NAME:____

Math 153H-01 **Quiz 11** **April 21, 2016**

No calculators, no notes, no books. Only pens, pencils and erasers are allowed.

1. Say if $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent or divergent.

Solution: We use the integral test. The function $f(x) = \frac{1}{x(\ln x)^2}$ is a positive decreasing function whose values at the points $x = 2, 3, 4, \ldots$ are exactly the terms of the series. Therefore the integral $\int_2^{\infty} f(x)dx$ converges if and only if the series converges. This integral is $\lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2}$. Now, $\int_2^b \frac{dx}{x(\ln x)^2}$ can be simplified by substituting $u = \ln x$: $\int_2^b \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln b} \frac{1}{u^2} = -u^{-1}|_{\ln 2}^{\ln b} = \frac{1}{\ln 2} - \frac{1}{\ln b}$. Therefore $\lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}(\frac{1}{\ln 2} - \frac{1}{\ln b}) = \frac{1}{\ln 2}$, which means the improper integral is convergent, and so the series is convergent.

2. Say if $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n+1}}{n}$ is convergent or divergent. If it is convergent, say if it is absolutely convergent or conditionally convergent.

Solution: This series is convergent by the alternating series test. In order to justify it, one should note that it is an alternating series, the absolute values of the terms are decreasing and they tend to zero at infinity. It is obviously alternating because $(-1)^n$ alternates and $\frac{\sqrt{n+1}}{n}$ is always positive. The fact that it is decreasing can be seen in several ways, one of which is that $\frac{\sqrt{n+1}}{n} = \frac{n+1}{n\sqrt{n+1}} = \frac{1}{\sqrt{n(n+1}} + \frac{1}{n\sqrt{n+1}}$ and both fractions on the right-hand side clearly decrease as n increases (since only the denominators increase). Those absolute values of the terms tend to zero because $\lim_{n\to\infty}(\frac{1}{\sqrt{n(n+1}} + \frac{1}{n\sqrt{n+1}}) = 0 + 0 = 0$.

The series is only conditionally convergent. One can use the ratio-comparison test to the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ to show that $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$ is divergent: $\lim_{n\to\infty} \frac{\sqrt{n+1}}{n} \cdot \sqrt{n} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} = \sqrt{1+0} = 1$. Since this limit is a finite positive number, one series converges if and only if the other converges, and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series which is known to diverge (p in this case is greater than -1).