Fourier Reconstruction

Aditya Viswanathan
Department of Electrical Engineering
Arizona State University
aditya.v@asu.edu
With Profs. Anne Gelb, Douglas Cochran and Rosemary Renaut

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Motivation – MR Imaging

Fourier Expansions
- Basis Expansions
- Gibbs Phenomenon
  - Manifestation
  - Resolution

Fourier Methods for Irregularly Sampled Data
- The Problem with non-harmonic Fourier Data
- Reconstruction Methods

MR Imaging

- NMR observed in atoms with odd number of protons or neutrons
- In the presence of an external magnetic field, their angular momenta ("spins") align to yield a net magnetic moment in the direction of the field
- They also rotate or *precess* at a frequency known as the Larmor frequency
- Now, the spins are excited by a secondary, momentary RF pulse
- The signal generated as the spins return to the equilibrium state is recorded
- The signal is proportional to concentration of the atoms in the imaged sample

MR Imaging – Localization

- Localization is achieved by only exciting those volumes at a particular Larmor frequency
- We vary the Larmor frequency across a specimen by applying field gradients known as frequency and phase encoding gradients

At a particular slice, say $z = z_0$, the acquired MR signal can be written as

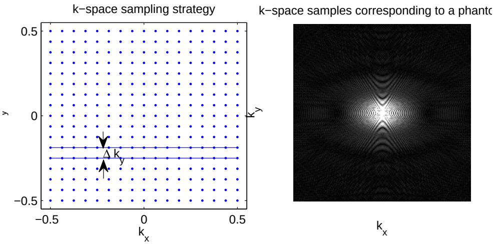
$$S(k_x, k_y) = \int \int \rho(x, y)\, e^{-i(k_x\, x + k_y\, y)}\, dx\, dy$$

where, $\rho(x, y)$ is a measure of the concentration of spins, and k_x, k_y take the form

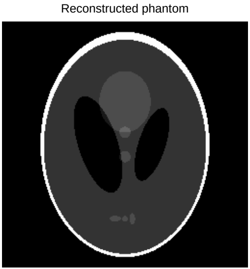
$$k_x = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau, \quad k_y = \frac{\gamma}{2\pi} \int_0^t G_y(\tau) d\tau$$

Here, G_x and G_y are the applied gradient fields and γ is a constant known as the Gyromagnetic ratio, which is unique to each atom. Further, it is convention to denote the signal acquisition space $\mathbf{k} = (k_x, k_y)$ as "k-space".

MR Imaging



(a) k-space samples acquired by the MR scanner



(b) Reconstructed Image

Figure: k-space acquisition

Issues in MR Scanning

- $k_x = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau \implies$ each measurement in k-space is acquired at a slightly different time
- If the patient moves in the course of the scan, reconstruction results can be poor
- We reconstruct piecewise-smooth images; eg., the human brain has the skull, tissue boundaries etc. Fourier reconstruction of piecewise-smooth functions suffers from the Gibbs phenomenon
- Non-Cartesian scanning trajectories are becoming increasingly popular – reconstruction is not straightforward

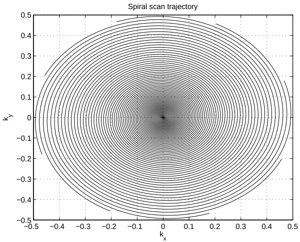


Figure: Non-Cartesian k-space acquisition (Data from Dr. Jim Pipe, Barrow Neurological Institute, Phoenix, AZ)

Basis Expansions

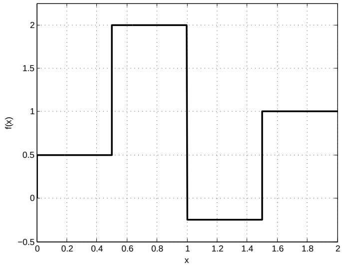


Figure: Function f

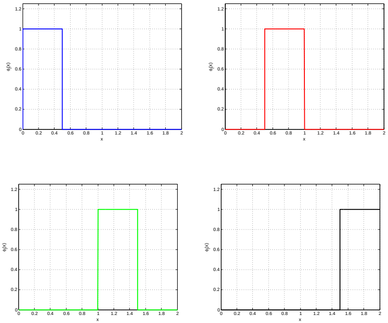


Figure: Basis functions ϕ_i

Analysis: $c_i = \langle f, \phi_i \rangle = \int_0^2 f(x)\phi_i(x)dx$

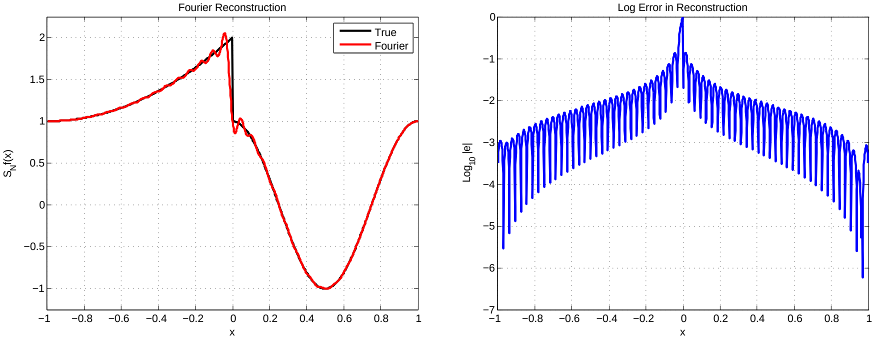
Synthesis: $f = \sum_i c_i \phi_i(x)$

The Gibbs Phenomenon

- Occurs in the Fourier reconstruction of piecewise-analytic functions
- result of reconstructing piecewise-analytic functions using smooth basis functions
- Two important consequences
  - Non-unifrom convergence – presence of non-physical oscillations in the vicinity of discontinuities
  - Reduced order of convergence – first order convergence even in smooth regions of the reconstruction
- Why is this important? smearing of sharp edges and oscillations in reconstruction cause problems in post-processing tasks like segmentation, edge detection etc.
- The reduced order of convergence means we need a lot of Fourier coefficients to get a good reconstruction

Example

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}, \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$



(a) Reconstruction (b) Reconstruction error

Figure: Gibbs Phenomenon, $N = 32$

Example

The Gibbs phenomenon does not go away by increasing the number of data points

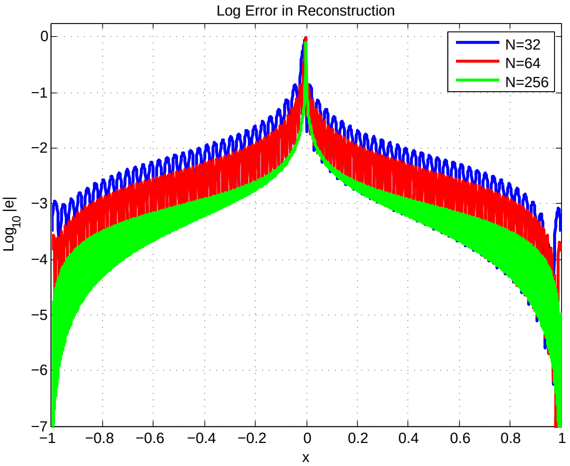
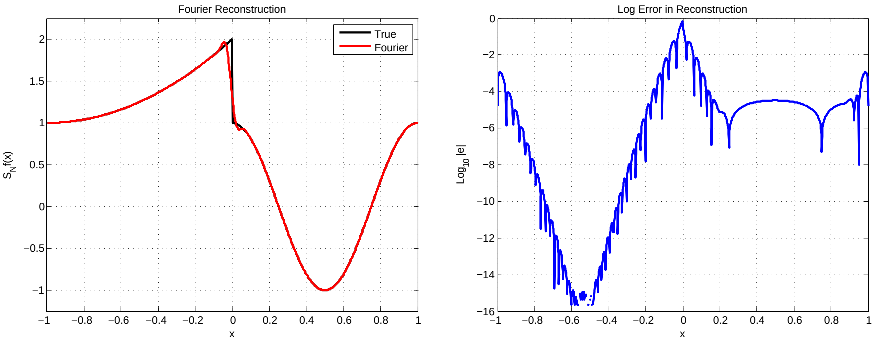


Figure: Reconstruction error for different N

Filtered Fourier Reconstructions

Filtering helps to ameliorate the effects of Gibbs, but does not eliminate it. In fact, it introduces a smearing artifact in the vicinity of a discontinuity.

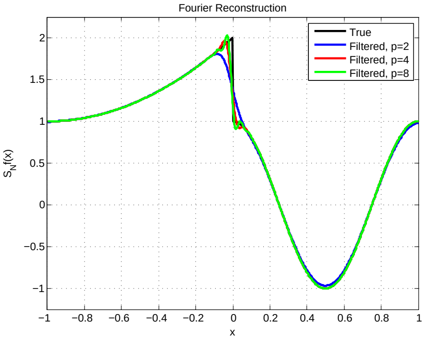


(a) Filtered Reconstruction

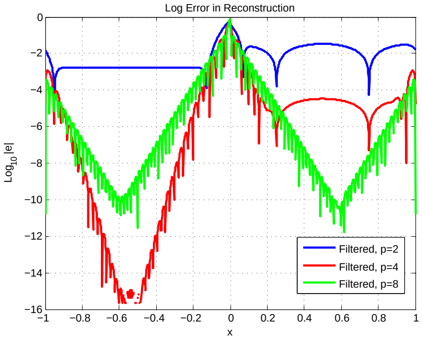
(b) Reconstruction error

Figure: Exponentially Filtered Reconstruction, $p = 2, N = 64$

Filtered Fourier Reconstructions



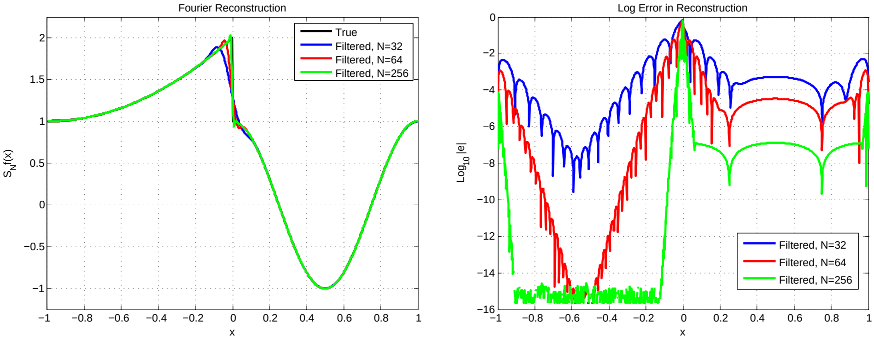
(a) Filtered Reconstruction



(b) Reconstruction error

Figure: Exponentially Filtered Reconstructions, $N = 64$

Filtered Fourier Reconstructions



(a) Filtered Reconstruction

(b) Reconstruction error

Figure: Exponentially Filtered Reconstructions, $p = 4$

Spectral Reprojection

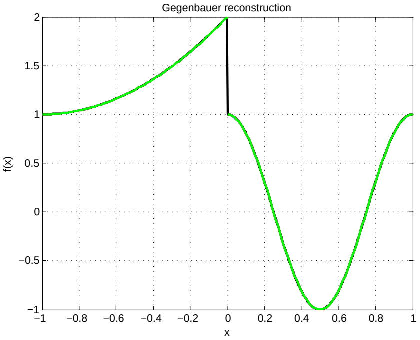
- Spectral reprojection schemes were formulated to resolve the Gibbs phenomenon. They involve reconstructing the function using an alternate basis, Ψ (known as a Gibbs complementary basis).
- Reconstruction is performed using the rapidly converging series

$$f(x) \approx \sum_{l=0}^{m} c_l \psi_l(x), \quad \text{where} \quad c_l = \frac{\langle f_N, \psi_l \rangle_w}{\|\psi_l\|_w^2},$$

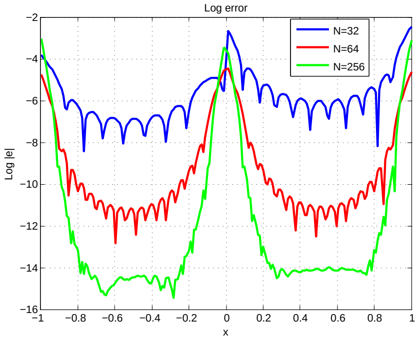
f_N is the Fourier expansion of f

- Reconstruction is performed in each smooth interval. Hence, we require jump discontinuity locations
- High frequency modes of f have exponentially small contributions on the low modes in the new basis

Spectral Reprojection



(a) Reconstruction



(b) Reconstruction error

Figure: Spectral Reprojection Reconstructions

Non-harmonic Fourier Data

- Periodic functions have a pure point spectra – $\mathcal{F}\{f\} = \hat{f}(k),\ k \in \mathbb{Z}$
- Compactly supported functions have a Fourier trasform

$$\mathcal{F}\{f\} = \hat{f}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i\omega x}dx, \quad \omega \in \mathbb{R}$$

- They can, however, be reconstructed from uniform samples of the Fourier transform taken at a sufficiently regular intervals
- We refer to the coefficients acquired at non-uniform intervals (non-integer nodes for a 2π-periodic function) as non-harmonic Fourier coefficients
- For many non-uniform sample locations, there are results to show that $\left\{e^{i\omega_k x}\right\}_{k=-N}^{N}$ does not form a basis
  $\implies$ we may not have sufficient or acceptable data to form a reconstruction

Non-harmonic Fourier Series

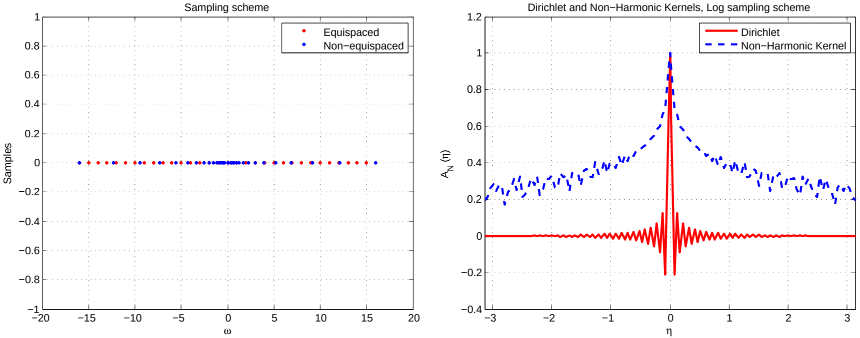
Let ω_k be non-equispaced data points with

$$\mathcal{F}\left\{f\right\}\big|_{\omega_k} = \hat{f}(\omega_k) = \frac{1}{2\pi}\int_{-a}^{a} f(x)e^{-i\omega_k x}dx, \quad \omega \text{ not necessarily } \in \mathbb{Z}$$

The "non-equispaced sum" yields,

$$\begin{aligned}
\tilde{f}(x) &= \sum_{k=0}^{N-1} \hat{f}(\omega_k)e^{i\omega_k x} \\
&= \sum_{k=0}^{N-1} \left(\int_{-\infty}^{\infty} f(u)e^{-i\omega_k u}du\right) e^{i\omega_k x} \\
&= \int_{-\infty}^{\infty} f(u) \left(\sum_{k=0}^{N-1} e^{i\omega_k (x-u)}\right) du \\
\tilde{f}(x) &= (f * A_N)(x)
\end{aligned}$$

Non-harmonic Fourier Series

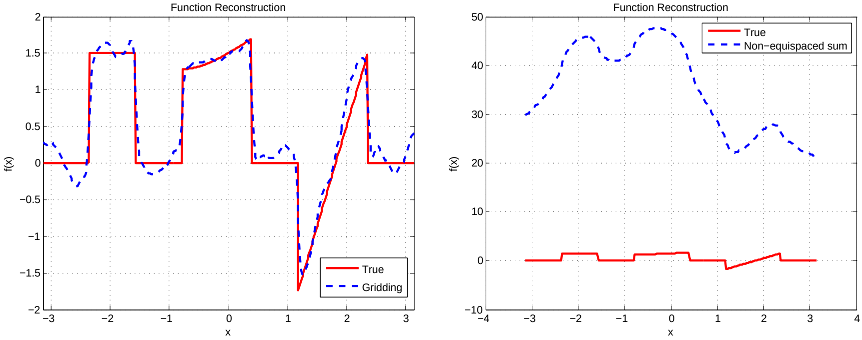


(a) Sampling scheme

(b) Non-harmonic kernel, $N = 128$

Figure: The non-harmonic Fourier kernel

Function Reconstructions



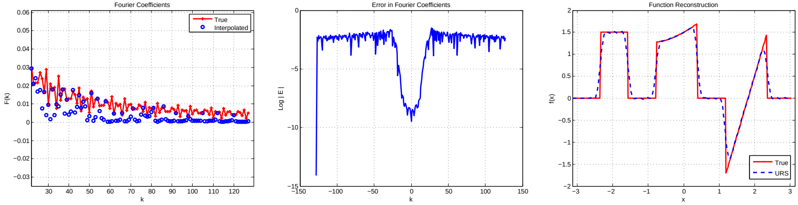
(a) Jittered Samples

(b) Samples acquired at log spacing

Figure: Reconstructions using the non-harmonic Fourier sum, $N = 128$

A Reconstruction Method for Non-uniform Fourier Data

- Recover equispaced coefficients from the non-uniform measurements using interpolation or by solving a system of equations
- Reconstruct using standard Fourier/filtered Fourier methods



(a) Recovered Fourier Coefficients – High modes

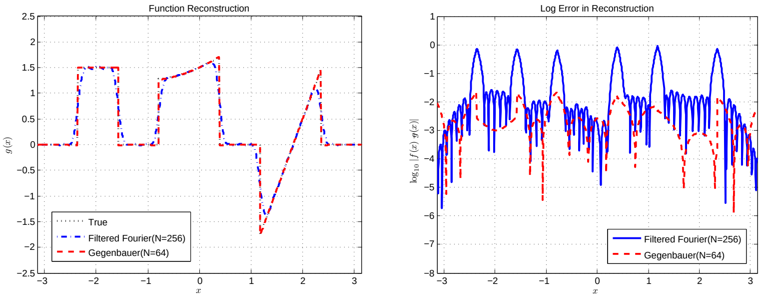
(b) Error in recovered coefficients

(c) Reconstruction

Figure: URS Solution, $N = 128$

Gegenbauer Reconstruction - Representative Results

Compute a Gegenbauer reconstruction using the accurately recovered equispaced Fourier coefficients



(a) Reconstruction (b) Reconstruction error

Figure: Gegenbauer reconstruction

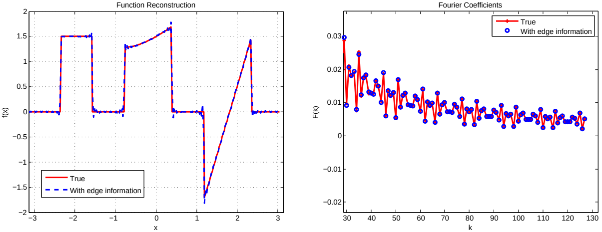
- Filtered Fourier reconstruction uses 256 coefficients
- Gegenbauer reconstruction uses 64 coefficients
- Parameters in Gegenbauer Reconstruction - $m = 2, \lambda = 2$

Methods Incorporating Edge Information

Let us assume that we have access to jump information – jump location and value

Compute the high frequency modes using the relation

$$\hat{f}(k) = \sum_{p \in \mathcal{P}} [f](\zeta_p) \, \frac{e^{-ik\zeta_p}}{2\pi i k}$$



(a) Reconstruction - Using edge information

(b) The high modes - Using edge information

Figure: Reconstruction of a test function using edge information

Summary

- We looked at Fourier reconstruction with MR imaging serving as a motivating application
- We discussed the Gibbs phenomenon and its mitigation
- We introduced spectral reprojection – a method to resolve the Gibbs phenomenon
- We discussed the reconstruction of functions from non-harmonic spectral data