Numerical Approximation Methods for Non-Uniform Fourier Data

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Joint work with

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Model Problem

Let f be defined in $\mathbb{R}$ with support in $[-\pi, \pi)$. Given

$$\hat{f}(\omega_k) = \left\langle f, e^{i\omega_k x} \right\rangle, \quad k = -N, \cdots, N,$$

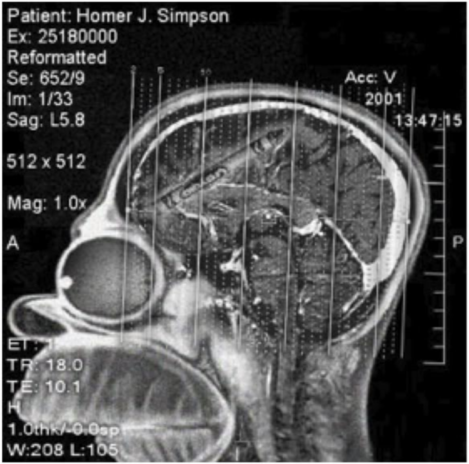
$$(\omega_k \text{ not necessarily} \in \mathbb{Z})$$

compute

- an approximation to the underlying function f,
- an approximation to the locations and values of jumps in the underlying function; i.e.,

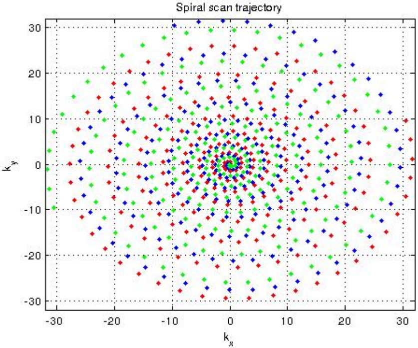
$$[f](x) := f(x^+) - f(x^-).$$

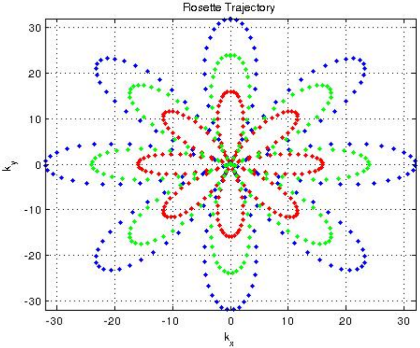
Motivating Application – Magnetic Resonance Imaging



Physics of MRI dictates that the MR scanner collect samples of the Fourier transform of the specimen being imaged.

Motivating Application – Magnetic Resonance Imaging





- Collecting non-uniform measurements has certain advantages; for example, they are easier and faster to collect, and, aliased images retain diagnostic qualities.

Challenges in Non-Uniform Reconstruction

- *Computational Issues*
  - The DFT is not defined for $\omega_k \neq k$; the FFT is not directly applicable.
  - Direct versus iterative solvers
- *Sampling Issues*
  Typical MR sampling patterns have non uniform sampling density; i.e., the high modes are sparsely sampled ($|\omega_k - k| > 1$ for k large).
- *Other Issues*
  Piecewise-smooth functions and Gibbs artifacts

Outline

Harmonic Fourier Reconstruction – A Review

Given

$$\hat{f}_k := \left\langle f, e^{ikx} \right\rangle, \quad k = -N, \cdots, N,$$

a periodic repetition of f may be reconstructed using the discrete partial sum

$$\bar{f}_j = \sum_{|k| \leq N} \hat{f}_k e^{ikx_j},$$

where x_j denotes the equispaced grid points

$$x_j = -\pi + j\,(2\pi/N)\,, \quad j = 0, \ldots, N-1.$$

- The discrete sum may be interpreted as a (trapezoidal) quadrature approximation of the inverse Fourier integral.
- The discrete Fourier sum may be evaluated efficiently using the FFT.

The Dirichlet Kernel – A Review

The approximation properties of the reconstruction may be described in terms of the Dirichlet kernel, since

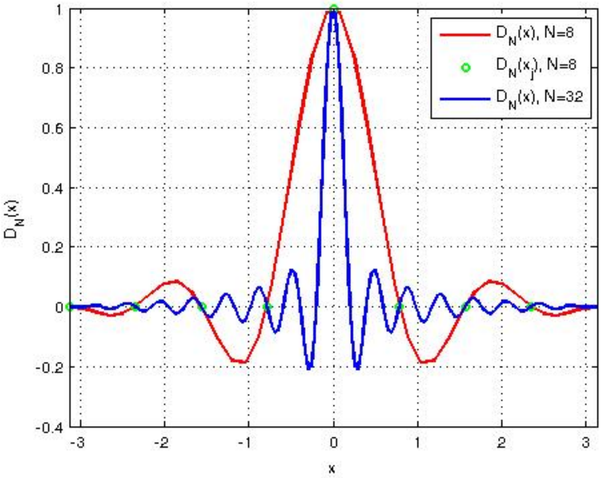
$$P_N f(x) = \sum_{|k|\leq N} \hat{f}_k e^{ikx} = (f * D_N)(x),$$

where

$$D_N(x) = \sum_{|k|\leq N} e^{ikx}.$$

- D_N is the bandlimited ($2N+1$ mode) approximation of the Dirac delta distribution.
- D_N completely characterizes the Fourier approximation $P_N f$.
- Filtered and jump approximations are similarly characterized by equivalent filtered and (filtered) conjugate Dirichlet kernels.

The Dirichlet Kernel – A Review



Non-Uniform Fourier Reconstruction

Extending the quadrature interpretation to the case of non-uniform Fourier modes, consider the non-uniform sum

$$\bar{f}_j = \sum_{|k| \leq N} \alpha_k \hat{f}(\omega_k) e^{i\omega_k x_j}, \quad j = 0, \ldots, N-1,$$

where α_k could be quadrature weights corresponding to a non-uniform trapezoidal quadrature rule.

- In the MR imaging community, these are referred to as *density compensation factors* (DCFs).
- For a suitable set of DCFs, the reconstruction procedure involves computing the above non-uniform sum efficiently (using, for example, a non-uniform FFT).

Non-Uniform FFTs (Kunis,Potts/Fessler/Dutt,Rokhlin . . .)

- Non-uniform FFTs (NFFTs) allow for the efficient computation of trigonometric polynomials involving non-uniform nodes and/or modes.
- They have a computational cost of $\mathcal{O}(N \log N + M)$, where N is the number of nodes and M is the number of modes.
- Most variants of the NFFT involve the use of an oversampled FFT and a *window* function which is simultaneously localized in time/space and frequency.
    - deconvolve the trigonometric polynomial with the window function in physical space
    - compute an oversampled FFT
    - convolve with the window function in Fourier space and evaluate this convolution at the non-uniform modes.

The Non-Uniform Kernel

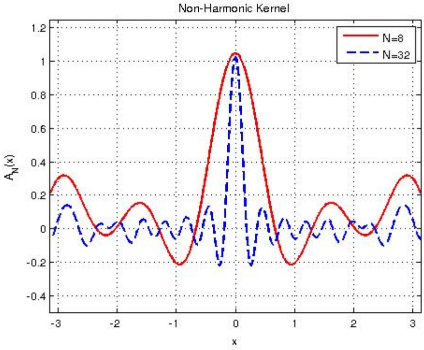
We may express the non-uniform sum as

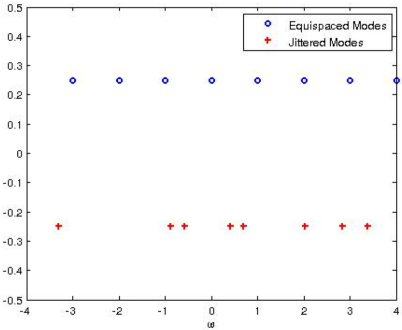
$$T_N f(x) = \sum_{|k| \leq N} \alpha_k \hat{f}(\omega_k) e^{i\omega_k x} = (f * A_N)(x), \quad \text{with}$$

$$A_N(x) = \sum_{|k| \leq N} \alpha_k e^{i\omega_k x}.$$

- A_N is the kernel associated with the non-uniform modes ω_k.
- Choice of α_k as well as the Fourier modes ω_k determine the resolution properties of the kernel.

The Non-Uniform Kernel

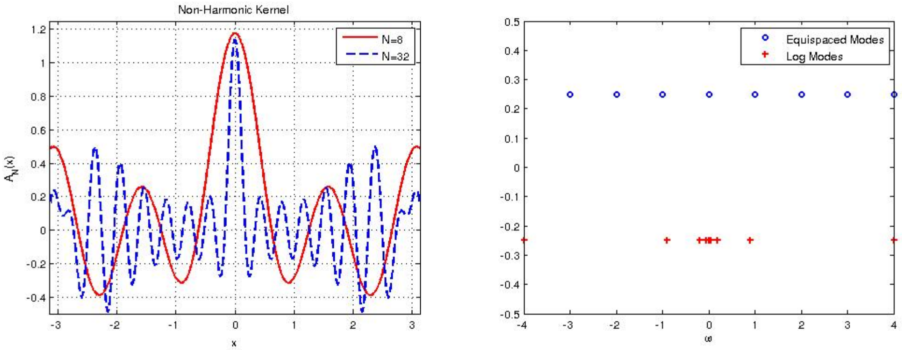




Jittered Modes

$$\omega_k = k \pm U[0, \mu], \quad \mu = 1.5$$

The Non-Uniform Kernel



Log Modes

ω_k logarithmically spaced

Outline

Designing Non-Uniform Reconstruction Kernels

- Recall that the non-uniform reconstruction is characterized by the non-harmonic kernel

$$A_N^\alpha(x) = \sum_{|k| \leq N} \alpha_k e^{i\omega_k x}.$$

- α_k are free design parameters which we choose such that A_N^α is compactly supported and a good reconstruction kernel (such as the Dirichlet kernel) in the interval of interest.

Design Problem – Formulation

Choose $\boldsymbol{\alpha} = \{\alpha_k\}_{-N}^{N}$ such that

$$\sum_{|k|\leq N} \alpha_k e^{i\omega_k x} \approx \begin{cases} \displaystyle\sum_{|\ell|\leq M} e^{i\ell x} & |x| \leq \pi \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

$$D\boldsymbol{\alpha} = \mathbf{b},$$

where

- $D_{\ell,j} = e^{i\omega_\ell x_j}$ denotes the (non-harmonic) DFT matrix, and
- $b_p = \frac{\sin(M+1/2)x_p}{\sin(x_p/2)} \cdot \Pi$ are the values of the Dirichlet kernel on the equispaced grid.

Design Problem – Formulation

Choose $\boldsymbol{\alpha} = \{\alpha_k\}_{-N}^{N}$ such that

$$\sum_{|k|\leq N} \alpha_k e^{i\omega_k x} \approx \begin{cases} \sum_{|\ell|\leq M} \sigma_\ell e^{i\ell x} & |x| \leq \pi \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

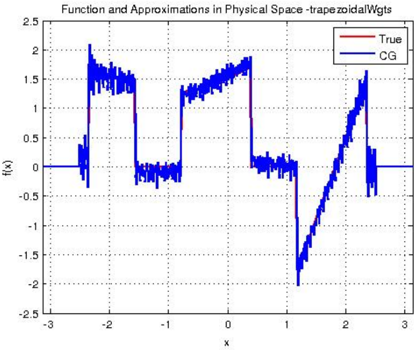
$$D\boldsymbol{\alpha} = \mathbf{b},$$

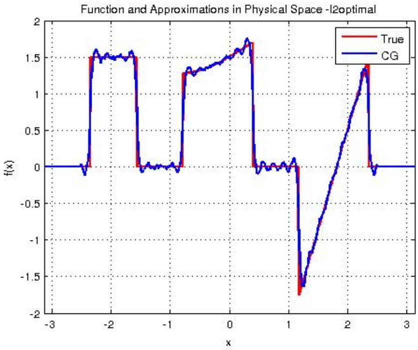
where

- $D_{\ell,j} = e^{i\omega_\ell x_j}$ denotes the (non-harmonic) DFT matrix, and
- b_p are the values of the (filtered) Dirichlet kernel on the equispaced grid.

Numerical Results

Reconstruction

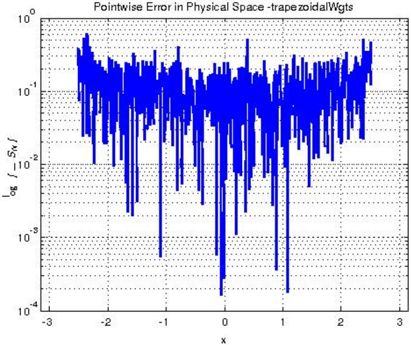


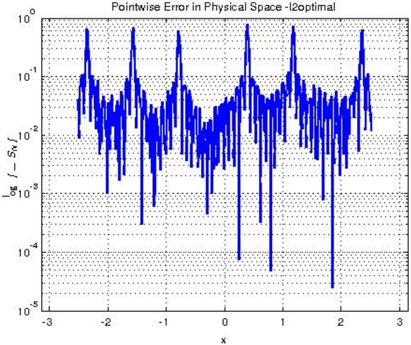


- ω_k logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR

Numerical Results

Reconstruction Error





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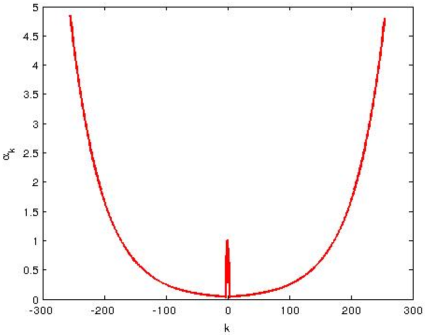
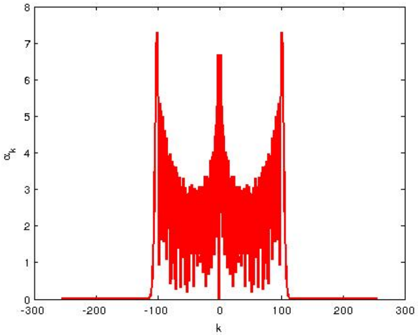
Numerical Results

DCF weights $\boldsymbol{\alpha}$





- ω_k logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR

Outline

Concentration Method (Gelb, Tadmor)

- Approximate the singular support of f using the *generalized conjugate partial Fourier sum*

$$S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}(k)\,\mathrm{sgn}(k)\,\sigma\left(\frac{|k|}{N}\right) e^{ikx}$$

- $\sigma_{k,N}(\eta) = \sigma(\frac{|k|}{N})$ are known as *concentration factors* which are required to satisfy certain admissibility conditions.
- Under these conditions,

$$S_N^\sigma[f](x) = [f](x) + \mathcal{O}(\epsilon), \quad \epsilon = \epsilon(N) > 0 \text{ being small}$$

i.e., $S_N^\sigma[f]$ concentrates at the singular support of f.

Concentration Factors

Factor	Expression
Trigonometric	$\sigma_T(\eta) = \frac{\pi \sin(\alpha\, \eta)}{Si(\alpha)}$ $Si(\alpha) = \int_0^{\alpha} \frac{\sin(x)}{x}\, dx$
Polynomial	$\sigma_P(\eta) = -p\, \pi\, \eta^p$ p is the order of the factor
Exponential	$\sigma_E(\eta) = C \eta \exp\left[\frac{1}{\alpha\, \eta\, (\eta - 1)}\right]$ C - normalizing constant α - order $C = \frac{\pi}{\int_{\frac{1}{N}}^{1-\frac{1}{N}} \exp\left[\frac{1}{\alpha\, \tau\, (\tau - 1)}\right] d\tau}$

Table: Examples of concentration factors

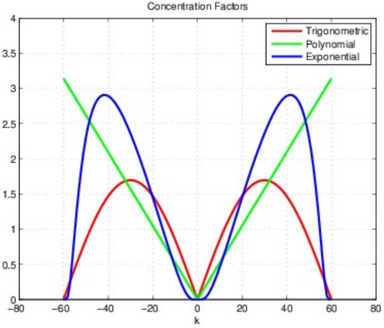
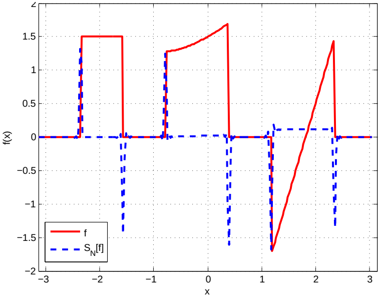
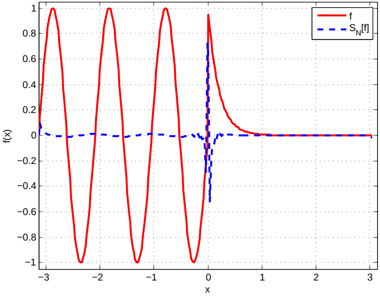


Figure:
Envelopes of Factors in k-space

Some Examples



(a) Trigonometric Factor



(b) Exponential Factor

Figure: Jump Function Approximation, $N = 128$

Designing Non-Harmonic Edge Detection Kernels

Choose $\boldsymbol{\alpha} = \{\alpha_k\}_{-N}^{N}$ such that

$$\sum_{|k|\leq N} \alpha_k e^{i\omega_k x} \approx \begin{cases} i \displaystyle\sum_{|\ell|\leq M} \mathsf{sgn}(l)\sigma(|l|/N)e^{i\ell x} & |x| \leq \pi \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

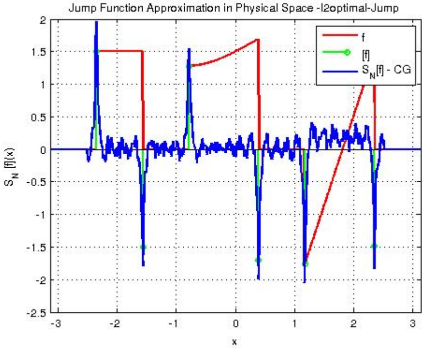
$$D\boldsymbol{\alpha} = \tilde{\mathbf{b}},$$

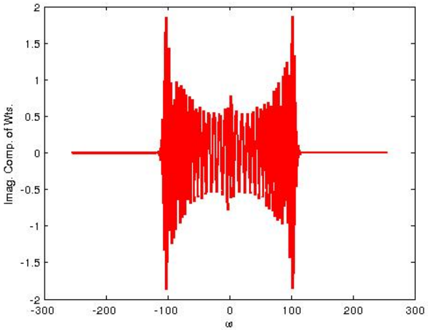
where

- $D_{\ell,j} = e^{i\omega_\ell x_j}$ denotes the (non-harmonic) DFT matrix, and
- $\tilde{b}_p$ are the values of the generalized conjugate Dirichlet kernel on the equispaced grid.

Numerical Results

Jump Approximation and Corresponding Weights





- ω_k logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR

Summary and Future Directions

1. Applications such as MR imaging require reconstruction from non-harmonic Fourier measurements.
2. Direct methods such as convolutional gridding are still of interest to the MR community.
3. A set of free parameters known as the density compensation factors (DCFs) allow us to design non-uniform reconstruction kernels with favorable characteristics.
4. To do – compare results with frame theoretic approaches, use banded DCFs to obtain better gridding approximations.

Selected References

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