NAME:____

Math 153H-01 **Quiz 4** **February 11, 2016**

No calculators, no notes, no books. Only pens, pencils and erasers are allowed.

1. Calculate $\int \frac{\cos x}{\sqrt{\sin^2 x+1}} dx$.

Solution: Substitute $u = \sin x$, and then $du = \cos x dx$, and so $\int \frac{\cos x}{\sqrt{\sin^2 x+1}} dx = \int \frac{du}{\sqrt{u^2+1}}$. There are two ways to solve this integral. One way is to substitute $t = \sinh^{-1} u$ and so $u = \sinh t$. Then $u^2 + 1 = \sinh^2 t + 1 = \cosh^2 t$. Note that $\cosh t$ is always positive, and so $\sqrt{u^2+1} = \cosh t$. Now $du = \cosh t dt$, and so $\int \frac{du}{\sqrt{u^2+1}} = \int \frac{\cosh t dt}{\cosh t} = \int dt = t + C = \sinh^{-1} u + C = \sinh^{-1}(\sin x) + C$.

Another way is to substitute $\theta = \tan^{-1} u$ and so $u = \tan\theta$. Then $u^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$. Note that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ over which interval $\tan\theta > 0$, and so $\sqrt{u^2+1} = \sec\theta$. Now $du = \sec^2\theta d\theta$, and so $\int \frac{du}{\sqrt{u^2+1}} = \int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$. Since $\tan\theta = u$ we have $\sec\theta = \sqrt{u^2+1}$. Note that $u + \sqrt{u^2+1}$ is always positive, so $\ln|\sec\theta+\tan\theta|+C = \ln(u + \sqrt{u^2+1}) + C = \ln(\sin x + \sqrt{\sin^2 x + 1}) + C$.

2. Solve the system $y'' = 4y$, $y(0) = 3$, $y'(0) = 6$. Express the solution in terms of hyperbolic functions.

Solution: The general solution to $y'' = 4y$ can be formulated either as $Ae^{2x} + Be^{-2x}$ or as $A\cosh(2x) + B\sinh(2x)$. [This does not mean that $Ae^{2x} + Be^{-2x} = A\cosh(2x) + B\sinh(2x)$, but rather that for every A and B one can find A' and B' such that $Ae^{2x} + Be^{-2x} = A'\cosh(2x) + B'\sinh(2x)$, and vice versa.] The second formulation is more convenient when we are given $y(0) = y_0$ and $y'(0) = v_0$, because then the solution is $y = y_0\cosh(\omega x) + \frac{v_0}{\omega}\sinh(\omega x)$, which in our case means $y = 3\cosh(2x) + 3\sinh(2x)$.

If one doesn't remember the formula, one should just write $y = A\cosh(2x) + B\sinh(2x)$. Then $y(0) = A\cosh 0 + B\sinh 0 = A = 3$, and $y'(0) = 2A\sinh 0 + 2B\cosh 0 = 2B = 6$ which means $B = 3$.