NAME:____

Math 153H-01 **Midterm Exam** **February 18, 2016**

No calculators, no notes, no books. Only pens, pencils and erasers are allowed. Make sure to write your final solution to each question clearly.

1. Solve $xy' = e^x - y$, $y(1) = 0$.
Solution:

$$y' = \frac{e^x}{x} - \frac{y}{x}$$

$$P(x) = \frac{-1}{x}$$

$$F(x) = -\ln x$$

$$w = ye^{-F(x)} = ye^{\ln x} = xy$$

$$w' = y + xy' = y + e^x - y = e^x$$

$$w = e^x + C$$

$$xy = e^x + C$$

$$y = \frac{e^x}{x} + \frac{C}{x}$$

2. Solve $x\frac{dy}{dx} = \sqrt{1-y^2}$, $y(1) = 0$.
Solution:

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{x}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x}$$

$$\arcsin y = \ln|x| + C$$

$$y = \sin(\ln|x| + C)$$

$$y(1) = 0 = \sin(\ln(1) + C) = \sin(0 + C) = \sin(C)$$

This means that C is $k\pi$ for some integer k. Since $y = \sin(\ln|x| + C) = \sin(\ln|x| + C + 2\pi)$, we get exactly two distinct solutions

$$y_1 = \sin(\ln|x|)$$

$$y_1 = \sin(\ln|x| + \pi)$$

3. Show how to derive a formula for $(\sinh^{-1} x)'$.
Solution:

$$y = \sinh^{-1} x$$
$$\sinh y = x$$
$$y' \cosh y = 1$$
$$y' = \frac{1}{\cosh y}$$

Now use the identity $\cosh^2 y = 1 + \sinh^2 y$ to get

$$y' = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

4. It takes 80,000 years for a certain radioactive substance to decrease to $\frac{1}{64}$ of its original amount. Find the half-life.

Solution: The decay formula is $y = A(\frac{1}{64})^{\frac{t}{80,000}}$ where t is given in years. We are looking for time t for which $y = \frac{1}{2}A$. In this case $\frac{1}{2} = (\frac{1}{64})^{\frac{t}{80,000}}$. Since $\frac{1}{2} = (\frac{1}{64})^{\frac{1}{6}}$, we get

$$\frac{t}{80,000} = \frac{1}{6}$$

$$t = \frac{80,000}{6} = 13,333.33\ldots$$

5. Find the solution of $y'' = -4y$, $y(0) = 1$, $y'(0) = 3$.
Solution:

$$y'' = -4y$$
$$y = A\cos(2x) + B\sin(2x)$$
$$y(0) = A\cos 0 + B\sin 0 = A = 1$$
$$y'(0) = -A\sin 0 + B\cos 0 = B = 3$$
$$y = \cos(2x) + 3\sin(2x)$$

6. Evaluate $\int_1^2 x^2(\ln x)^2 dx$.
Solution: Integration by parts. First compute $\int x^2 \ln x dx$.

$$\int x^2 \ln x dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

Now

$$\int x^2(\ln x)^2 dx = \frac{x^3}{3}(\ln x)^2 - \int \frac{x^3}{3}\cdot 2\frac{\ln x}{x} dx = \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int x^2 \ln x dx$$
$$= \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}(\frac{x^3}{3}\ln x - \frac{x^3}{9}) + C$$
$$= \frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2x^3}{27} + C$$

So

$$\int_1^2 x^2(\ln x)^2 dx = \frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2x^3}{27}|_1^2 = 8\cdot(\frac{1}{3} - \frac{2}{9} + \frac{2}{27}) = \frac{40}{27}$$

7. Calculate $\int \sin(2x)\cos x dx$.

Solution: Recall $\sin(2x) = 2\sin x \cos x$, so

$$\int \sin(2x)\cos x dx = \int 2\sin x \cos^2 x dx$$

Now substitute $u = \cos x$ and so $du = -\sin x dx$

$$\int 2\sin x \cos^2 x dx = -2\int u^2 du = -\frac{2u^3}{3} + C = -\frac{2\cos^3 x}{3} + C$$

8. Calculate $\int \frac{1}{x\sqrt{x^2-1}} dx$.

Solution: Recall $\tan^2\theta + 1 = \sec^2\theta$ and $\frac{d\sec\theta}{\theta} = \sec\theta\tan\theta$. Substitute $\theta = \sec^{-1} x$. There are two cases to consider now - $x > 1$ and $x < -1$. If $x > 1$ then $0 < \theta < \frac{\pi}{2}$, in which case $\tan\theta > 0$, and so $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{x^2-1}$. Then

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \int \frac{\sec\theta\tan\theta d\theta}{\tan\theta\sec\theta} = \int d\theta = \theta + C = \sec^{-1} x + C$$

If $x < -1$ then $\frac{\pi}{2} < \theta < \pi$, in which case $\tan\theta < 0$, and so $\tan\theta = -\sqrt{\sec^2\theta - 1} = \sqrt{x^2-1}$. Then

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \int \frac{\sec\theta\tan\theta d\theta}{-\tan\theta\sec\theta} = \int d - \theta = -\theta + C = -\sec^{-1} x + C$$