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Math 153H-01 **Midterm Exam** **March 24, 2016**

No calculators, no notes, no books. Only pens, pencils and erasers are allowed. Make sure to write your final solution to each question clearly.

1. Solve $\int \frac{dx}{(x^2+1)^2}$. The final answer cannot include trigonometric or hyperbolic functions. Solution: Substitute $x = \tan\theta$. Then $\frac{dx}{d\theta} = \sec^2\theta$. Furthermore, $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Therefore $\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta}{\sec^4\theta} d\theta = \int \cos^2\theta d\theta = \int (\frac{1}{2} + \frac{1}{2}\cos(2\theta) d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C = \frac{1}{2}(\tan^{-1} x + \frac{x}{x^2+1}) + C$.

2. Find the volume of the solid of revolution obtained by revolving the region between the curve $f(x) = x^3$ and the x-axis on the interval $[0,1]$ about the y-axis by the shell method. Solution: By the shell method the volume is $\int_0^1 (2\pi x \cdot x^3) dx = \int_0^1 (2\pi x^4) dx = \frac{2\pi}{5} x^5 |_0^1 = \frac{2\pi}{5}$.

3. Solve $\int \frac{(x+1)dx}{x^3-x^2+x-1}$.

Solution: $x^3-x^2+x-1 = (x-1)(x^2+1)$. Write $\frac{x+1}{x^3-x^2+x-1} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+CX-Bx-C}{x^3-x^2+x-1}$. Then we obtain the system

$$A + B = 0$$

$$C - B = 1$$

$$A - C = 1$$

This means $B = -A$, $C = A - 1$ and $A - 1 - (-A) = 1$. So $2A = 2$, i.e. $A = 1$, $B = -1$ and $C = 0$.

Therefore $\int \frac{(x+1)dx}{x^3-x^2+x-1} = \int (\frac{1}{x-1} - \frac{x}{x^2+1})dx = \ln|x-1| - \frac{1}{2}\ln|x^2+1| + C$

4. A certain solid has a circular base of radius 2 and the cross-sections with respect to the x-axis are squares. Find the volume of this solid.

Solution: We set the axis such that the circular base lies in the xy-plane with the center located at the origin. The equation of this circle in the xy-plane is $x^2 + y^2 = 4$. The cross section at point x is a square with edge length $2\sqrt{4-x^2}$. Therefore the area of the cross-section at point x is $4(4-x^2) = 16 - 4x^2$. The volume is then $\int_{-2}^{2}(16-4x^2)dx = 16x - \frac{4}{3}x^3|_{-2}^{2} = 2 \cdot (32 - \frac{32}{3}) = \frac{128}{3}$

5. A spring has a natural length of 10 centimeters. A force of 9 newtons is required in order to hold it at length 13 centimeters. How much work is required to stretch from 13 centimeters to 15?
Solution: The force acting on the spring is $F = -kx$ where x is its location with respect to its natural length and k is a constant positive number. The force required to hold it at 0.03 meters longer than its natural length is 9 newtons, which means that this is the magnitude of the force acting on the force (to bring it back to its natural size), so $9 = k \cdot 0.03$, i.e. $k = 300$. The work required to stretch the spring from 0.03 to 0.05 is then $\int_{0.03}^{0.05} 300x dx = 150x^2|_{0.03}^{0.05} = 150 \cdot (\frac{25}{10000} - \frac{9}{10000}) = 150 \cdot \frac{16}{10000} = \frac{2400}{10000} = \frac{24}{100} = 0.24J.$

6. Find the center of mass of the region under the curve $f(x) = \sqrt{1-x^2}$ and the x-axis on the interval $[-1, 1]$.
Solution: Because of the symmetry about the y-axis, the x-coordinate of the center of mass is 0. It is enough to compute the y-coordinate which is given by the formula:

$$\bar{y} = \frac{\int_{-1}^{1} \frac{1}{2} f(x)^2 dx}{\int_{-1}^{1} f(x) dx}.$$

Now $\int_{-1}^{1} \frac{1}{2} f(x)^2 dx = \int_{-1}^{1} \frac{1}{2}(1-x^2)dx = \frac{1}{2}x - \frac{1}{6}x^3|_{-1}^{1} = \frac{2}{3}$, and $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \sqrt{1-x^2}dx$ which is the area of half of a circle of radius 1, i.e. $\frac{\pi}{2}$. Therefore $\bar{y} = \frac{\frac{2}{3}}{\frac{pi}{2}} = \frac{4}{3\pi}$.

7. Calculate the surface area of a ball of radius 1.
Solution: We consider the surface area as the area of the surface of revolution obtained by revolving the function $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$ by the x-axis. This area is $\int_{-1}^{1} 2\pi f(x)\sqrt{1+(f'(x))^2}dx = \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+(\frac{x}{\sqrt{1-x^2}})^2}dx = \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}dx = \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}dx = \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}dx = \int_{-1}^{1} 2\pi dx = 4\pi$.

8. A particle moves around the elliptic track $4x^2 + y^2 = 4$ according to the equations $x = \cos t$ and $y = 2\sin t$. Find its maximal and minimal speeds and relevant times.
Solution: the speed at any given time t is $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{\sin^2 t + 4\cos^2 t} = \sqrt{1 + 3\cos^2 t}$. Since $\cos^2 t$ is always positive with minimum at odd multiples of $\frac{\pi}{2}$ and maximum at even multiples of $\frac{\pi}{2}$, these are the times of minimum and maximum velocities. The maximal velocity is therefore $\sqrt{1+3} = \sqrt{4} = 2$ and the minimal is $\sqrt{1+0} = \sqrt{1} = 1$.