \( s_{x_0+y} = 1.04^y s_{x_0} \).
Example 10.2

For first case

\[ s_{avg} = 75000 \cdot \frac{1}{3} \frac{s_{62} + s_{63} + s_{64}}{s_{34}} \]

\[ = \frac{75000}{3} \cdot (1.04^{28} + 1.04^{29} + 1.04^{30}) = 234019 \] (301)

For second case

\[ s_{avg} = 100000 \cdot \frac{1}{3} \frac{s_{62} + s_{63} + s_{64}}{s_{54.5}} \]

\[ = \frac{100000}{3} \cdot (1.04^{7.5} + 1.04^{8.5} + 1.04^{9.5}) = 139639 \] (302)

Now read Example 10.3 for more practice and Section 10.4 for setting the DC contribution.
We can define a multiple decrement model for a pension plan via states

\[ Y(t) = \begin{cases} 
0 & \text{if (x) is a member at age } x + t \\
1 & \text{if (x) has withdrawn by time } x + t \\
2 & \text{if (x) has retired due to disability by age } x + t \\
3 & \text{if (x) has retired due to age at } x + t \\
4 & \text{if (x) has died in service by age } x + t 
\end{cases} \]

**Figure:** Pension Plan Flow Chart. In a DC plan, benefit on exit is the same. However, in a DB plan different benefits may be payable on different forms of exit.
Example 10.5

A pension plan member is entitled to a lump sum benefit on death in service of four times the salary paid in the year up to death. Assuming the multiple decrement model with

\[
\begin{align*}
\mu_{x}^{01} &= \mu_{x}^{w} \\
\mu_{x}^{02} &= \mu_{x}^{i} = 0.001 \\
\mu_{x}^{03} &= \mu_{x}^{r} \\
\mu_{x}^{04} &= \mu_{x}^{d} = A + Bc^{x} \\
&= 0.00022 + (2.7 \times 10^{-6}) \cdot 1.124^{x}
\end{align*}
\]
Example 10.5

Assume

$$\mu_w^x = \begin{cases} 
0.1, & \text{if } x < 35 \\
0.05, & \text{if } 35 \leq x < 45 \\
0.02, & \text{if } 45 \leq x < 60 \\
0, & \text{if } x \geq 60 
\end{cases}$$

$$\mu_r^x = \begin{cases} 
0, & \text{if } x < 60 \\
0.1, & \text{if } 60 < x < 65 
\end{cases}$$

and

$$\Pr[(x) \text{ retires at (60)} \mid \text{survives in employment to (60)}] = 0.3$$

$$\Pr[(x) \text{ retires at (65)} \mid \text{survives in employment to (65)}] = 1$$
Example 10.5

Calculate, for a member aged 35, the probability of retiring at age 65. Notice the similarities to Example 8.12.
Calculate, for a member aged 35, the probability of retiring at age 65. Notice the similarities to Example 8.12.

For \( t \in (0, 10) \), we have

\[
\begin{align*}
tp_{35}^{00} & = \exp \left[ - \int_0^t \left( \mu_{35+s}^w + \mu_{35+s}^i + \mu_{35+s}^r + \mu_{35+s}^d \right) ds \right] \\
& = \exp \left[ - \int_0^t \left( 0.05 + 0.001 + 0 + A + Bc^{35+s} \right) ds \right] \\
& = \exp \left[ -0.05122t + \frac{2.7 \times 10^{-6}}{\ln (1.124)} 1.124^{35} (1.124^t - 1) \right] \\
\end{align*}
\]

It follows that \( 10p_{35}^{00} = 0.597342 \)
Example 10.5

For \( t \in [10, 25) \), we compute

\[
\frac{tp_{35}^{00}}{10p_{35}^{00}} = \mathbb{P} \left[ (35, 0) \rightarrow (35 + t, 0) \mid (35, 0) \rightarrow (45, 0) \right]
\]

\[
= \exp \left[ -\int_{0}^{t-10} \left( \mu_{45+s}^w + \mu_{45+s}^i + \mu_{45+s}^r + \mu_{45+s}^d \right) ds \right]
\]

\[
= \exp \left[ -\int_{0}^{t-10} \left( 0.02 + 0.001 + 0 + A + Bc^{45+s} \right) ds \right]
\]

\[
= \exp \left[ -0.02122(t - 10) + \frac{2.7 \times 10^{-6}}{\ln(1.124)} 1.124^{45} (1.124^{t-10} - 1) \right]
\]

\[
\Rightarrow 25^- p_{35}^{00} = \frac{25p_{35}^{00}}{10p_{35}^{00}} \cdot 10p_{35}^{00} = 0.712105 \cdot 0.597342 = 0.425370
\]

and \( 25^+ p_{35}^{00} = 0.7 \cdot 25^- p_{35}^{00} = 0.297759 \)
Example 10.5

For \( t \in (25, 30) \), we compute

\[
\frac{tp_{35}^{00}}{25+p_{35}^{00}} = \mathbb{P} \left[ (35, 0) \to (35 + t, 0) \mid (35, 0) \to (60^+, 0) \right] \\
= \exp \left[ - \int_0^{t-25} \left( \mu_{60+s}^w + \mu_{60+s}^i + \mu_{60+s}^r + \mu_{60+s}^d \right) ds \right] \\
= \exp \left[ -0.10122(t - 25) + \frac{2.7 \times 10^{-6}}{\ln(1.124)} \times 1.124^{60} (1.124^{t-25} - 1) \right] 
\]

(307)

It follows that the probability of retirement at exact age 65 is

\[
30-p_{35}^{00} = \left( \frac{tp_{35}^{00}}{25+p_{35}^{00}} \right) (25+p_{35}^{00}) = 0.175879 
\]

(308)

Now calculate: \( \mathbb{P}_{35}[\text{withdrawal}], \mathbb{P}_{35}[\text{retirement}], \mathbb{P}_{35}[\text{disability retirement}] \) and \( \mathbb{P}_{35}[\text{death in service}] \).
We can represent the multiple decrement model for pensions in tabular form. Begin by defining a minimum integer entry age $x_0$ and corresponding arbitrary radix (cohort) $l_{x_0}$. and now organize a table with entries

\[
\begin{align*}
  w_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{01} \\
  i_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{02} \\
  r_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{03} \\
  d_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{04} \\
  l_{x_0+k} &= l_{x_0} k p_{x_0}^{00}
\end{align*}
\]

and it follows that

\[
\begin{align*}
  w_{x_0+k} + i_{x_0+k} + r_{x_0+k} + d_{x_0+k} &= l_{x_0} k p_{x_0}^{00} \left( p_{x_0+k}^{01} + p_{x_0+k}^{02} + p_{x_0+k}^{03} + p_{x_0+k}^{04} \right) \\
  &= l_{x_0} k p_{x_0}^{00} \left( 1 - p_{x_0+k}^{00} \right) \\
  &= l_{x_0} \left( k p_{x_0}^{00} - k+1 p_{x_0}^{00} \right) = l_{x_0+k} - l_{x_0+k+1}.
\end{align*}
\]
It follows that we can use the service table to answer questions like

\[ P_{35} \text{ [withdraws]} = \sum_{k=0}^{24} \frac{w_{35+k}}{l_{35}} \]

\[ P_{35} \text{ [retires in ill health]} = \sum_{k=0}^{29} \frac{i_{35+k}}{l_{35}} \]  \( (311) \)

\[ P_{35} \text{ [retires for age reasons]} = \sum_{k=0}^{30} \frac{r_{35+k}}{l_{35}} \]

\[ P_{35} \text{ [dies in service]} = \sum_{k=0}^{29} \frac{d_{35+k}}{l_{35}} \]

For long-horizon investments with uncertain returns (forecasts may only be valid for a small horizon), using tabular methods with UDD approximation is common in pension valuation.
Employees in a pension plan pay contributions of 6% of their previous month’s salary at each month end until age 60. Calculate the EPV at entry of contributions for a new entrant aged 35, with a starting salary rate of 100000 using the model $\mu_1^x = \lambda$, $\mu_2^x = \gamma$, $\mu_3^x = 0$ and $\mu_4^x = \mu$ for $x \in (35, 60)$. Assume a constant force of interest $\delta$ and a salary scale function $s_y = e^{\epsilon y}$ for $y \in (35, 60)$. 
Per month, the contribution amount is a scaling of $0.06 \times \frac{100000}{12} = 500$. It follows that

\[
\mathbb{E} [\text{PV(Contributions)}] = 500 \sum_{k=1}^{300} p_{35}^{00} e^{\frac{k}{12}} e^{-\frac{k}{12}}
\]

\[
= 500 \sum_{k=1}^{300} e^{-(\mu + \gamma + \lambda)}^{\frac{k}{12}} e^{\frac{k}{12}} e^{-\frac{k}{12}}
\]

\[
= 500 e^{x} \left( \frac{1 - e^{300x}}{1 - e^{x}} \right)
\]

\[
x = \frac{\epsilon - \mu - \gamma - \lambda - \delta}{12}
\]
For a DB plan, the basic annual pension benefit is equal to $n \cdot S_{Fin} \cdot \alpha$, where $n$ is the total number of years of service, $S_{Fin}$ is the average salary in a specified period before retirement (ie. three years preceding exit) and $\alpha$ is the accrual rate, usually between 0.01 and 0.02.
Estimate the EPV of the **accrued age retirement pension benefit** for a member aged 55 with 20 years of service, whose salary in the year prior to the valuation date was 50000.

- Assume that mid-year age retirements happen at exactly halfway into the year. (Recall Claims Acceleration.)
- Assume the final average salary is defined as the earnings in the three years before retirement.
- Assume $\alpha = 0.015$.
- Calculate this EPV by using elements of a corresponding service table.
Note that for this problem,

\[ n \cdot S \cdot \alpha = (20)(50000)(0.015) = 15000 \]

\[ z_y = \frac{s_{y-3} + s_{y-2} + s_{y-1}}{3} \]  \hspace{1cm} (313)

as well as

\[ \mathbb{E} \left[ \text{Projected Final Salary} \mid \text{Retirement at age } y \right] = 50000 \frac{z_y}{s_{54}} =: \hat{S}_{Din} \]  \hspace{1cm} (314)
\[
\begin{align*}
\therefore \mathbb{E} \left[ PV(Benefits) \right] &= 15000 \left( \frac{r_{60} - z_{60}}{l_{55}} \frac{s_{54}}{\nu^5 \ddot{a}_{60}^{(12)}} + \frac{r_{65} - z_{65}}{l_{55}} \frac{s_{54}}{\nu^{10} \ddot{a}_{65}^{(12)}} \right) \\
&+ 15000 \frac{r_{60^+}}{l_{55}} \frac{z_{60.5}}{s_{54}} \nu^{5.5} \ddot{a}^{(12)}_{60.5} \\
&+ 15000 \sum_{k=1}^{4} \frac{r_{60+k}}{l_{55}} \frac{z_{60.5+k}}{s_{54}} \nu^{5.5+k} \ddot{a}^{(12)}_{60.5+k} \\
&+ 15000 \sum_{k=1}^{3} \frac{s_{y-k}}{3} \\
&= \frac{s_{y-3} + s_{y-2} + s_{y-1}}{3}
\end{align*}
\]

One can program this using numerical software, using linear interpolation for mid-year quantities. Read Examples 10.8, 10.9 for a discussion on withdrawal pension.
Plan Funding

Assuming..

- All employer contributions to a fund are paid the start of the year.
- There are no employee contributions.
- Any benefits payable during the year are paid exactly half-way though the year.

We define the **normal contribution** due at the start of the year $t$ to $t + 1$ for a member aged $x$ at time $t$ as $C_t$. 
Assuming:

- All employer contributions to a fund are paid the start of the year.
- There are no employee contributions.
- Any benefits payable during the year are paid exactly half-way through the year.

We define the **normal contribution** due at the start of the year $t$ to $t + 1$ for a member aged $x$ at time $t$ as $C_t$.

Using reserving principles studied earlier, we have the equation

$$ t V + C_t = \mathbb{E} [PV(\text{Benefits for mid-year exits})] + \nu_1 p_x^{00} t+1 V $$

(316)
Example 10.10

Assume a pension plan with the following valuation methods:

- Accrual rate: 1.5%
- Final salary plan
- Pension based on earnings in the year before age retirement
- Normal retirement at age 65
- The pension benefit is a life annuity payable monthly in advance
- There is no benefit due on death in service
- No exits other than by death before normal retirement age
Example 10.10

Calculate the value of the accrued pension benefit and normal contribution due at the start of the year using a projected unit funding (PUC), where interest is set at 5% per year, salaries increase at 4% per year and assume a constant mortality \( \mu \) before and after retirement.
Example 10.10

Calculate the value of the accrued pension benefit and normal contribution due at the start of the year using a projected unit funding (PUC), where interest is set at 5% per year, salaries increase at 4% per year and assume a constant mortality \( \mu \) before and after retirement.

\[
S_{Fin} = 50000 \frac{s_{64}}{s_{49}} = 50000(1.04)^{15} = 90047
\]

\[
0V = 0.015 \cdot 20 \cdot S_{Fin} \cdot 15p_{50} \cdot v^{15} \cdot a_{65}^{(12)}
\]

\[
= 27014.10 \cdot e^{-15\mu} \cdot v^{15} \cdot \frac{1}{12} \sum_{k=0}^{\infty} v^{k_{12}} \cdot e^{-\mu \frac{k_{12}}{12}}
\]

\[
= \frac{27014.10}{12e^{15\mu} \cdot 1.05^{15} \cdot \left(1 - \sqrt[12]{\frac{1}{1.05e^{\mu}}}\right)}
\]
Example 10.10

It follows that our equation for $C$ is

$$1V = 0.015 \cdot 21 \cdot S_{Fin} \cdot 14p_{51} \cdot v^{14} \cdot \ddot{a}_{65}^{(12)}$$

$$= \frac{21}{20} \frac{1}{\nu p_{50}} \left( 0.015 \cdot 20 \cdot S_{Fin} \cdot 15p_{50} \cdot v^{15} \cdot \ddot{a}_{65}^{(12)} \right)$$

$$= \frac{21}{20} \frac{1}{\nu p_{50}} (0V).$$

$$\therefore C = \nu \cdot p_{50} \cdot 1V - 0V = \frac{21}{20} 0V - 0V = \frac{0V}{20} = 0V$$

Consider the traditional unit credit funding approach, and see how this affects our previous calculation. Also, read over Example 10.11 which allows for benefits payable on exit during the year.
HW: 10.1, 10.2, 10.5, 10.7, 10.8, 10.11, 10.12
Law of Total Variance

Recall that for two random variables $X, Y$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have the Tower Property

\[
\mathbb{E}\left[\mathbb{E}[X \mid Y]\right] = \mathbb{E}[X]
\]
\[
\mathbb{E}\left[\mathbb{E}[Y \mid X]\right] = \mathbb{E}[Y]
\]

and so it follows that

\[
\mathbb{V}[X] = \mathbb{E}\left[ X^2 \right] - \left( \mathbb{E}[X] \right)^2
\]
\[
= \mathbb{E}\left[ \mathbb{E}\left[ X^2 \mid Y \right] \right] - \left( \mathbb{E}\left[ \mathbb{E}[X \mid Y] \right] \right)^2
\]
\[
= \mathbb{E}\left[ \mathbb{V}[X \mid Y] + \mathbb{E}[X \mid Y]^2 \right] - \left( \mathbb{E}\left[ \mathbb{E}[X \mid Y] \right] \right)^2
\]
\[
= \mathbb{E}\left[ \mathbb{V}[X \mid Y] \right] + \mathbb{V}\left[ \mathbb{E}[X \mid Y] \right]
\]
Assuming a sequence of \textbf{i.i.d.} Random Variables \( \{X_k\}_{k=1}^n \), one measure of the risk associated to the average

\[
\bar{X}_n := \frac{1}{n} \sum_{k=1}^n X_k
\]

is the total variance

\[
\rho_n(X) := \text{Var}[\bar{X}_n]
\]
Correspondingly, we say that such a risk is Diversifiable if \( \lim_{n \to \infty} \rho_n(X) = 0 \), and not diversifiable otherwise.

Note that if \( \{X_k\}_{k=1}^n \) are dependent but otherwise identically distributed with correlation coefficient \( \rho \), mean \( \mu \) and variance \( \sigma^2 \), then

\[
\rho_n(X) = \frac{n\sigma^2 + n(n - 1)\rho\sigma^2}{n^2} \to \rho\sigma^2 \neq 0 \quad (323)
\]

For a history of variance as a risk measure in Modern Portfolio Theory and the corresponding use of diversification, click here and references within.
Recall that for any random variable $Y$ and i.i.d. sequence $\{X_k\}_{k=1}^n$ with identical copy $X$

$$
\mathbb{V} \left[ \frac{1}{n} \sum_{k=1}^{n} X_k \right] = \mathbb{E} \left[ \mathbb{V} \left[ \frac{1}{n} \sum_{k=1}^{n} X_k \mid Y \right] \right] + \mathbb{V} \left[ \mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} X_k \mid Y \right] \right]
$$
Recall that for any random variable $Y$ and i.i.d. sequence $\{X_k\}_{k=1}^n$ with identical copy $X$

$$V\left[\frac{1}{n} \sum_{k=1}^n X_k\right] = \mathbb{E}\left[V\left[\frac{1}{n} \sum_{k=1}^n X_k \mid Y\right]\right] + V\left[\mathbb{E}\left[\frac{1}{n} \sum_{k=1}^n X_k \mid Y\right]\right]$$

$$= \frac{1}{n} \mathbb{E}\left[V\left[X \mid Y\right]\right] + V\left[\mathbb{E}\left[X \mid Y\right]\right]$$

(324)
Recall that for any random variable $Y$ and i.i.d. sequence $\{X_k\}_{k=1}^n$ with identical copy $X$

$$V\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \mathbb{E}\left[V\left[\frac{1}{n}\sum_{k=1}^{n} X_k \mid Y\right]\right] + V\left[\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} X_k \mid Y\right]\right]$$

$$= \frac{1}{n}\mathbb{E}\left[V\left[X \mid Y\right]\right] + V\left[\mathbb{E}\left[X \mid Y\right]\right]$$  \hspace{1cm} (324)

It follows that $\rho_n(X) \to 0$ as long as $V\left[\mathbb{E}\left[X \mid Y\right]\right] = 0$. 
Note that by the Central Limit Theorem,

\[
\lim_{n \to \infty} \mathbb{P}\left[ |\bar{X}_n - \mu| \geq k \right] = \lim_{n \to \infty} \mathbb{P}\left[ \left| \frac{\sum_{k=1}^{n} X_k - n\mu}{\sqrt{n}\sigma} \right| \geq \frac{k\sqrt{n}}{\sigma} \right]
\]

\[
= \lim_{n \to \infty} 2\Phi\left( -\frac{k\sqrt{n}}{\sigma} \right) = 0.
\]

This says that for uncorrelated r.v.’s, since the variance of the aggregate mean is linear in \(n\), we have the deviation of the aggregate mean from the individual mean asymptotically disappears.
Example of Diversifiable Risk

Consider the case where we have an i.i.d. sequence \( \{ X_k \} \) with

\[
X_k \in \{0, 1\}
\]

\[
P[X_k = 1] = tp_x \cdot (1 - sp_{x+t}).
\] (326)

It follows that

\[
\bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k
\] (327)

models the sample probability of deaths of a population of \( n \) alive at age \( x \) where death occurs between age \( x + t \) and age \( x + t + s \). This is of course a binomial random variable with

\[
p = tp_x \cdot (1 - sp_{x+t}).
\]
Example of Diversifiable Risk

We can see that for a copy $X$ of the sequence,

$$
V\left[ \bar{X}_n \right] = \frac{V\left[ \sum_{k=1}^{n} X_k \right]}{n^2}
$$

$$
= \frac{nV[X]}{n^2}
$$

$$
= \frac{1}{n} \cdot \left[ t p_x \cdot (1 - s p_{x+t}) \right] \cdot \left[ 1 - t p_x \cdot (1 - s p_{x+t}) \right]
$$

$$
\rightarrow 0
$$

and so the risk is diversifiable.
Example of Non-Diversifiable Risk

Consider now the case where the $X_k$ model the loss associated with a member of an insured population. If each member has loss function $X_k$ and the premiums are charged in keeping with the EPP, then we expect that $\mathbb{E}[X_k] = 0$ for all $k \in \{1, \ldots, n\}$.

Correlated loan losses, such as those in a large portfolio, can be modeled in a similar way. Check out Vasicek's model, beautifully explained in these slides by Stephen M. Schaefer of the London Business School.
Consider now the case where the $X_k$ model the loss associated with a member of an insured population. If each member has loss function $X_k$ and the premiums are charged in keeping with the EPP, then we expect that $\mathbb{E}[X_k] = 0$ for all $k \in \{1, \ldots, n\}$.

If, however, the forecasted yield rate used is a random variable $Y$, then if $\mathbb{E}[X_k \mid Y] \neq 0$ we have non-diversifiable risk as

$$\mathbb{V}[\bar{X}_n] \to \mathbb{V}[\mathbb{E}[X \mid Y]] \neq 0.$$  \hfill (329)

Correlated loan losses, such as those in a large portfolio, can be modeled in a similar way. Check out Vasicek’s model, beautifully explained in these slides by Stephen M. Schaefer of the London Business School.
An insurer issues a number of identical special 1-year term life insurance policies.

Each policy has a death benefit of 1000 payable at the end of the year of death, on condition that:

- The policyholder dies during the year; and
- A stock index ends the year below its value at the start of the year.

Both conditions must be satisfied for the death benefit to be paid.
Furthermore, you are given:

- Future lifetimes of the policyholders are independent
- $q_x = 0.05$ for all $x$.
- The probability that the stock index ends the year below its value at the start of the year is 0.1 for all years.
- Future lifetimes of the policyholders and the value of the stock index are independent.
- The annual effective rate of interest rate is 3%.
- $X_N$ denotes the total present value of benefits for $N$ policies.

Calculate

$$\lim_{N \to \infty} \frac{\sqrt{V[X_N]}}{N}$$

(330)
Define $F = \{\text{Fund drops below current level in the next year}\}$ . Then

\[
P[F] = 0.1
\]

\[
\mathbb{E}[X_N \mid F] = N \times \frac{1000}{1 + i} q_x = 48.54N
\]

\[
\mathbb{E}[X_N \mid F^c] = 0
\]

\[
\therefore \mathbb{E}[X_N] = \mathbb{E}[X_N \mid F] \times P[F] + \mathbb{E}[X_N \mid F^c] \times P[F^c]
\]

\[
= 4.854N
\]
Also,

\[
V[X_N | F] = N \times \left( \frac{1000}{1 + i} \right)^2 q_x(1 - q_x) = 44773.31N
\]

\[
V[X_N | F^c] = 0
\]

\[
E \left[ V[X_N | 1_{\{F\}}] \right] = V[X_N | F] \times P[F] + V[X_N | F^c] \times P[F^c]
\]

\[
= 4477.331N
\]

\[
V \left[ E[X_N | 1_{\{F\}}] \right] = \left( E[X_N | F] \right)^2 \times P[F] + \left( E[X_N | F^c] \right)^2 \times P[F^c]
\]

\[
- E \left[ X_N \right]^2
\]

\[
= \left( 48.54N \right)^2 \times 0.1 - (4.854N)^2 = 212.05N^2
\]

\[
\Rightarrow \lim_{N \to \infty} \frac{\sqrt{V[X_N]}}{N} = \lim_{N \to \infty} \frac{\sqrt{212.05N^2 + 4477.331N}}{N} = \sqrt{212.05} = 14.56
\]

(332)
Homework Questions

HW: 11.1, 11.2, 11.3, 11.5, 11.7
Recall the need for policy values when negative future cash flows were expected. In this lecture, we cover the idea of reserves, which is the actual amount of money held by the insurer to cover future liabilities associated with contracts.
Reserves

The insurer may decide to set aside assets in reserve as equal to the net premium policy values on a certain (reserve) basis.

\[ R_t = t V_t = SA_{x:n} + t \cdot n - t \cdot p_{x:n} = SA_{x:n} \cdot \left( A_{x:n} + t \cdot n - t \cdot a_{x:n} + t \cdot n - t \right) \]
The insurer may decide to set aside assets in reserve as equal to the net premium policy values on a certain (reserve) basis.

For example, consider an \( n \)–year term insurance contract issued to a life \( x \) with sum insured \( S \). Since we use the net premium basis to compute fixed premiums, it follows that

\[
P = S \frac{A_{x:n}^1}{\bar{a}_{x:n}}
\]
The insurer may decide to set aside assets in reserve as equal to the net premium policy values on a certain (reserve) basis.

For example, consider an $n-$year term insurance contract issued to a life $x$ with sum insured $S$. Since we use the net premium basis to compute fixed premiums, it follows that

\[ P = S \frac{A_{x:n}^1}{\ddot{a}_{x:n}} \]

\[ \Rightarrow R_t = t V = S A_{x+t:n-t}^1 - P \ddot{a}_{x+t:n-t} \]

\[ = S A_{x:n}^1 \cdot \left( \frac{A_{x+t:n-t}^1}{A_{x:n}^1} - \frac{\ddot{a}_{x+t:n-t}}{\ddot{a}_{x:n}} \right) \]

(333)
Reserves

The insurer may decide to set aside assets in reserve as equal to the net premium policy values on a certain (reserve) basis.

For example, consider an \( n \)-year term insurance contract issued to a life \( x \) with sum insured \( S \). Since we use the net premium basis to compute fixed premiums, it follows that

\[
P = S \frac{A_{x:n}}{\bar{a}_{x:n}}
\]

\[
\Rightarrow R_t = tV = SA_{x+t:n-t}^1 - P\bar{a}_{x+t:n-t}
\]

\[
= SA_{x:n}^1 \cdot \left( \frac{A_{x+t:n-t}^1}{A_{x:n}} - \frac{\bar{a}_{x+t:n-t}}{\bar{a}_{x:n}} \right)
\]

The cost of setting up, from \( t - 1 \) to \( t \), the reserve amount of \( tV \) is at time \( t \) equal to \( tV \cdot p_{x+t-1} \) when valued at time \( t \).

i.e. the proportion of contracts that survive to the end of the year.
At time $t$, just before and just after, we have quantities that are assets and costs.

- At time $(t - 1)_+$, we have the cost $E_t$ associated from $t - 1$ to $t$.
- Between $(t - 1)_+$ and $t_-$, we have the payout $S$ settled at time $t_-$ with expected value $S \cdot q_{x+t-1}$.
- At time $(t - 1)_+$, we have the asset $t_{-1}V$ which grows at the interest rate $i$ to value $(1 + i) \cdot t_{-1}V$ at time $t_-$.
- At time $(t)_-$, we have the cost $tV \cdot p_{x+t-1}$. 
Correspondingly, we can set up an equation for the profit at time $t$, denoted by $Pr_t$:

$$Pr_t = \left( t_{-1} V + P - E_t \right)(1 + i) - Sq_{x+t-1} - t Vp_{x+t-1}$$  \hspace{1cm} (334)

The Profit Vector

$$\vec{Pr} := \left( Pr_0, \ldots, Pr_n \right)$$  \hspace{1cm} (335)

is comprised of elements that represent the expected profit at the end of the year given that the policy is in effect at the start of the year.
The Profit Signature is the vector $\vec{\Pi}$ comprised of elements

$$\Pi_t := t-1p_x Pr_t$$

that represent the expected profit at the end of the year given that the policy was in effect at age $x$. 

(336)
Recall that for any set of cash flows $C_t$, the internal rate of return $\text{IRR}$ (if it uniquely exists) is the interest rate $j$ such that

$$\sum_{t=0}^{n} \frac{C_t}{(1 + j)^t} = 0. \quad (337)$$

In accordance with the IRR, the insurer may set a minimum hurdle or risk discount rate $r$ such that the contract is satisfiably profitable if $\text{IRR} > r$. 
If the IRR does not exist, the insurer may seek to measure the profitability via the Net Present Value computed using the risk discount rate:

\[ NPV := \sum_{t=0}^{n} \frac{\Pi_t}{(1 + r)^t} \]  (338)
Another measure is the ratio of NPV to $\mathbb{E}[PV(\text{Premiums})]$:

$$\text{Profit Margin} := \frac{NPV}{\mathbb{E}[PV(\text{Premiums})]}$$  \hspace{1cm} (339)

as is the discounted payback period DPP:

$$DPP := \min \left\{ m : \sum_{t=0}^{m} \frac{\Pi_t}{(1+r)^t} \geq 0 \right\}$$  \hspace{1cm} (340)

which represents the time until the insurer starts to make a profit.

A question naturally arises of how to jointly measure interest risk and profit. One may even compute the marginal changes in the profit measures with respect to change in risk discount factor $r$. 
Exercise 12.13 (Profit Testing Basis)

A special 10-year endowment insurance is issued to a healthy life aged 55. The benefits under the policy are

- 50000 if at the end of a month the life is disabled, having been healthy at the start of the month,
- 100000 if at the end of a month the life is dead, having been healthy at the start of the month,
- 50000 if at the end of a month the life is dead, having been disabled at the start of the month,
- 50000 if the life survives as healthy to the end of the term.

On withdrawal at any time, a surrender value equal to 80% of the net premium policy value is paid, and level monthly premiums are payable throughout the term while the life is healthy.
Other elements of the profit testing basis are as follows:

- **Interest**: 7% per year.
- **Expenses**: 5% of each gross premium, including the first, together with an additional initial expense of 1000.
- **The benefit on withdrawal**: is payable at the end of the month of withdrawal and is equal to 80% of the sum of the reserve held at the start of the month and the premium paid at the start of the month.
- **Reserves are set equal to the net premium policy values.**
Extra modeling assumptions:

- The gross premium and net premium policy values are calculated using the same survival model as for profit testing except that withdrawals are ignored, so that $\mu^0_x = 0$ for all $x$.
- The net premium policy values are calculated using an interest rate of 5% per year.

The monthly gross premium is calculated using the equivalence principle on the following basis:

- Interest: 5.25% per year.
- Expenses: 5% of each premium, including the first, together with an additional initial expense of 1000.
Exercise 12.13

(a) Calculate the monthly premium on the net premium policy value basis.

(b) Calculate the reserves at the start of each month for both healthy lives and for disabled lives.

(c) Calculate the monthly gross premium.

(d) Calculate the monthly profit earned, using the profit testing basis, under assumption that the life is healthy \( (Pr_t^{(0)}) \) or that the life is disabled \( (Pr_t^{(1)}) \) at the beginning of the month. **Hint:** Consider equation (334) for both healthy and disabled lives.

(e) Calculate the internal rate of return.

(f) Calculate the NPV, the profit margin (using the EPV of gross premiums), the NPV as a percentage of the acquisition costs, and the discounted payback period for the contract, in all cases using a risk discount rate of 15% per year.
Exercise 12.13

The model for state transition in this model follows the flow chart below:

0 (Healthy) → 1 (Disabled)

↓ ↓
3 (Withdrawn) 2 (Dead)

Figure: Multiple State Model
Exercise 12.13

The associated rate matrix for Profit Testing is

$$\tilde{Q}(t) = \begin{pmatrix} -0.035 & 0.01 & 0.015 & 0.01 \\ 0 & -0.03 & 0.03 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ (341)

and the associated rate matrix for gross premium and net premium policy values is

$$Q(t) = \begin{pmatrix} -0.025 & 0.01 & 0.015 \\ 0 & -0.03 & 0.03 \\ 0 & 0 & 0 \end{pmatrix}$$ (342)
Exercise 12.13 - Diagonalization of $\tilde{Q}$

$U = \begin{pmatrix} -1 & 0.894427 & 0.245036 & 0.438025 \\ 0 & 0.447214 & -0.0439573 & 0.634545 \\ 0 & 0 & -0.0439573 & 0.634545 \\ 0 & 0 & 0.967519 & -0.0532764 \end{pmatrix}$

$D = \begin{pmatrix} -0.035 & 0 & 0 & 0 \\ 0 & -0.03 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$U^{-1} = \begin{pmatrix} -1 & 2 & -1.28571 & 0.285714 \\ 0 & 2.23607 & -2.23607 & 0 \\ 0 & 0 & 0.0871109 & 1.03753 \\ 0 & 0 & 1.58197 & 0.0718735 \end{pmatrix}$
Exercise 12.13 - Diagonalization of $\tilde{Q}$

It follows that if we do allow for withdrawal, our transition probability matrix $P(t) = \exp(tQ)$ has the solution

$$
\tilde{P}(t) = \begin{pmatrix}
    e^{-0.035t} & t\tilde{p}_{55}^{01} & t\tilde{p}_{55}^{02} & t\tilde{p}_{55}^{03} \\
    0 & e^{-0.03t} & 1 - e^{-0.03t} & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

where

\begin{align*}
    t\tilde{p}_{55}^{01} &= 2e^{-0.03t} - 2e^{-0.035t} \\
    t\tilde{p}_{55}^{02} &= 0.71428571 + 1.285714287e^{-0.035t} - 2e^{-0.03t} \\
    &= \frac{5}{7} + \frac{9}{7}e^{-0.035t} - 2e^{-0.03t} \\
    t\tilde{p}_{55}^{03} &= 0.285714287 - 0.285714287e^{-0.035t} \\
    &= \frac{2}{7} - \frac{2}{7}e^{-0.035t}
\end{align*}
Exercise 12.13 - No Withdrawal

If we do not allow for withdrawal, however, our transition probability matrix $P(t) = \exp(tQ)$ has the simpler solution

$$
P(t) = \begin{pmatrix}
e^{-0.025t} & 2e^{-0.025t} - 2e^{-0.03t} & 1 + 2e^{-0.03t} - 3e^{-0.025t} \\
0 & e^{-0.03t} & 1 - e^{-0.03t} \\
0 & 0 & 1
\end{pmatrix}.
$$

(346)
Exercise 12.13 - Monthly Net Premium

For the equation of value, we determine for the monthly net premium $P'$

\[
\mathbb{E}[\text{Premium Income}] = P' \sum_{k=0}^{119} \frac{p_{55}^{00}}{12} v^{k_{12}}
\]

\[
\mathbb{E}[\text{Benefit}] = 50000 v^{10} 10 p_{55}^{00}
\]

\[
+ 50000 \sum_{k=0}^{119} \left( \frac{k_{12}}{12} p_{55}^{00} + \frac{k_{12}}{12} p_{55}^{01} + \frac{k_{12}}{12} p_{55}^{12} \right) v^{k+1_{12}}
\]

\[
+ 100000 \sum_{k=0}^{119} \frac{k_{12}}{12} p_{55}^{00} + \frac{k_{12}}{12} p_{55}^{02} v^{k+1_{12}}
\]

Using our solution for transition probabilities that don’t allow for withdrawals, a discount rate of $v = \frac{1}{1.05}$ and solving the resulting geometric series for the EPV’s above, we return $P' = 452$. 

(347)
Recall that \( tV^{(i)} = \mathbb{E}[\text{Loss} \mid Y(t) = i] \)

Given the parameters of our contract, we have the boundary values

\[
10 - V^{(0)} = 50000 \\
10 - V^{(1)} = 0
\]  

and the recursive equations

\[
tV^{(0)} = -P' + \frac{1}{12} p_{55}^{00} t \nu^{\frac{1}{12}} V^{(0)} + \frac{1}{12} p_{55}^{01} t \nu^{\frac{1}{12}} (50000 + t + \frac{1}{12} V^{(1)}) \\
+ 100000 \nu^{\frac{1}{12}} \frac{1}{12} p_{55}^{02} t
\]

\[
tV^{(1)} = \frac{1}{12} p_{55}^{11} t \nu^{\frac{1}{12}} V^{(1)} + 50000 \nu^{\frac{1}{12}} \frac{1}{12} p_{55}^{12} t
\]

A matrix (array) recursion method can be encoded via spreadsheet or other numerical software to iterate backwards from \( t = 10 \).
Exercise 12.13 - Monthly Gross Premium

For the equation of value, we determine for the monthly gross premium $P$

$$\mathbb{E}[	ext{Premium Income}] = 0.95P \sum_{k=0}^{119} p_{55}^{00} v_{12}^{k}$$

$$\mathbb{E}[	ext{Benefit}] = 50000v_{10}^{10} p_{55}^{00}$$

$$+ 50000 \sum_{k=0}^{119} \left( p_{55}^{00} p_{55}^{01} + p_{55}^{01} p_{55}^{12} + p_{55}^{00} p_{55}^{12} \right) v_{12}^{k+1}$$

$$+ 100000 \sum_{k=0}^{119} p_{55}^{00} p_{55}^{02} v_{12}^{k}$$

$$+ 1000.$$

Using our solution for transition probabilities that don’t allow for withdrawals, a new discount rate of $v = \frac{1}{1.0525}$, and solving the resulting geometric series for the EPV’s above, we return $P = 484.27$. 

(350)
Once we have the arrays \( \left( Pr_t^{(0)}, Pr_t^{(1)} \right) \) from part (d), we can further compute

- \( \Pi_t = t - \frac{1}{12} \hat{p}_{55}^{00} Pr_t^{(0)} + t - \frac{1}{12} \hat{p}_{55}^{01} Pr_t^{(1)} \).

- An IRR \( j \) such that \( \sum_{k=0}^{120} \frac{\Pi_k}{(1+j)^{\frac{k}{12}}} = 0 \).

- The net present value \( NPV = \sum_{k=0}^{120} \frac{\Pi_k}{(1.15)^{\frac{k}{12}}} \).

- The discounted payback period \( m \), where \( m \) is the smallest value such that \( \sum_{k=0}^{m} \frac{\Pi_k}{(1.15)^{\frac{k}{12}}} \geq 0 \).

- The profit margin \( P \sum_{k=0}^{119} \frac{\hat{p}_{55}^{00}}{(1.15)^{\frac{k}{12}}} \).
Homework Questions

- **Finish** Exercise 12.13 by filling in the numbers in the previous slide.
- **Read** Example 12.1.
- **HW:** 12.1, 12.2, 12.3, 12.4, 12.6
Modern insurance contracts can include some form of guarantee.

These are known in America as **Variable Annuities** and **Segregated Funds** in Canada.

The accumulating premiums the policyholder pays is invested on the policyholder’s behalf.

These premiums form the **policyholder’s fund**, from which regular management charges are deducted by the insurer and paid into the **insurer’s fund** to cover expenses and insurance charges.
Equity Linked Insurance

On survival to the end of the contract term the benefit may be just the policyholder’s fund and no more, or there may be a **guaranteed minimum maturity benefit (GMMB)**. There may also be a **guaranteed minimum death benefit (GMDB)**.
Equity Linked Insurance

On survival to the end of the contract term the benefit may be just the policyholder’s fund and no more, or there may be a guaranteed minimum maturity benefit (GMMB). There may also be a guaranteed minimum death benefit (GMDB).

There are very real consequences to the differences between financial pricing and actuarial reserving. A short but excellent analysis can be found in the paper by Bangwon Ko and Elias S. W. Shiu on Financial Pricing and Actuarial Reserving.

Also consider A Heavy Traffic Approach to Modeling Large Life Insurance Portfolios (Stochastic modeling of actuarial reserve, with Ito integration of a time-changed Brownian Bridge.)

We follow the example set by Shiu and Ko now.
Fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a standard Brownian motion \(W\) that lives on this space.

Consider now a term contact with term \(T\) and let \(\alpha\) denote the \textit{management charges factor} along with \(\beta\) representing the policyholder’s \textit{participation factor}.

Furthermore, assume mean and standard deviation parameters \((\mu, \sigma)\) respectively and the corresponding \textit{Geometric Brownian Mutual Fund Asset}

\[
S_t = S_0 e^{\mu t + \sigma W_t}. \tag{351}
\]
Using this as the model of the asset returns upon which premiums are invested, the policyholder wishes to purchase a contract that pays a maturity benefit credited at a rate of return which is the greater of

- the customer’s risk discount rate $r$, where $r < \mu$ or
- the participation rate of the stock index returns of $S$.

Symbolically, for a current premium $P$ invested in the, the contract payout value at maturity is

$$V(T) = (1 - \alpha)P \max \left\{ e^{rT}, 1 + \beta \left( \frac{S_T}{S_0} - 1 \right) \right\}.$$  \hspace{1cm} (352)
Assume that the policyholder is able to fully participate in the returns from the fund (i.e. $\beta = 1$.)

Then

$$V(T) = (1 - \alpha)P \frac{S_T}{S_0} + (1 - \alpha)P \max \left\{ e^{rT} - \left( \frac{S_T}{S_0} \right), 0 \right\}$$

$$:= V_1(T) + V_2(T).$$

Here, $V_1(T)$ is the *net premium*, or payoff, for investing in the index fund and $V_2(T)$ is the *guaranteed option* payoff if the index fund under-performs relative to the risk discount rate $r$.

How does one reserve to meet the obligations of $V_2(T)$.
One can see that the probability of a payout, that $V_2(T) \neq 0$ is for large $T$

$$
\mathbb{P}[V_2(T) \neq 0] = \mathbb{P}[rT > \mu T + \sigma W_T] = \Phi\left(\frac{r - \mu}{\sigma} \sqrt{T}\right) \approx 0.
$$

(354)
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Since it is a low probability event that we have to prepare for a payout $V_2(T)$ and since we can directly replicate the payoff $V_1(T)$ by initially purchasing $\frac{(1-\alpha)P}{S_0}$ units of the index fund, an actuary may be tempted to not reserve for the uncertain portion of the guarantee, $V_2(T)$, if the contract has a relatively long term $T$. 
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Is this a wise decision?
Stochastic Reserving for non-diversifiable risk

Given a random loss \( L \), we define the quantile reserve, also known as the **Value at Risk** with parameter \( \alpha \), as the amount which with probability \( \alpha \) will not be exceeded by the loss.

Symbolically, if \( L \) has a continuous distribution function \( F_L \), then the \( \alpha \)-quantile reserve is \( Q_\alpha \) where

\[
P[L \leq Q_\alpha] = \alpha.
\]
One feature that is missing in VaR is the description of what the loss could be if it does exceed the quantile $Q_\alpha$. In this case, the Conditional Tail Expectation ($CTE_\alpha$) is defined as

$$CTE_\alpha = \mathbb{E}[L \mid L \geq Q_\alpha].$$  \hspace{1cm} (356)

A risk manager should not rely on static measures of risk involved with a portfolio of liabilities. Rather, the CTE or VaR reserve should be regularly updated to incorporate market information as it arrives. This allows reserves which are held in less-risky (and possibly more liquid) funds to be invested higher return and higher risk assets if current market information dictates that CTE reserves can be reduced.
Another set of problems that actuaries share with retirement planners and financial economists alike is the lack of planning many folks do for their retirement.

Consider the (2014) article ¹ *The Crisis in Retirement Planning* by Robert C. Merton, a Nobel Laureate and a founding father of mathematical finance.

In this work, Prof. Merton discusses the move from DB to DC plans, and the weight of managing one’s retirement account in the light of this cultural shift.

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Hedging Retirement Risks? Ideas from a Nobel Laureate.

- Defining the strength of one’s DC pension, seen as an investment account, Merton (2014) argues for maximizing the probability of hitting a desired income level in retirement, as opposed to maximizing total capital in savings.

- In more recent work, Merton has proposed the development of new financial instruments to hedge more of this DC risk. One such instrument are known as **Standard of Living indexed, Forward-starting, Income-only Securities**.

- These are government issued default-free bonds which offer “…certainty about two characteristics critical for DC retirement portfolios: (i) a commitment to pay over a particular time horizon (how/when one is paid); and (ii) a specific cash flow (what is paid). DC investors require a guaranteed cash flow that protects their real purchasing power in retirement.”

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Homework Questions

- Read Example 14.1 and Section 14.4
- HW: 14.1 – 14.5