Constructing Approximation Kernels for Non-Harmonic Fourier Data

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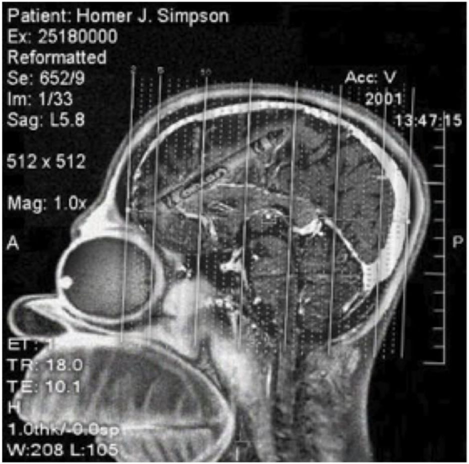
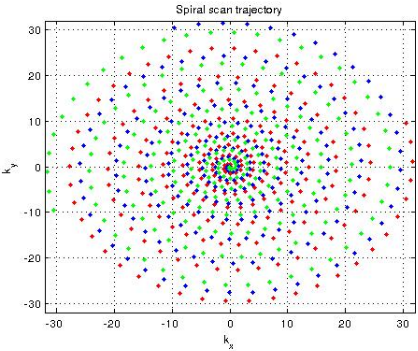
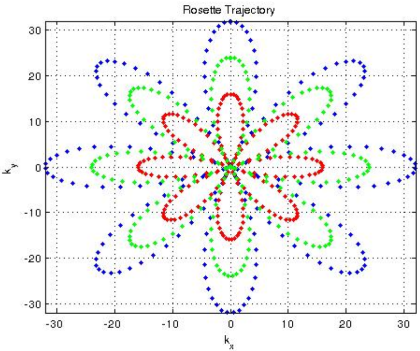
Motivating Application – Magnetic Resonance Imaging



Physics of MRI dictates that the MR scanner collect samples of the Fourier transform of the specimen being imaged.

Motivating Application – Magnetic Resonance Imaging





- Collecting non-uniform measurements has certain advantages; for example, they are easier and faster to collect, and, aliased images retain diagnostic qualities.
- Reconstructing images from such measurements accurately and efficiently is, however, challenging.

Model Problem

Let f be defined in $\mathbb{R}$ with support in $[-\pi, \pi)$. Given

$$\hat{f}(\omega_k) = \left\langle f, e^{i\omega_k x} \right\rangle, \quad k = -N, \cdots, N, \quad \omega_k \text{ not necessarily} \in \mathbb{Z},$$

compute

- an approximation to the underlying (possibly piecewise-smooth) function f,
- an approximation to the locations and values of jumps in the underlying function; i.e., $[f](x) := f(x^+) - f(x^-)$.

Issues

- Sparse sampling of the high frequencies, i.e., $|\omega_k - k| > 1$ for k large.
- The DFT is not defined for $\omega_k \neq k$; the FFT is not directly applicable.

Outline

Uniform Re-sampling (Rosenfeld)

- We consider *direct* methods of recovering f and $[f]$ from $\hat{f}(\omega_k)$.
- Due to our familiarity with harmonic Fourier reconstructions and the applicability of FFTs, we will consider a two step process:
    1. Approximate the Fourier coefficients at equispaced modes
    2. Compute a standard (filtered) Fourier partial sum

Basic Premise

f is compactly supported in physical space. Hence, the *Shannon sampling theorem* is applicable in Fourier space; i.e.,

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} \mathrm{sinc}(\omega - k)\hat{f}_k, \quad \omega \in \mathbb{R}.$$

Uniform Re-sampling – Implementation

We truncate the problem as follows

$$\hat{\mathbf{f}}(\omega_{\mathbf{k}}) \approx \sum_{|\ell|\leq M} \underbrace{\mathsf{sinc}(\omega_k - \ell)}_{A \,\in\, \mathbb{R}^{2N+1\times 2M+1}} \hat{\mathbf{f}}_\ell, \quad k = -N, \cdots, N$$

The equispaced coefficients are approximated using $\overline{\mathbf{f}}_\ell = A^\dagger \hat{\mathbf{f}}(\omega_{\mathbf{k}})$, where $A^\dagger$ is the Moore-Penrose pseudo-inverse of A.

- A and its properties characterize the resulting approximation.
- Regularization may be used (truncated SVD, Tikhonov regularization) in the presence of noise.
- $A^\dagger$ is (unfortunately) a dense matrix in general, with the computation of $\overline{\mathbf{f}}$ requiring $\mathcal{O}\left(N^2\right)$ operations.

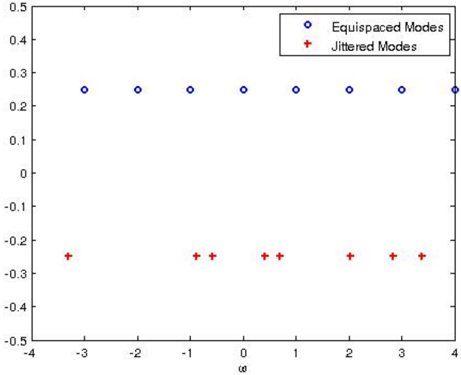
Uniform Re-sampling – Sampling Patterns

Consider the sampling pattern

$$\omega_k = k \pm U[0,\mu], \quad k = -N, \cdots, N$$

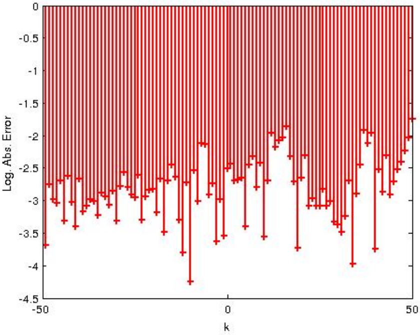
where $U[a,b]$ denotes an iid uniform distribution in $[a,b]$ with equiprobable positive/negative *jitter*.

Jitter μ	$\kappa(A)$
0.1	1.371
0.5	27.806
1.0	1.690×10^3
5.0	1.137×10^8
10.0	1.875×10^9

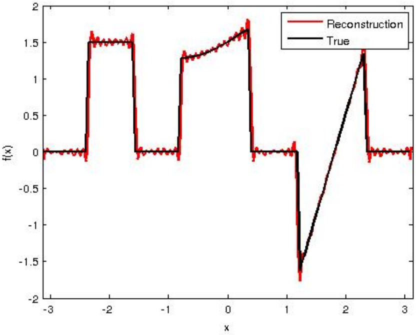


Uniform Re-sampling – An Example

Reconstruction using *jittered* samples ($\mu = 0.5$).



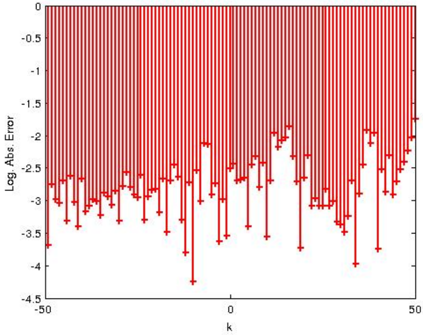
Error – Fourier Modes



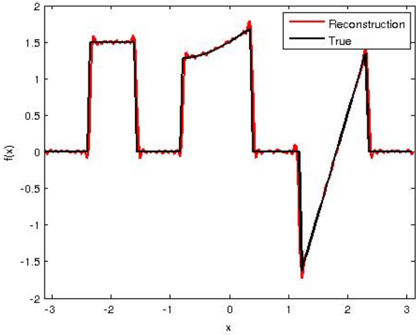
Recontruction

Uniform Re-sampling – An Example

Reconstruction using *jittered* samples ($\mu = 0.5$).



Error – Fourier Modes



Reconstruction

From Uniform Re-sampling to Convolutional Gridding

Recall that for uniform re-sampling, we use the relation

$$\hat{f}(\omega) = \sum_k \mathsf{sinc}(\omega - k)\hat{f}_k = (\hat{f} * \mathsf{sinc})(\omega)$$

Since the Fourier transform pair of the sinc function is the box/rect function (of width 2π and centered at zero), we have

$$f \cdot \Pi \longleftrightarrow \hat{f} * \mathsf{sinc}$$

Now consider replacing the sinc function by a bandlimited function $\hat{\phi}$ such that $\hat{\phi}(|\omega|) = 0$ for $|\omega| > q$ (typically a few modes wide). We now have

$$f \cdot \phi \longleftrightarrow \hat{f} * \hat{\phi}$$

Convolutional Gridding (Jackson/Meyer/Nishimura and others)

- Gridding is an inexpensive *direct* approximation to the uniform re-sampling procedure.
- Given measurements $\hat{f}(\omega_k)$, we compute an approximation to $\hat{f} * \hat{\phi}$ at the equispaced modes using

$$(\hat{f} * \hat{\phi})(\ell) \approx \sum_{|\ell - \omega_k| \leq q} \alpha_k \hat{f}(\omega_k) \hat{\phi}(\ell - \omega_k), \quad \ell = -M, \dots, M.$$

- Now that we are on equispaced modes, use a (F)DFT to reconstruct an approximation to $f \cdot \phi$ in physical space.
- Recover f by dividing out ϕ.
- α_k are density compensation factors (DCFs) and determine the accuracy of the reconstruction.

Analysis of the Convolution Gridding Sum

The gridding approximation can be written as

$$
\begin{aligned}
f_{cg}(x) &= \frac{\sum_{\ell \le M} \left(\sum_{|\ell-\omega_k| \le q} \alpha_k \hat{f}(\omega_k) \hat{\phi}(\ell - \omega_k) \right) e^{i\ell x}}{\phi(x)} \\
&= \frac{\sum_k \sum_\ell \alpha_k \left(\int f(\xi) e^{-i\omega_k \xi} d\xi \right) \left(\int \phi(\eta) e^{-i(\ell-\omega_k)\eta} d\eta \right) e^{i\ell x}}{\phi(x)} \\
&= \frac{\int \int f(\xi)\phi(\eta) \underbrace{\left(\sum_k \alpha_k e^{i\omega_k(\eta-\xi)} \right)}_{\color{red}{A_N^\alpha(\eta-\xi)}} \underbrace{\left(\sum_\ell e^{i\ell(x-\eta)} \right)}_{\color{blue}{D_N(x-\eta)}} d\xi d\eta}{\phi(x)} \\
&= \frac{\int (f * A_N^\alpha)(\eta)\, \phi(\eta)\, D_N(x-\eta) d\eta}{\phi(x)} \\
&= \frac{\left([\{f * A_N^\alpha\} \cdot \phi] * D_N \right)(x)}{\phi(x)}
\end{aligned}
$$

The Dirichlet Kernel – A Review

Given

$$\hat{f}_k := \left\langle f, e^{ikx} \right\rangle, \quad k = -N, \cdots, N,$$

a periodic repetition of f may be reconstructed using the Fourier partial sum

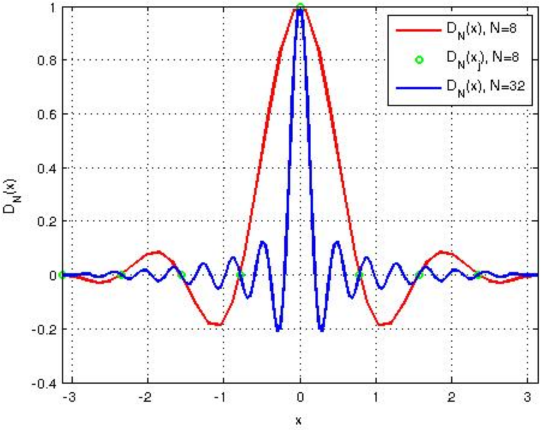
$$P_N f(x) = \sum_{|k| \leq N} \hat{f}_k e^{ikx} = (f * D_N)(x),$$

where

$$D_N(x) = \sum_{|k| \leq N} e^{ikx}$$

is the Dirichlet kernel. D_N is the bandlimited ($2N + 1$ mode) approximation of the Dirac delta distribution.

The Dirichlet Kernel – A Review



- D_N completely characterizes the Fourier approximation $P_N f$.
- Filtered and jump approximations are similarly characterized by equivalent filtered and (filtered) conjugate Dirichlet kernels.

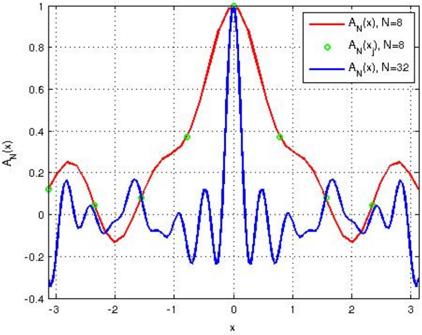
The Non-Harmonic Kernel

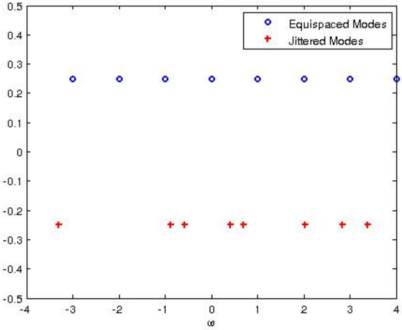
Consider the non-harmonic kernel

$$A_N(x) = \sum_{|k| \leq N} e^{i\omega_k x}$$

- A_N is non-periodic.
- The non-harmonic kernel is a poor approximation to the Dirac delta distribution.
- Depending on the mode distribution, A_N may be non-decaying.
- Filtering is of no help under these circumstances.

The Non-Harmonic Kernel

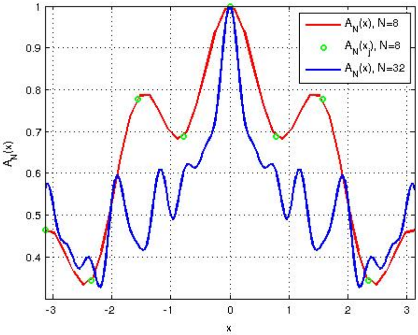


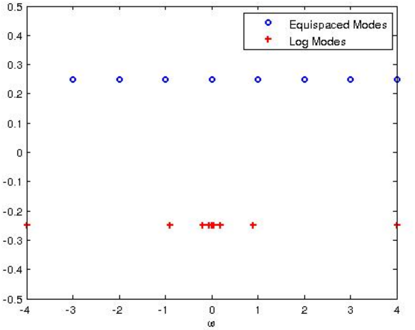


Jittered Modes

$$\omega_k = k \pm U[0, \mu], \quad \mu = 1.5$$

The Non-Harmonic Kernel





Log Modes

ω_k logarithmically spaced

Outline

Designing Gridding Kernels

- Recall that the convolutional gridding sum can be written in the form

$$f_{cg}(x) = \frac{\left([\{f * A_N^\alpha\} \cdot \phi] * D_N\right)(x)}{\phi(x)}$$

where the (weighted) non-harmonic kernel

$$A_N^\alpha(x) = \sum_{|k| \leq N} \alpha_k e^{i\omega_k x}.$$

- α_k are free design parameters which we choose such that A_N^α is compactly supported and a good reconstruction kernel (such as the Dirichlet kernel) in the interval of interest.

Design Problem – Formulation

Choose $\boldsymbol{\alpha} = \{\alpha_k\}_{-N}^{N}$ such that

$$\sum_{|k|\leq N} \alpha_k e^{i\omega_k x} \approx \begin{cases} \sum_{|\ell|\leq M} e^{i\ell x} & |x| \leq \pi \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

$$D\boldsymbol{\alpha} = \mathbf{b},$$

where

- $D_{\ell,j} = e^{i\omega_\ell x_j}$ denotes the (non-harmonic) DFT matrix, and
- $b_p = \frac{\sin(M+1/2)x_p}{\sin(x_p/2}) \cdot \Pi$ are the values of the Dirichlet kernel on the equispaced grid.

Design Problem – Formulation

Choose $\boldsymbol{\alpha} = \{\alpha_k\}_{-N}^{N}$ such that

$$\sum_{|k|\le N} \alpha_k e^{i\omega_k x} \approx \begin{cases} \sum_{|\ell|\le M} \sigma_\ell e^{i\ell x} & |x| \le \pi \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

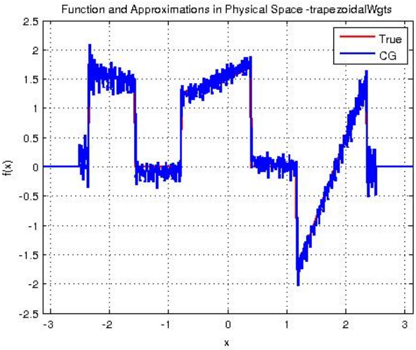
$$D\boldsymbol{\alpha} = \mathbf{b},$$

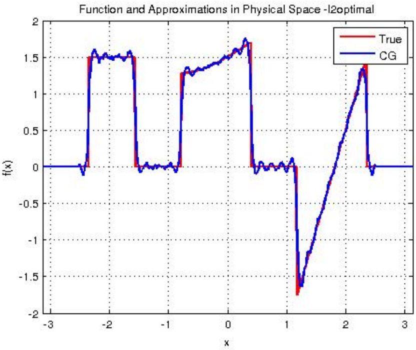
where

- $D_{\ell,j} = e^{i\omega_\ell x_j}$ denotes the (non-harmonic) DFT matrix, and
- b_p are the values of the (filtered) Dirichlet kernel on the equispaced grid.

Numerical Results

Reconstruction

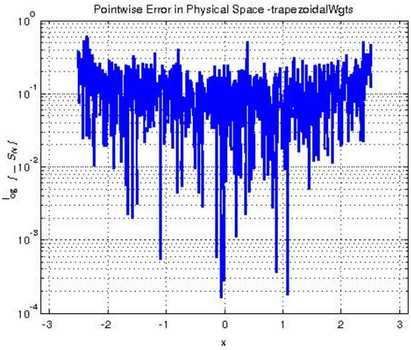


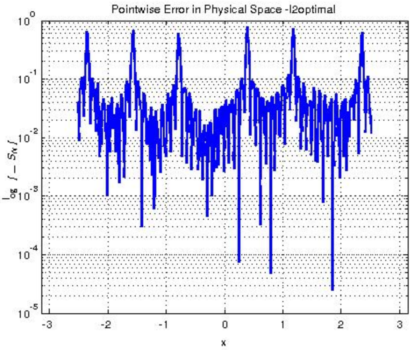


- ω_k logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR

Numerical Results

Reconstruction Error

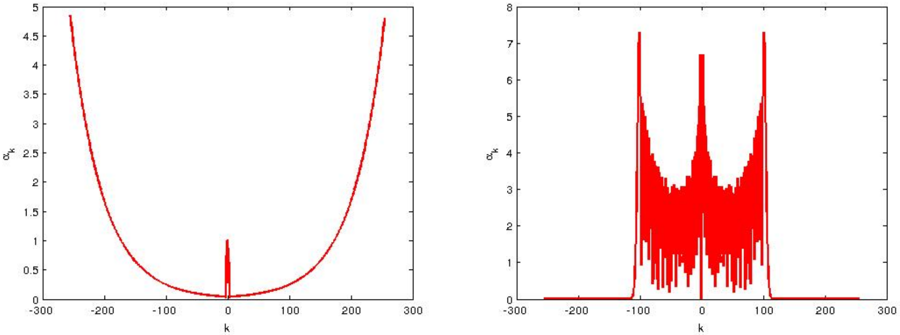




- ω_k logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR

Numerical Results

DCF weights $\boldsymbol{\alpha}$



- ω_k logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR

Outline

Concentration Method (Gelb, Tadmor)

- Approximate the singular support of f using the *generalized conjugate partial Fourier sum*

$$S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}(k)\,\mathsf{sgn}(k)\,\sigma\left(\frac{|k|}{N}\right)\, e^{ikx}$$

- $\sigma_{k,N}(\eta) = \sigma(\frac{|k|}{N})$ are known as *concentration factors* which are required to satisfy certain admissibility conditions.
- Under these conditions,

$$S_N^\sigma[f](x) = [f](x) + \mathcal{O}(\epsilon), \quad \epsilon = \epsilon(N) > 0 \text{ being small}$$

i.e., $S_N^\sigma[f]$ concentrates at the singular support of f.

Concentration Factors

Factor	Expression
Trigonometric	$\sigma_T(\eta) = \frac{\pi \sin(\alpha\,\eta)}{Si(\alpha)}$ $Si(\alpha) = \int_0^\alpha \frac{\sin(x)}{x}\,dx$
Polynomial	$\sigma_P(\eta) = -p\,\pi\,\eta^p$ p is the order of the factor
Exponential	$\sigma_E(\eta) = C\eta\exp\left[\frac{1}{\alpha\,\eta\,(\eta-1)}\right]$ C - normalizing constant α - order $C = \frac{\pi}{\int_{\frac{1}{N}}^{1-\frac{1}{N}} \exp\left[\frac{1}{\alpha\,\tau\,(\tau-1)}\right] d\tau}$

Table: Examples of concentration factors

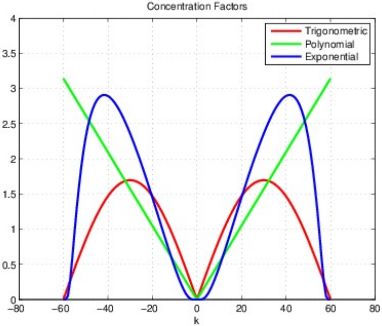
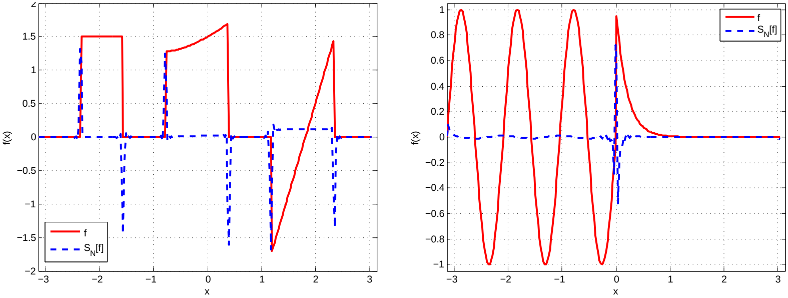


Figure:
Envelopes of Factors in k-space

Some Examples



(a) Trigonometric Factor (b) Exponential Factor

Figure: Jump Function Approximation, $N = 128$

Designing Non-Harmonic Edge Detection Kernels

Choose $\boldsymbol{\alpha} = \{\alpha_k\}_{-N}^{N}$ such that

$$\sum_{|k|\le N} \alpha_k e^{i\omega_k x} \approx \begin{cases} i \sum_{|\ell|\le M} \mathsf{sgn}(l)\sigma(|l|/N)e^{i\ell x} & |x| \le \pi \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

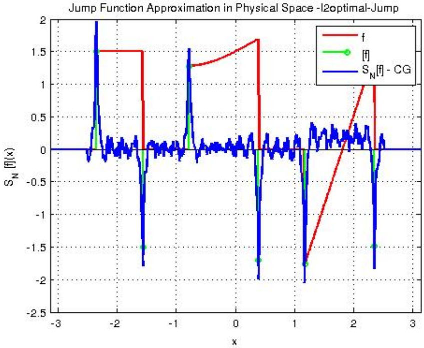
$$D\boldsymbol{\alpha} = \tilde{\mathbf{b}},$$

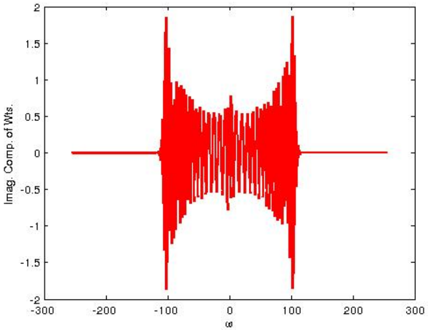
where

- $D_{\ell,j} = e^{i\omega_\ell x_j}$ denotes the (non-harmonic) DFT matrix, and
- $\tilde{b}_p$ are the values of the generalized conjugate Dirichlet kernel on the equispaced grid.

Numerical Results

Jump Approximation and Corresponding Weights





- ω_k logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR

Summary and Future Directions

1. Applications such as MR imaging require reconstruction from non-harmonic Fourier measurements.
2. Assuming the function of interest is compactly supported, the underlying relation between non-harmonic and harmonic Fourier data is the Shannon sampling theorem (sinc interpolation).
3. Convolutional gridding is an efficient approximation to sinc-based resampling.
4. A set of free parameters known as the density compensation factors (DCFs) allow us to design gridding kernels with favorable characteristics.
5. To do – compare results with frame theoretic approaches, use banded DCFs to obtain better gridding approximations.

Selected References

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