On Fourier Reconstruction from Non-Uniform Spectral Data

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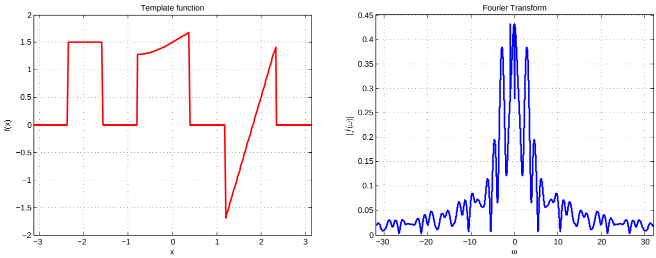
Trondheim, Norway

Motivating Example



(a) Template Function (b) Fourier Transform

Figure: A motivating example

- Fourier samples violate the quadrature rule for discrete Fourier expansion
- Computational issue – no FFT available
- Mathematical issue – given these coefficients, can we/how do we reconstruct the function?
- Resolution – what accuracy can we achieve given a finite (usually small) number of coefficients?

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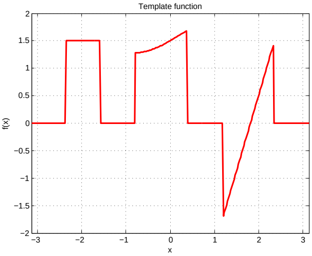

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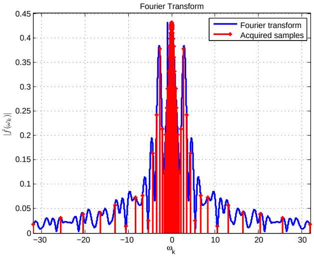

(b) Fourier Coefficients, $N = 32$

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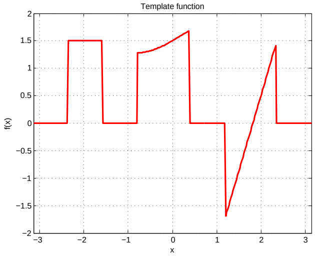

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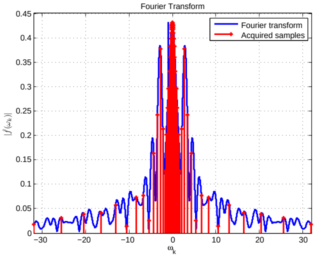

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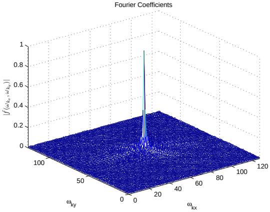
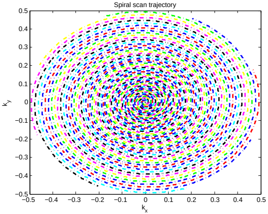
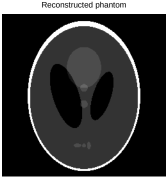
Application – Magnetic Resonance Imaging



(a) Acquired Fourier Samples



(b) Sampling Trajectory



(c) Reconstructed Image

Figure: MR Imaging

Non-Cartesian sampling trajectories have some advantages

- greater resistance to motion artifacts
- instrumentation concerns – ease in generating gradient waveforms

In this Talk

We will discuss

- Issues with non-harmonic Fourier reconstruction
- Conventional reconstruction methods
- Accuracy vs Sampling Density
- Spectral Re-projection methods
- Incorporating edge information in the reconstruction

Outline

Problem Formulation

- Let f be defined on $\mathbb{R}$ and supported in $(-\pi, \pi)$
- It has a Fourier transform representation, $\hat{f}(\omega)$, defined as

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i\omega x} dx, \quad \omega \in \mathbb{R}$$

Objective

Recover f given a finite number of its non-harmonic Fourier coefficients,

$$\hat{f}(\omega_k), \quad k = -N, ..., N \quad \omega_k \text{ not necessarily } \in \mathbb{Z}$$

- We will refer to $\{\omega_k\}_{-N}^{N}$ as the sampling pattern/trajectory
- We will be particularly interested in sampling patterns with variable sampling density
- We will pay special attention to piecewise-smooth f

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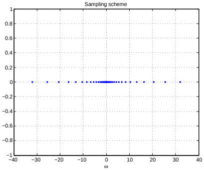

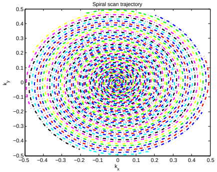

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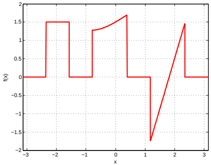

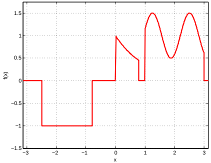

Non-harmonic Fourier Series

- The data we are given correspond to $\langle f, e^{i\omega_k x}\rangle,\ k = -N, ..., N$
- $\{e^{i\omega_k x}\}$ may not constitute a basis for arbitrary sampling patterns
- Classical works by Paley-Weiner, Kadec and others show that for $\{e^{i\omega_k x}\}$ to be a basis

$$\sup_k |\omega_k - k| < \frac{1}{4}$$

- Even if they constitute a basis, the dual basis is not $\{e^{-i\omega_k x}\}$, but a numerically unstable Lagrange-type polynomial
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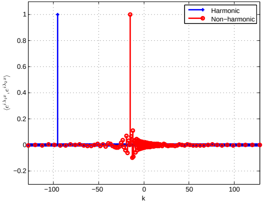

Figure: Plot of inner products $\langle e^{i\omega_m x}, e^{i\omega_n x}\rangle$

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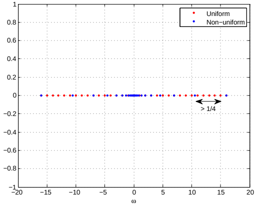

Figure: Representative sampling scheme

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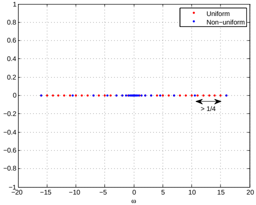

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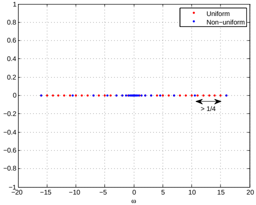

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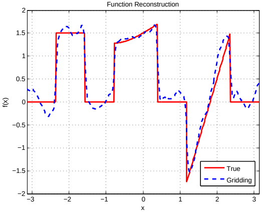
Non-harmonic Fourier Reconstruction – Naive Methods

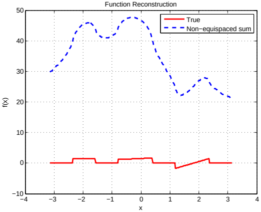
Reconstruction methods which do not work

- setting non-harmonic coefficients to zero
- linear or other general interpolation schemes for coefficients
- "Non-harmonic" Fourier partial sum

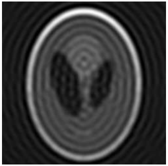
$$S_N \tilde{f}(x) := \sum_{k=-N}^{N} \hat{f}(\omega_k) e^{i\omega_k x}$$



(a) "Jittered" Sampling



(b) "Log" Sampling



(c) "Spiral" Sampling

Figure: Non-harmonic Fourier sum Reconstruction, $N = 128$

Outline

Conventional Reconstruction Methods

Several approaches available to perform reconstruction

- Convolutional gridding – most popular
- Uniform resampling
- Iterative Methods

– "Fix" the quadrature rule while evaulating the non-harmonic sum

$$S_N \tilde{f}(x) = \sum_{k=-N}^{N} \alpha_k \hat{f}(\omega_k) e^{ikx}$$

– α_k are density compensation factors
– Evaluated using a "non-uniform" FFT

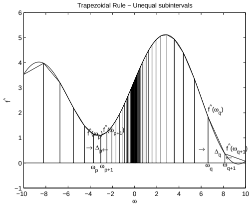


Figure: Evaluating the non-uniform Fourier sum

Although there are distinct difference in methodology and computational cost, reconstruction accuracy is similar in most schemes. We will look at uniform re-sampling (URS) to obtain an intuitive understanding of the problems in reconstruction

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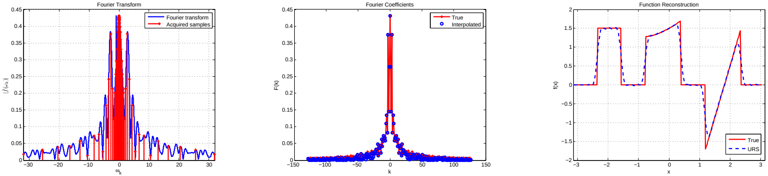

(a) Non-harmonic modes (b) Obtain uniform modes (c) (Filtered) Fourier sum

Figure: Uniform Resampling

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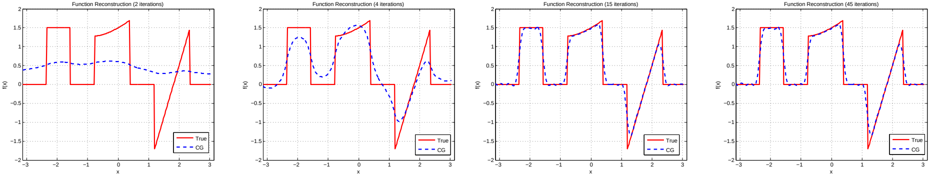

(a) after iteration 2 (b) after iteration 4 (c) after iteration 15 (d) after iteration 45

Figure: Iterative Reconstruction

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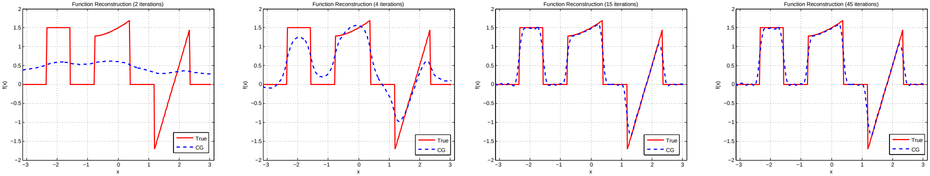

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Uniform Re-sampling

Reconstruction is accomplished in two steps:

1. recover equispaced coefficients $\hat{f}(k)$
2. partial Fourier reconstruction using the FFT algorithm

Since f is compactly supported, we use the sampling theorem to relate $\hat{f}(\omega_k)$ and $\hat{f}(k)$.

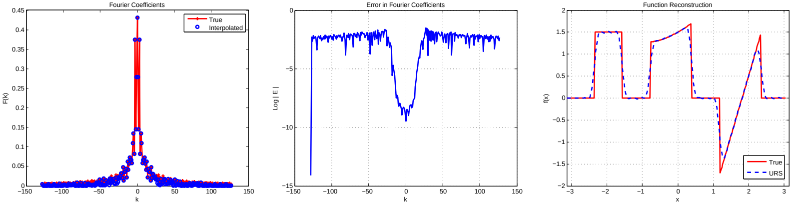
$$\hat{f}(\omega) = \sum_{p=-\infty}^{\infty} \hat{f}(p)\, \mathrm{sinc}(\omega - p), \quad \omega \in \mathbb{R}, p \in \mathbb{N}$$

- To recover $\hat{f}(k)$, we have to invert the above system, i.e., solve

$$Ax = b, \quad A_{ij} = \mathrm{sinc}(\omega_i - j), \quad b = \left\{ \hat{f}(\omega_k) \right\}_{k=-N}^{N}, \quad x = \left\{ \hat{f}(p) \right\}_{p=-M}^{M}$$

- Any number of methods to do so - iterative methods, pseudoinverse-based methods with regularization ...
- In problems like MRI, pseudoinverse-based methods are preferred for computational purposes - the pseudoinverse can be precomputed for a given sampling scheme

Representative Results



(a) Recovered Fourier coefficients (b) Error in recovered coefficients (c) Filtered reconstruction

Figure: URS solution, $N = 128$

- Solved a square 128×128 system
- Inverted the system by computing the pseudoinverse
- Pseudoinverse was computed using TSVD, with a SVD threshold of 10^{-5}

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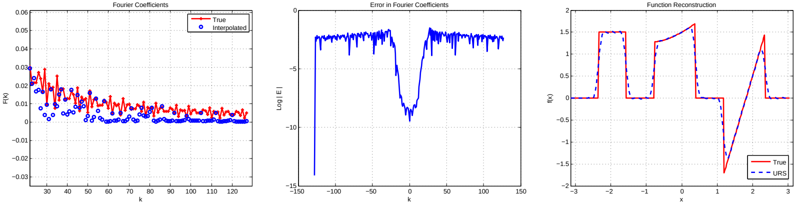

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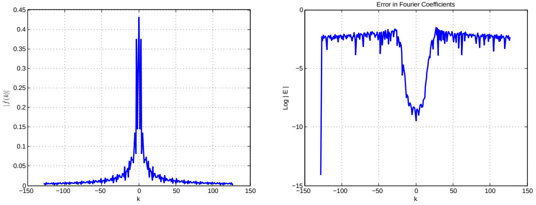
Error vs Sampling Density

Let $\hat{f}(k)$ denote the true coefficients and $\hat{g}(k)$ the recovered equispaced coefficients. The error in the reconstruction can be written as

$$e(x) = \sum_{|k|>N} \hat{f}(k)e^{ikx} + \sum_{|k|\leq N} \left(\hat{f}(k) - \hat{g}(k)\right) e^{ikx}$$

- this term decreases as N increases
- this term increases as N increases



(a) Fourier coefficients (b) Coefficient error

Figure: Error in uniform re-sampling

For a given sampling trajectory and function, there is a critical value N_{crit} beyond which adding coefficients does not improve the accuracy.
While filtering decreases the error, the underlying problem is not solved.

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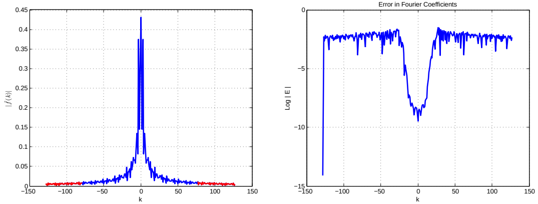

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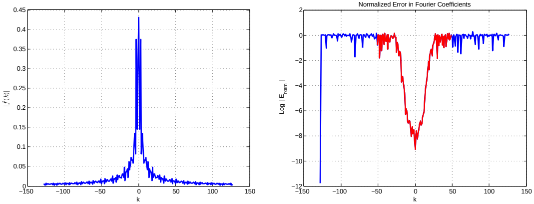

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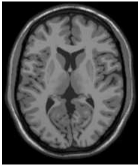
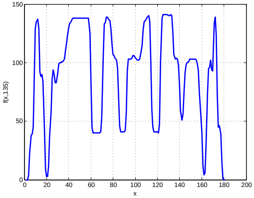
Piecewise-Smooth Functions



(a) Brain scan



(b) Cross-section of a scan

Figure: Peicewise-smooth nature of medical images

- Due to the Gibbs phenomenon, we have non-physical oscillations at discontinuities, and, more importantly, reduced order of convergence (first order). Hence, we require a large number of coefficients to get acceptable reconstructions.
- However, by formulation of the sampling scheme and recovery procedure, the coefficients recovered at large ω are inaccurate.

$\Longrightarrow$ we need more coefficients, but the coefficients we get are inaccurate!

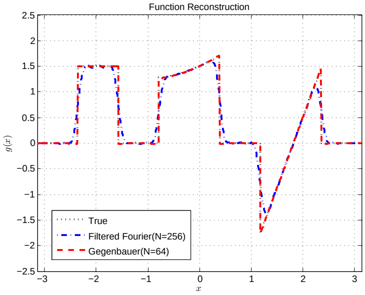
Spectral Re-projection

- Spectral reprojection schemes were formulated to resolve the Gibbs phenomenon. They involve reconstructing the function using an alternate basis, Ψ (known as a Gibbs complementary basis).
- Reconstruction is performed using the rapidly converging series

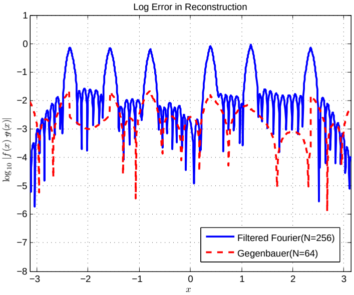
$$f(x) \approx \sum_{l=0}^{m} c_l \psi_l(x), \quad \text{where} \quad c_l = \frac{\langle f_N, \psi_l \rangle_w}{\|\psi_l\|_w^2},$$ f_N is the Fourier expansion of f

- Reconstruction is performed in each smooth interval. Hence, we require jump discontinuity locations
- High frequency modes of f have exponentially small contributions on the low modes in the new basis

Gegenbauer Reconstruction - Results



(a) Reconstruction



(b) Reconstruction error

Figure: Gegenbauer reconstruction

- Filtered Fourier reconstruction uses 256 coefficients
- Gegenbauer reconstruction uses 64 coefficients
- Parameters in Gegenbauer Reconstruction - $m = 2, \lambda = 2$

Getting Jump Data from Fourier Coefficients

Let f contain a single jump at $x = \zeta$.

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\zeta^-} f(x)e^{-ikx}dx + \int_{\zeta^+}^{\pi} f(x)e^{-ikx}dx\right) \\
&= \frac{1}{2\pi}\left(f(x)\,\frac{e^{-ikx}}{-ik}\Big|_{-\pi}^{\zeta^-} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx + f(x)\,\frac{e^{-ikx}}{-ik}\Big|_{\zeta^+}^{\pi} - \int_{\zeta^+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\right) \\
&= \frac{1}{2\pi}\Bigg(\frac{f(\zeta^-)e^{-ik\zeta} - f(-\pi)e^{ik\pi}}{-ik} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx \\
&\qquad + \frac{f(\pi)e^{-ik\pi} - f(\zeta^+)e^{-ik\zeta}}{-ik} - \int_{\zeta^+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\Bigg) \\
&= \left(f(\zeta^+) - f(\zeta^-)\right)\frac{e^{-ik\zeta}}{2\pi ik} + \frac{f(-\pi)e^{ik\pi} - f(\pi)e^{-ik\pi}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right)
\end{aligned}
$$

Since f is periodic, $f(-\pi) = f(\pi)$ and the second term vanishes.

$$
\hat{f}(k) = [f](\zeta)\,\frac{e^{-ik\zeta}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right)
$$

Obtaining Edge Information

- Solve the following equation

$$\hat{f}(k) = \sum_{p \in \mathcal{P}} [f](\zeta_p) \, \frac{e^{-ik\zeta_p}}{2\pi i k}$$

- Use the concentration method on the recovered coefficients

$$S_N^{\sigma}[g](x) = i \sum_{k=-N}^{N} \hat{g}(k) \, \mathrm{sgn}(k) \, \sigma\left(\frac{|k|}{N}\right) e^{ikx}$$

- Solve for the jump function directly from the non-harmonic Fourier data

$$\begin{aligned} \min \quad & \| \, [f] \, \|_1 \\ \text{s.t.} \quad & \mathcal{F}\{[f]\}\big|_{\omega_k} = i \, \mathrm{sgn}(\omega) \, \sigma\left(\tfrac{|\omega|}{N}\right) \, \hat{f}\Big|_{\omega_k} \end{aligned}$$

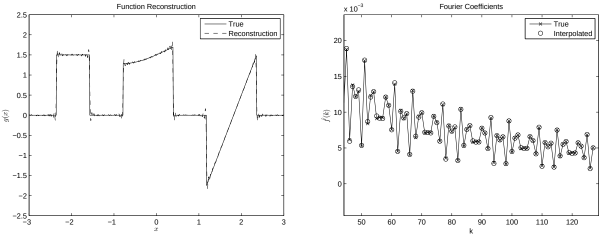
Methods Incorporating Edge Information

Suppose we have access to discontinuity locations, η_j and magnitudes, $[f](\eta_j)$.

$$\text{Let} \quad \hat{g}(k) = \sum_j [f](\eta_j) \frac{e^{-ik\eta_j}}{2\pi ik}, \quad y = \{\hat{g}(k)\}_{k=-N}^{N}$$

$$\text{Solve} \quad \min \|x - y\|_2 \quad \text{subject to} \quad \|Ax - b\|_2 < \sigma$$

Notation



(a) Reconstruction - Using edge information

(b) The high modes - Using edge information

Figure: Reconstruction of a test function using edge information

Compare

Summary

- We introduced the Fourier reconstruction problem for non-uniform spectral data
- We discussed the inherent problems associated with non-uniform Fourier data
- We briefly looked at conventional reconstruction methods
- We studied the error characteristics and relation to sampling density
- We looked at spectral re-projection and methods incorporating edge information to obtain better reconstructions