NAME:____________________________________

Math 153H-01 **Quiz 6** **March 2, 2016**

No calculators, no notes, no books. Only pens, pencils and erasers are allowed.

1. Find the center of mass of the region under the graph of $\sqrt{1-x^2}$ on $[0,1]$.

Solution: $f(x)=\sqrt{1-x^2}$. $\int_0^1 f(x)dx$ is one quarter of the area of a circle of radius 1, i.e. $\int_0^1 f(x)dx=\frac{\pi}{4}$.

$\int f(x)xdx = \int x\sqrt{1-x^2}dx$. Substitute $u=1-x^2$ and then $du=-2xdx$. Then $\int x\sqrt{1-x^2}dx=\int -\frac{1}{2}u^{\frac{1}{2}}du=-\frac{1}{3}u^{\frac{3}{2}}=-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$. Therefore $\int_0^1 f(x)xdx=\frac{1}{3}$.

$\int \frac{1}{2}f(x)^2dx=\int\frac{1}{2}(1-x^2)dx=\frac{1}{2}x-\frac{1}{6}x^3+C$. Therefore $\int_0^1\frac{1}{2}f(x)^2dx=\frac{1}{3}$.

Now $\bar{x}=\frac{\int_0^1 f(x)xdx}{\int_0^1 f(x)dx}=\frac{4}{3\pi}$ and $\bar{y}=\frac{\int_0^1\frac{1}{2}f(x)^2dx}{\int_0^1 f(x)dx}=\frac{4}{3\pi}$.

2. Compute the volume of the solid obtained by revolving the circle of radius 1 and center $(3,0)$ about the y-axis by the shell method.

Solution: For each $x\in[2,4]$, consider the line segment between $(x,\sqrt{1-(x-3)^2})$ and $(x,-\sqrt{1-(x-3)^2})$. This line segment is of length $2\sqrt{1-(x-3)^2}$. By revolving the circle about the y-axis, this line segment turns into a shell of area $4\pi x\sqrt{1-(x-3)^2}$. The volume of the resulting solid is the integral of those areas $\int_2^4 4\pi x\sqrt{1-(x-3)^2}dx$. Substitute $u=x-3$ and then the integral becomes $\int_{-1}^1 4\pi(u+3)\sqrt{1-u^2}dx=\int_{-1}^1 12\pi\sqrt{1-u^2}dx+\int_{-1}^1 4\pi u\sqrt{1-u^2}dx$. Since $4\pi u\sqrt{1-u^2}$ is an anti-symmetric function, $\int_{-1}^1 4\pi u\sqrt{1-u^2}dx=0$. Therefore we need to compute $\int_{-1}^1 12\pi\sqrt{1-u^2}dx=12\pi\int_{-1}^1\sqrt{1-u^2}du$. The integral $\int_{-1}^1\sqrt{1-u^2}du$ is one half the area of a circle of radius 1, and so $\int_{-1}^1\sqrt{1-u^2}du=\frac{1}{2}\pi$. Hence the volume is $\int_{-1}^1 12\pi\sqrt{1-u^2}dx=6\pi^2$.