Exercise. Compute the volume of a doughnut with outer radius R and hole radius r.

Solution using slice method. One can construct this doughnut by revolving the area between the curves $y = \frac{1}{2}(R+r) + \sqrt{\frac{1}{4}(R-r)^2 - x^2}$ and $y = \frac{1}{2}(R+r) - \sqrt{\frac{1}{4}(R-r)^2 - x^2}$ on the interval $[-\frac{1}{2}(R-r), \frac{1}{2}(R-r)]$ about the x-axis. The volume is the difference between $\int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} \pi(\frac{1}{2}(R+r)+\sqrt{\frac{1}{4}(R-r)^2 - x^2})^2 dx$ and $\int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} \pi(\frac{1}{2}(R+r)-\sqrt{\frac{1}{4}(R-r)^2 - x^2})^2 dx$, which is

$$V = \int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} 2\pi(R+r)\sqrt{\frac{1}{4}(R-r)^2 - x^2 dx}$$

Note that $\int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} \sqrt{\frac{1}{4}(R-r)^2 - x^2 dx}$ is half the area of a circle of radius $\frac{1}{2}(R-r)$, which is $\frac{1}{8}\pi(R-r)^2$, and so

$$V = 2\pi(R+r) \cdot \frac{1}{8}\pi(R-r)^2 = \frac{1}{4}\pi^2(R+r)(R-r)^2.$$

Solution using the shell method. We consider again the area between the curves above. For each y between $R-r$ and R we look at the line segment inside that region. This line segment turns into a shell by revolving the region by the x-axis. The area of that shell is the length of this line segment times y times 2π. The length of the line segment is $2\sqrt{\frac{1}{4}(R-r)^2 - (y - \frac{1}{2}(R+r))^2}$, and so the area of the shell is $4\pi y\sqrt{\frac{1}{4}(R-r)^2 - (y - \frac{1}{2}(R+r))^2}$. Now the volume is

$$V = \int_{R-r}^{R} 4\pi y\sqrt{\frac{1}{4}(R-r)^2 - (y - \frac{1}{2}(R+r))^2 dy}$$

Substitute $u = y - \frac{1}{2}(R+r)$ and so

$$V = \int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} 4\pi(u + \frac{1}{2}(R+r))\sqrt{\frac{1}{4}(R-r)^2 - u^2 du}$$

$$V = \int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} 2\pi(R+r)\sqrt{\frac{1}{4}(R-r)^2 - u^2 du} + \int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} 4\pi u\sqrt{\frac{1}{4}(R-r)^2 - u^2 du}$$

Now, the function $4\pi u\sqrt{\frac{1}{4}(R-r)^2 - u^2}$ is anti-symmetric, which means that $\int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} 4\pi u\sqrt{\frac{1}{4}(R-r)^2 - u^2}du = 0$, and so

$$V = \int_{-\frac{1}{2}(R-r)}^{\frac{1}{2}(R-r)} 2\pi(R+r)\sqrt{\frac{1}{4}(R-r)^2 - u^2}du$$

and by the same half-circle argument as before (surprise!) you get the same answer $V = \frac{1}{4}\pi^2(R+r)(R-r)^2$.