# Pseudoholomorphic Strips in Symplectisations II: Fredholm Theory and Transversality

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**Abstract**

This paper is part of a larger program, the investigation of the chord problem in three dimensional contact geometry. The main tool will be pseudoholomorphic strips in the symplectisation of a three dimensional contact manifold with two totally real submanifolds $L_0, L_1$ as boundary conditions. The submanifolds $L_0$ and $L_1$ do not intersect transversally. In this paper we will develop a nonlinear Fredholm theory that guarantees the existence of a family of embedded pseudo-holomorphic strips near a given one with suitable properties. 

## Contents



## 1 Introduction

This paper continues the investigation of the partial differential equation studied in [2], and it is the second one in a series of papers [2, 3, 4] dedicated to the investigation of the chord problem in three dimensional contact geometry. Let $(M, \lambda)$ be a closed three-dimensional contact manifold, i.e. $\lambda$ is a 1-form on $M$ such that $\lambda \wedge d\lambda$ is a volume form on $M$. The contact structure associated to $\lambda$ is the 2-dimensional vector bundle $\xi = \ker \lambda \to M$, which is a symplectic vector bundle with symplectic structure $d\lambda|_{\xi \oplus \xi}$. There is a distinguished vector field associated to a contact form, the Reeb vector field $X_\lambda$, which is defined by the equations

$$i_{X_\lambda} d\lambda \equiv 0\,, \quad i_{X_\lambda}\lambda \equiv 1\,.$$

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We denote by $\pi_\lambda : TM \to \xi$ the projection along the Reeb vector field. The chord problem is dealing with the existence of so-called 'characteristic chords'. These are trajectories $x$ of the Reeb vector field which hit a given Legendrian knot $\mathcal{L} \subset (M, \lambda)$ at two different times $t = 0$, $T > 0$. We also ask for $x(0) \neq x(T)$, otherwise the chord would actually be a periodic orbit. Recall that a Legendrian knot is an embedded 1-sphere which is everywhere tangent to the contact planes $\xi$. For more remarks on the chord problem see the introduction of the first paper [2]. In [4] we will use solutions to a nonlinear version of the Cauchy Riemann equation ('pseudoholomorphic curves') to address the chord problem. In order to formulate the partial differential equation, we have to consider certain almost complex structures on $\mathbf{R} \times M$. We pick a complex structure $J : \xi \to \xi$ such that $d\lambda \circ (\mathrm{Id} \times J)$ is a bundle metric on $\xi$. We then define an almost complex structure on $\mathbf{R} \times M$ by demanding $\tilde{J} \equiv J$ on $\xi$ and sending $\partial/\partial t$ (the generator of the $\mathbf{R}$-component) onto the Reeb vector field. Then $\tilde{J}(p)$ has to map $X_\lambda(p)$ onto $-\partial/\partial t$. Moreover, we assume that $\mathcal{L}$ is a homologically trivial Legendrian knot, i.e. there is an embedded compact surface $\mathcal{D}$ with $\partial\mathcal{D} = \mathcal{L}$. A point $p \in \mathcal{D}$ is called singular if $T_p\mathcal{D} = \ker \lambda(p)$. If the surface is oriented (by a volume form $\sigma$) and if $j : \mathcal{D} \hookrightarrow M$ is the inclusion, then we define a vector field $Z$ on $\mathcal{D}$ by $i_Z\sigma = j^*\lambda$. This vector field vanishes precisely in the singular points. The flow lines of $Z$ determine a singular foliation of the surface $\mathcal{D}$ which does not depend on the particular choice of the volume form or the contact form. This singular foliation is also called the characteristic foliation of $\mathcal{D}$ (induced by $\ker \lambda$). Let $p \in \mathcal{D}$ be a singular point and denote by $Z'(p) : T_p\mathcal{D} \to T_p\mathcal{D}$ the linearization of the vector field $Z$ in $p$. Let $\lambda_1, \lambda_2$ be the eigenvalues of $Z'(p)$. We say that $p$ is non-degenerate if none of the eigenvalues lie on the imaginary axis. A non-degenerate singular point $p$ is called elliptic if $\lambda_1\lambda_2 > 0$ and hyperbolic if $\lambda_1\lambda_2 < 0$. In the elliptic case the critical point $Z(p) = 0$ is either a source or a sink, and in the hyperbolic case it is a saddle point.

Choosing $\mathcal{D}$ appropriately we may assume that there are only non-degenerate singular points, in particular there are only finitely many. We denote the surface without the singular points by $\mathcal{D}^*$. We can now formulate the boundary value problem we are interested in:

$$
\begin{cases}
\tilde{u} = (a, u) : S \longrightarrow \mathbf{R} \times M \\
\partial_s \tilde{u} + \tilde{J}(\tilde{u})\partial_t \tilde{u} = 0 \\
\tilde{u}(s, 0) \subset \mathbf{R} \times \mathcal{L} \\
\tilde{u}(s, 1) \subset \{0\} \times \mathcal{D}^{**} \\
0 < E(\tilde{u}) < +\infty
\end{cases}
\tag{1.1}
$$

Here $S := \mathbf{R} \times [0, 1]$ and $\mathcal{D}^{**}$ is the spanning surface $\mathcal{D}$ without some open neighborhood $U$ of the set of singular points $\Gamma$. The energy $E(\tilde{u})$ is defined by

$$E(\tilde{u}) := \sup_{\phi\in\Sigma} \int_S \tilde{u}^* d(\phi\lambda) < +\infty$$

where $\Sigma := \{\phi \in C^\infty(\mathbf{R}, [0,1]) \mid \phi' \geq 0\}$. Solutions to (1.1) exist locally near elliptic singular points on the boundary:

THEOREM 1.1 *Let* $(M, \lambda)$ *be a three dimensional contact manifold. Moreover, let* $\mathcal{L}$ *be a Legendrian knot which bounds an embedded surface* $\mathcal{D}'$ *so that the characteristic foliation has only finitely many singular points. Then there is another embedded surface* $\mathcal{D}$ *which is a smooth* $C^0$*-small perturbation of* $\mathcal{D}'$ *having the same boundary and the same singular points as* $\mathcal{D}'$ *and a* $d\lambda$*-compatible complex structure* $J : \ker\lambda \to \ker\lambda$ *so that the following is true: Near each elliptic singular point* $e \in \partial\mathcal{D} = \mathcal{L}$ *there are embedded solutions* $\tilde{u}_\tau$ , $0 < \tau < 1$ *to the boundary value problem* (1.1) *with the properties*:

- $\tilde{u}_\tau(S) \cap \tilde{u}_{\tau'}(S) = \emptyset$ *if* $\tau \neq \tau'$,
- $\tilde{u}_\tau \to e$ *uniformly with all derivatives as* $\tau \to 0$,
- *the family* $\tilde{u}_\tau$ *depends smoothly on the parameter* $\tau$.

PROOF: See [2]. □

In local Darboux-coordinates near the elliptic point $e$, where $\lambda = dy + x d\theta$ and

$$\mathcal{D} = \left\{(\theta, x, y) \in \mathbf{R}^3 \mid y = -\tfrac{1}{2}x\theta\right\}, \quad \mathcal{L} = \{(\theta, 0, 0)\},$$

we were able to find solutions explicitly given by

$$\begin{aligned} \tilde{u}_\tau(s,t) = \Bigg( &-\frac{\tau^2\cos(\frac{\pi t}{2})}{2[\cos(\frac{\pi t}{2}) + \cosh(\frac{\pi s}{2})]}, \frac{\tau\sinh(\frac{\pi s}{2})}{\cos(\frac{\pi t}{2}) + \cosh(\frac{\pi s}{2})}, \\ &\frac{-\tau\sin(\frac{\pi t}{2})}{\cos(\frac{\pi t}{2}) + \cosh(\frac{\pi s}{2})}, \frac{\tau^2\sin(\frac{\pi t}{2})\sinh(\frac{\pi s}{2})}{2[\cos(\frac{\pi t}{2}) + \cosh(\frac{\pi s}{2})]^2}\Bigg). \end{aligned} \tag{1.2}$$

The topic of this paper and the next one [3] is the question whether local solution families above can be extended. It will turn out in [4] that the space of solutions starting out with a local family $\tilde{u}_\tau$ above is one-dimensional and not compact, with non-compactness (in many situations) caused by the existence of a characteristic chord. The main result of this paper states that existence of a suitable embedded solution implies the existence of a whole 1-parameter family of embedded solutions near by:

THEOREM 1.2 *Let* $\tilde{u}_0$ *be an embedded solution of* (1.1) *so that its Maslov index* $\mu(\tilde{u}_0)$ *vanishes. Assume moreover, that* $|\tilde{u}_0(s,t) - p_\pm|$ *decays either like* $e^{-\pi|s|}$ *or like* $e^{-\frac{\pi}{2}|s|}$ *for large* $|s|$ *in local coordinates near the points* $p_\pm := \lim_{s\to\pm\infty}\tilde{u}_0(s,t)$ *and that* $p_- \neq p_+$. *Then there is a smooth family* $(\tilde{v}_\tau)_{-1<\tau<1}$ *of embedded solutions of* (1.1) *with the following properties*:

- $\tilde{v}_0 = \tilde{u}_0$,

- *The solutions* $\tilde v_\tau$ *have the same Maslov index and the same decay rates as* $\tilde u_0$*,*
- *The sets*
$$U_\pm := \bigcup_{-1<\tau<1} \left\{ \lim_{s\to\pm\infty} \tilde v_\tau(s,t)\right\}$$
*are open neighborhoods of the points* $p_\pm$ *in* $\mathcal{L}$*.*

*If* $|\tilde u_0(s,t)-p_\pm|$ *decays like* $e^{-\frac{\pi}{2}|s|}$ *for both* $s\to+\infty$ *and* $s\to-\infty$ *then we have in addition*

- $\tilde v_\tau(S)\cap \tilde v_{\tau'}(S)=\varnothing$ *if* $\tau\neq\tau'$*.*

The Maslov index $\mu(\tilde u_0)$ is a topological quantity associated to the boundary condition $\{0\}\times\mathcal{D}$. For details see Definition 3.4. We have shown in [2] that $|\tilde u_0(s,t)-p_\pm|$ has to decay like $e^{-\frac{\pi}{2}\kappa|s|}$, where $\kappa$ is a positive integer (see Theorem 1.5 below). We will show in the paper [3] that the only relevant solutions $\tilde u_0$ are the ones with Maslov index zero and decay rate $\kappa=1$ at both ends. In [3] we will show independently of the results of this paper that solutions with nonzero Maslov index and $\kappa\geq 3$ do not come up in the situations we are interested in. This argument however does not apply to $\kappa=2$. We will only use the part of Theorem 1.2 concerning $\kappa=2$ later in [3] to prove that solutions with $\kappa=2$ on either end cannot occur as well.

Before we briefly summarize the results of the paper [2], a few remarks about the proof of Theorem 1.2 are in order: Why do we insist on embedded solutions? The “usual” approach, finding a generic almost complex structure $\tilde J$ so that the linearization of $\partial_s+\tilde J\partial_t$ becomes surjective (as for example in A. Floer’s theory of Lagrangian intersections [9]), is not appropriate here because the image $u(S)$ of the $M$-part of a solution $\tilde u=(a,u)$ always hits an elliptic singular point on $\mathcal{L}$, but we wanted $\tilde J$ fixed there because of the local existence result, Theorem 1.1. Instead, we consider only embedded solutions $\tilde u_0$ and describe near by solutions as graphs over $\tilde u_0$, i.e., as sections in the normal bundle of $\tilde u_0$. We derive the corresponding PDE for the normal bundle in Section 3. In contrast to A. Floer’s Lagrange intersection theory, the end points of the solutions $\tilde u_0$ are not fixed, they are allowed to float along the 1-dimensional intersection set of $\{0\}\times\mathcal{D}$ and $\mathbf{R}\times\mathcal{L}$. Since this intersection is not transverse we have to work in weighted Sobolev spaces in order to obtain a nonlinear Fredholm operator. Because the solutions $\tilde u_0$ converge to single points for $s\to\pm\infty$, the growth/decay rates of the coefficients for the PDE in the normal direction have to be estimated very carefully. The behavior of the coefficients determines for which weighted Sobolev spaces the nonlinear operator can be defined. On the other hand, the Fredholm index of the linearized operator depends on the decay rates of the solution $\tilde u_0$ that we started with. It will turn out that there is a happy end to the story if and only if $|\tilde u_0(s,t)-p_\pm|$ decays at either end like $e^{-\frac{\pi}{2}|s|}$ or like $e^{-\pi|s|}$. The index of the linearization equals $+1$ and the linearization will be surjective for whatever almost complex structure we started with, resulting in a 1-dimensional family of solutions. A faster decay rate of $\tilde u_0$ at one of the ends would result in a negative Fredholm index, i.e., the linearization would have a non trivial cokernel and the implicit function theorem would not be applicable.

Since we will use the results and the notation from the paper [2], we briefly summarize what is needed. First, we may modify the surface $\mathcal{D}$ near its boundary in order to achieve some normal form in local coordinates.

PROPOSITION 1.3 *Let $(M, \lambda)$ be a three-dimensional contact manifold. Further, let $\mathcal{L}$ be a Legendrian knot and $\mathcal{D}$ an embedded surface with $\partial\mathcal{D} = \mathcal{L}$ so that all the singular points are non-degenerate. We denote the finitely many singular points on the boundary by $e_k$, $1 \le k \le N$ (ordered by orienting $\mathcal{L}$). Then there is an embedded surface $\mathcal{D}'$ having the same boundary as $\mathcal{D}$ which differs from $\mathcal{D}$ only by a smooth $C^0$-small perturbation supported near $\mathcal{L}$ having the same singular points as $\mathcal{D}$ so that the following holds: There is a neighborhood $U$ of $\mathcal{L}$ and a diffeomorphism $\Phi : U \to S^1 \times \mathbf{R}^2$ so that*

- $\Phi^*(dy + x d\theta) = \lambda|_U$, $(\theta, x, y) \in S^1 \times \mathbf{R}^2$,
- $\Phi(\mathcal{L}) = S^1 \times \{(0, 0)\}$,
- $\Phi(e_k) = (\theta_k, 0, 0)$, $0 \le \theta_1 < \cdots < \theta_N < 1$,
- $\Phi(U \cap \mathcal{D}') = \{(\theta, a(\theta)r, b(\theta)r) \in S^1 \times \mathbf{R}^2 \mid \theta, r \in [0, 1]\}$,

*where $a, b$ are smooth 1-periodic functions with*:

- *$b(\theta_k) = 0$ and $b(\theta)$ is nonzero if $\theta \ne \theta_k$,*
- *$a(\theta_k) < 0$ if $e_k$ is a positive singular point, $a(\theta_k) > 0$ if $e_k$ is a negative singular point,*
- *if $e_k$ is elliptic then $-1 < \frac{b'(\theta_k)}{a(\theta_k)} < 0$,*
- *if $e_k$ is hyperbolic then the quotient $\frac{b'(\theta_k)}{a(\theta_k)}$ is either strictly smaller than $-1$ or positive,*
- *$a$ has exactly one zero in each of the intervals $[\theta_k, \theta_{k+1}]$, $k = 1, \ldots, N-1$ and $[\theta_N, 1] \cup [0, \theta_1]$,*
- *If $e_k$ is an elliptic singular point and if $|\theta - \theta_k|$ is sufficiently small then we have $b(\theta) = -\frac{1}{2} a(\theta)(\theta - \theta_k)$.*

PROOF: See [2]. □

We showed in [2] that solutions to equation (1.1) converge to points $(0, p_\pm) \in \{0\} \times \mathcal{L} \cap \mathcal{D}^{**}$ as $s \to \pm\infty$. It is sometimes convenient to modify the coordinates given by the above proposition near the points $p_\pm$ in order to make the surface $\mathcal{D}$ flat. Away from the boundary singular points $e_k$ we introduce the coordinate transformation

$$\mathbf{R} \times S^1 \times \mathbf{R}^2 \ni (\tau, \theta, x, y) \longmapsto \left(\tau, \theta, x - \frac{a(\theta)}{b(\theta)} y, y\right) = (\tau, \theta, x - q(\theta) y, y) \,. \tag{1.3}$$

We then obtain the following coordinates on suitable neighborhoods $V_\pm$ of the points $p_\pm \in \mathcal{L}$:

$$\begin{aligned} \psi_\pm : \mathbf{R}^4 \supset B_\varepsilon(0) \xrightarrow{\sim} V_\pm \subset \mathbf{R} \times M\,, \quad \psi_\pm(0) = p_\pm\,, \\ \psi_\pm(\mathbf{R}^2 \times \{0\} \times \{0\}) = (\mathbf{R} \times \mathcal{L}) \cap V_\pm\,, \\ \psi_\pm(\{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}^\pm) = (\{0\} \times \mathcal{D}) \cap V_\pm\,. \end{aligned} \tag{1.4}$$

Using the coordinates $(\tau, \theta, x, y)$ for $\mathbf{R}^4$, the contact form on $\{0\} \times \mathbf{R}^3$ is then given by

$$\hat{\lambda}_\pm = \psi_\pm^* \lambda = dy + (x + q(\theta)y)\, d\theta\,, \quad q(\theta) := \frac{a(\theta)}{b(\theta)}\,,$$

with Reeb vector field

$$X_{\hat{\lambda}_\pm} = \frac{\partial}{\partial y} - q(\theta)\frac{\partial}{\partial x}$$

(recall that the functions $a, b$ determine how the surface $\mathcal{D}$ is wrapping itself around the knot $\mathcal{L}$, see Proposition 1.3). Let $v_\pm(s,t) := (\psi_\pm^{-1} \circ \tilde{u}_0)(s,t)$ be the representative of a solution $\tilde{u}$ of (1.1) in the above coordinates for large $|s|$. Our differential equation (1.1) has the following form in the above coordinates:

$$\begin{aligned} v = (\tau, \theta, x, y) : [s_0, \infty) \times [0,1] \longrightarrow \mathbf{R}^4\,, \\ \partial_s v + M(v)\partial_t v = 0\,, \\ v(s,0) \in L_0 = \mathbf{R}^2 \times \{0\} \times \{0\}\,, \\ v(s,1) \in L_1 = \{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}\,, \end{aligned} \tag{1.5}$$

where $M$ is a suitable $4 \times 4$-matrix valued function with $M^2 = -\mathrm{Id}$. We have shown in [2]:

THEOREM 1.4 *There exist numbers $\rho, s' > 0$ so that we have the following estimate for each multi index $\alpha \in \mathbf{N}^2$, $|\alpha| \geq 0$ and $s \geq s'$:*

$$\sup_{t \in [0,1]} |\partial^\alpha v(s,t)| \leq c_\alpha e^{-\rho(s-s')}\,,$$

*where $c_\alpha$ are suitable positive constants.*

The main result of [2] is the following asymptotic formula for nonconstant solutions $v$ of (1.5) having finite energy:

THEOREM 1.5 *For sufficiently large $s_0$ and $|s| \geq s_0$ we have the following asymptotic formulas for nonconstant solutions $v$ of* (1.5) *having finite energy*

$$v(s,t) = e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau}(e_\pm(t) + r_\pm(s,t))\,, \tag{1.6}$$

*where $\alpha_\pm : [s_0, \infty) \to \mathbf{R}$ are smooth functions satisfying $\alpha_+(s) \to \lambda_+ < 0$ and $\alpha_-(s) \to \lambda_- > 0$ as $s \to \pm\infty$ with $\lambda_\pm \in \mathbf{Z}\frac{\pi}{2}$ being eigenvalues of the selfadjoint* *operators*

$$A_{\pm\infty} : L^2([0,1],\mathbf{R}^4) \supset H^{1,2}_L([0,1],\mathbf{R}^4) \longrightarrow L^2([0,1],\mathbf{R}^4)\,,$$
$$\gamma \longmapsto -M_{\pm\infty}\dot{\gamma}\,, \quad M_{\pm\infty} := \lim_{s\pm\to\infty} M(v(s,t))\,.$$

*Moreover, $e_\pm(t)$ is an eigenvector of $A_{\pm\infty}$ belonging to the eigenvalue $\lambda_\pm$ with $e_\pm(t) \neq 0$ for all $t \in [0,1]$, and $r_\pm$ are smooth functions so that $r_\pm$ and all their derivatives converge to zero uniformly in $t$ as $s \to \pm\infty$.*

PROOF: See [2]. □

The domain of the operators $A_{\pm\infty}$ above is the following dense subspace of $L^2([0,1],\mathbf{R}^4)$:

$$H^{1,2}_L([0,1],\mathbf{R}^4) := \left\{\gamma \in H^{1,2}([0,1],\mathbf{R}^4) \mid \gamma(0) \in L_0\,, \gamma(1) \in L_1\right\},$$

where

$$L_0 := \mathbf{R}^2 \times \{0\} \times \{0\} \quad \text{and} \quad L_1 := \{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}\,.$$

In view of the Sobolev embedding theorem this definition makes sense. There is also the following refinement of the above asymptotic formula:

THEOREM 1.6 *Let $v$ be as in Theorem 1.5. Then there is a constant $\delta > 0$ such that for each integer $l \geq 0$ and each multi-index $\beta \in \mathbf{N}^2$*

$$\sup_{0\le t\le 1}\left|D^\beta r_\pm(s,t)\right|, \quad \left|\frac{d^l}{ds^l}(\alpha_\pm(s) - \lambda_\pm)\right| \le c_{\beta,l}e^{-\delta|s|}$$

*with suitable constants $c_{\beta,l} > 0$.*

PROOF: See [2]. □

We recall from [2] that we chose the complex structure $J : \xi \to \xi$ near the Legendrian knot $\mathcal{L}$ as follows (in the local coordinates given by Proposition 1.3):

$$\begin{aligned} J(\theta,x,y)\cdot(1,0,-x) &:= (0,-1,0)\,,\\ J(\theta,x,y)\cdot(0,1,0) &:= (1,0,-x)\,. \end{aligned} \tag{1.7}$$

This choice will facilitate computations later on. In the coordinates (1.3) the almost complex structure $\hat{J}$ on $\mathbf{R}^4$ induced by $\tilde{J}$ is given by

$$\begin{aligned} \hat{J}(\tau,\theta,x,y)\hat{e}_1(\theta,x,y) &= -\hat{e}_2(\theta,x,y)\,,\\ \hat{J}(\tau,\theta,x,y)\hat{e}_2(\theta,x,y) &= \hat{e}_1(\theta,x,y)\,,\\ \hat{J}(\tau,\theta,x,y)(1,0,0,0) &= (0,0,-q(\theta),1)\,,\\ \hat{J}(\tau,\theta,x,y)(0,0,-q(\theta),1) &= (-1,0,0,0)\,, \end{aligned} \tag{1.8}$$

or

$$\hat{J}(\tau,\theta,x,y) =$$
$$\begin{pmatrix} 0 & -(x+q(\theta)y) & 0 & -1 \\ 0 & yq'(\theta) & 1 & q(\theta) \\ -q(\theta) & -1+yq'(\theta)((x+q(\theta)y)q(\theta)-yq'(\theta)) & (x+q(\theta)y)q(\theta)-yq'(\theta) & q(\theta)((x+q(\theta)y)q(\theta)-yq'(\theta)) \\ 1 & -(x+q(\theta)y)yq'(\theta) & -(x+q(\theta)y) & -(x+q(\theta)y)q(\theta) \end{pmatrix}.$$

If $\lambda_\pm$ is an odd integer multiple of $\pi/2$ then the asymptotic formula of Theorem 1.5 looks as follows:

$$\begin{aligned} v_\pm(s,t) = &-\kappa_\pm e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau}\big(\cos(\lambda_\pm t), -q_\pm(0)\cos(\lambda_\pm t), 0, \sin(\lambda_\pm t)\big) \\ &+ e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau}\varepsilon_\pm(s,t)\,. \end{aligned} \tag{1.9}$$

In the following we will denote by $\varepsilon(s,t)$ any $\mathbf{R}^4$- or real-valued function which converges to zero with all its derivatives uniformly in $t$ as $s\to\pm\infty$ if we are not interested in the particular function. In order to simplify notation we will often drop the subscript $\pm$. Using the fact that $\alpha'(s)\to 0$ as $|s|\to\infty$ (proved in [2]), we obtain the following asymptotic formulas for the derivatives of $v(s,t)$ ($\kappa,\kappa_\pm$ are suitable nonzero constants):

$$\begin{aligned} \partial_s v(s,t) = &\, e^{\int_{s_0}^s \alpha(\tau)d\tau} \\ &\cdot\big[-\kappa\big(\lambda\cos(\lambda t), -\lambda q(0)\cos(\lambda t), 0, \lambda\sin(\lambda t)\big) + \varepsilon(s,t)\big]\,, \end{aligned} \tag{1.10}$$

$$\begin{aligned} \partial_t v(s,t) = &\, e^{\int_{s_0}^s \alpha(\tau)d\tau} \\ &\cdot\big[-\kappa\big(-\lambda\sin(\lambda t), \lambda q(0)\sin(\lambda t), 0, \lambda\cos(\lambda t)\big) + \varepsilon(s,t)\big]\,. \end{aligned} \tag{1.11}$$

If we use the coordinates given by Proposition 1.3 without making the boundary conditions “flat” as in (1.3) then the appropriate versions of (1.9) and (1.10) are the following. If $\lambda_\pm$ is an odd integer multiple of $\pi/2$ we have:

$$\begin{aligned} &v_\pm(s,t) \\ &= -\kappa_\pm e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau}\big(\cos(\lambda_\pm t), -q_\pm(0)\cos(\lambda_\pm t), q_\pm(0)\sin(\lambda_\pm t), \sin(\lambda_\pm t)\big) \\ &\qquad + e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau}\varepsilon_\pm(s,t) \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} &\partial_s v_\pm(s,t) = e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau} \\ &\cdot\big[-\kappa_\pm\big(\lambda_\pm\cos(\lambda_\pm t), -\lambda_\pm q_\pm(0)\cos(\lambda_\pm t), \lambda_\pm q_\pm(0)\sin(\lambda_\pm t), \lambda_\pm\sin(\lambda_\pm t)\big) \\ &\qquad + \varepsilon_\pm(s,t)\big]\,. \end{aligned} \tag{1.13}$$

For $\lambda_\pm\in\mathbf{Z}\pi$ we have

$$v_\pm(s,t) = \kappa_\pm e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau}\big(0, \cos(\lambda_\pm t), -\sin(\lambda_\pm t), 0\big) + e^{\int_{s_0}^s \alpha_\pm(\tau)d\tau}\varepsilon_\pm(s,t) \tag{1.14}$$

and

$$\text{(1.15)}\quad \partial_s v_{\pm}(s,t) = e^{\int_{s_0}^{s} \alpha_{\pm}(\tau)d\tau} \cdot \big[\kappa_{\pm}\big(0, \lambda_{\pm}\cos(\lambda_{\pm}t), -\lambda_{\pm}\sin(\lambda_{\pm}t), 0\big) + \varepsilon_{\pm}(s,t)\big].$$

## 2 Some Properties of the Solutions

We derive some properties of solutions to equation (1.1) which follow from the maximum principle.

PROPOSITION 2.1 *Let* $\tilde{u} = (a, u) : S \longrightarrow \mathbf{R} \times M$ *be a nonconstant solution of the boundary value problem* (1.1) (*we may replace* $\{0\} \times \mathcal{D}^{**}$ *by* $\{0\} \times \mathcal{D}$).

- *Then the path* $s \longmapsto u(s,1)$ *is transverse to the characteristic foliation, i.e.,* $\partial_s u(s,1) \notin \ker \lambda(u(s,1))$. *We actually have*
$$0 < \partial_t a(s,1) = -\lambda(u(s,1))\partial_s u(s,1) \quad \text{for all } s \in \mathbf{R}.$$
- *We have* $a(s,t) < 0$ *whenever* $0 \le t < 1$,
- *The pseudoholomorphic strip never hits* $\{0\} \times \mathcal{L}$; *i.e.,*
$$\tilde{u}(S) \cap (\{0\} \times \mathcal{L}) = \varnothing,$$
*in particular,*
$$\lim_{s\to\pm\infty} \tilde{u}(s,t) \notin \tilde{u}(S).$$

PROOF: We have

$$\Delta a = \frac{1}{2}\big(|\pi_\lambda \partial_s u|_J^2 + |\pi_\lambda \partial_t u|_J^2\big) \ge 0,$$

$$\Delta a \text{ is not identically zero}, \qquad a(s,1) \equiv 0,$$

$$\partial_t a(s,0) = -\lambda(u(s,0))\partial_s u(s,0) = 0 \quad \text{since } u(s,0) \in \mathcal{L},$$

and

$$|a(s,\,.\,)| \to 0 \quad \text{for } |s| \to \infty.$$

If we had $\Delta a \equiv 0$ then also $\int_S u^* d\lambda = 0$, and we have shown in [2] that this would imply that $\tilde{u}$ is constant.

We first show that $a \le 0$. Assume in the contrary that $a$ is positive somewhere. Then there is a positive maximum $M = a(s_0, t_0) > 0$. Apply now the weak maximum principle [10, theorem 2.2] to the function $a$ on $[s_0 - N, s_0 + N] \times [0,1]$, where $N$ is a sufficiently large positive number, to conclude $t_0 = 0$ since $a(s,1) \equiv 0$. Now apply the strong maximum principle [10, lemma 3.4] to the point $(s_0, 0)$ to conclude either $-\partial_t a(s_0, 0) > 0$ or $a \equiv$ const, which are both impossible in view of the boundary condition and $M > 0$. Hence we have shown that $a \le 0$.

Next, we will prove that $a < 0$ on $\mathbf{R} \times [0,1)$. Assume that $a(s_0, t_0) = 0$ for some $0 \le t_0 < 1$. If $t_0 > 0$ then $(s_0, t_0)$ would be an interior maximum. Applying the weak maximum principle once again we conclude that $a \equiv 0$ in contradiction to $\Delta a \not\equiv 0$. If $t_0 = 0$ then the strong maximum principle would imply that $\partial_t a(s_0, 0) < 0$ or $a \equiv$constant which are both impossible. Hence we have shown that $a < 0$ on $\mathbf{R} \times [0, 1)$.

Pick now $(s, 1)$ and note that $a(s, 1) = 0$ is a maximum for $a$. Using the strong maximum principle and the fact that $a \not\equiv 0$ we conclude that

$$0 < \partial_t a(s, 1) = -\lambda(u(s, 1))\partial_s u(s, 1) \quad \text{for all } s \in \mathbf{R}.$$

This also implies that the curve $s \mapsto u(s, 1)$ avoids the singular points on $\mathcal{D}$. Assume now that $(s, t) \in S$ is a point for which $\tilde{u}(s, t) = (a, u)(s, t) \in \{0\} \times \mathcal{L}$. Then $t = 1$. On the other hand, the curve $\mathbf{R} \ni s \longmapsto u(s, 1)$ is an immersion into $\mathcal{D}$. If the curve hits the boundary then it must be tangent to $\mathcal{L}$ at the intersection point. But this is impossible because $\mathcal{L} \cap \mathcal{D}$ is a disjoint union of leaves of the characteristic foliation and singular points. □

The following is an easy consequence of Proposition 2.1 and the asymptotic formula. We know from the asymptotic formula, Theorem 1.5, that solutions $\tilde{u}(s, t)$ have to converge to their limits $p_\pm = \lim_{s\to\pm\infty} \tilde{u}(s, t)$ exponentially fast, i.e.,

$$|p_\pm - \tilde{u}(s, t)| \leq ce^{\lambda_\pm s},$$

where $\lambda_+ < 0$ , $\lambda_- > 0$ are integer multiples of $\frac{\pi}{2}$.

COROLLARY 2.2 *Let $\tilde{u}$ be a non constant solution to the boundary value problem* (1.1) *and let $\lambda_\pm = n_\pm \frac{\pi}{2}$ be the exponential decay rates as in Theorem* 1.5. *If $n_-$ is odd then $n_- = +1$ and if $n_+$ is odd then $n_+ = -1$.*

PROOF: Writing $\tilde{u} = (a, u)$ we obtain for odd $n_\pm$ the following asymptotic formulas for $a(s, t)$ using equation (1.9):

$$a(s, t) = -\kappa_\pm e^{\int_{s_0}^{s} \alpha_\pm(\tau)d\tau}\big(\cos(\lambda_\pm t) + \varepsilon_\pm(s, t)\big)$$

where $\varepsilon_\pm$ are functions which tend to zero uniformly in $t$ as $s \to \pm\infty$. We proved in Proposition 2.1 that $a(s, t)$ is never positive. If $|\lambda_\pm|$ were greater than $\frac{\pi}{2}$ then $a(s, t)$ would have to change signs according to the above asymptotic formula. □

*Remark.* If $|\lambda_\pm| = \frac{\pi}{2}$ then the constants $\kappa_\pm$ in the asymptotic formula have to be positive.

The next proposition provides an a priori bound on the energy. Whenever the energy is finite, it is bounded by a constant only depending on the geometry of the problem. Let $\sigma$ be a volume form on $\mathcal{D}$, write $d\lambda = g_\lambda \cdot \sigma$ for a suitable function $g_\lambda$ and define the $\lambda$-volume of $\mathcal{D}$ by

$$\mathrm{vol}_\lambda(\mathcal{D}) := \int_{\mathcal{D}} |g_\lambda| \cdot \sigma .$$

PROPOSITION 2.3 *Assume that $\tilde{u}$ is a solution of the problem* (1.1) *so that the path $s \mapsto u(s, 1) \in \mathcal{D}$ represents a trivial homology class in $H_1(\mathcal{D}, \partial\mathcal{D})$. Then*

$$E(\tilde{u}) \leq \mathrm{vol}_\lambda(\mathcal{D}) .$$

PROOF: We may of course assume that $\tilde{u}$ is not constant. Then (with $Q, \tilde{v}, \gamma$ as in the proof of proposition 2.1)

$$\begin{aligned} E(\tilde{u}) = \sup_{\phi\in\Sigma} \int_Q \tilde{v}^* \, d(\phi\lambda) &= \sup_{\phi\in\Sigma} \int_{\partial Q} \tilde{v}^*(\phi\lambda) \\ &= \sup_{\phi\in\Sigma} \left( - \int_{-1}^{+1} \phi(b(s,1)) \cdot v(\,.\,,1)^*\lambda \right) \\ &= \sup_{\phi\in\Sigma} \left( -\phi(0) \int_\gamma \lambda \right) = - \int_\gamma \lambda \end{aligned}$$

(recall that $b(s,1) \equiv 0$ , $v(s,0) \in \mathcal{L}$ and $v(\pm 1,\,.\,) \equiv \text{const}$). Let now $\gamma_0$ be a path in $\mathcal{L}$ connecting $\gamma(+1)$ with $\gamma(-1)$ and denote by $\mathcal{D}_\gamma \subset \mathcal{D}$ the portion of $\mathcal{D}$ bounded by $\gamma([-1,+1]) \cup \gamma_0([-1,+1])$. Then by Stokes' theorem

$$E(\tilde{u}) = -\int_\gamma \lambda = \int_{\gamma_0} \lambda - \int_{\mathcal{D}_\gamma} d\lambda = -\int_{\mathcal{D}_\gamma} d\lambda \le \int_{\mathcal{D}_\gamma} |g_\lambda|\sigma \le \mathrm{vol}_\lambda(\mathcal{D})\,.$$

□

*Remark.* The above assumption on the path $s \mapsto u(s,1) \in \mathcal{D}$ will always be satisfied in this and the subsequent papers [3, 4].

## 3 Setting up the Nonlinear Equation in the Normal Direction

The aim of this section is to find a suitable analytic framework to describe solutions of the boundary value problem (1.1) which are close to a given embedded solution $\tilde{u}_0$. We have shown in Corollary 2.2 that the functions $\alpha_\pm(s)$ obtained by applying the asymptotic formula (Theorem 1.5) to $\tilde{u}_0$ converge to $\lambda_\pm$ with $\lambda_\pm \in \{\pm\frac{\pi}{2}\} \cup \mathbf{Z}\pi$ as $s \to \pm\infty$. We will have to take into account that the ends of all the possible solutions do not remain fixed. Since we are interested in solutions which can be viewed as a graph over $\tilde{u}_0$, we split the complex vector bundle $(\tilde{u}_0^*(\mathbf{R} \times M), \tilde{J}(\tilde{u}_0)) \to S$ into the direct sum of $T\tilde{u}_0(S)$ and a normal component $\nu$, which will both be complex line bundles with respect to $\tilde{J}(u_0)$. We then investigate the differential equation for the normal component of a potential nearby solution $\tilde{u}$. The normal bundle $\nu$ has to satisfy several conditions, for example the splitting should be compatible with the boundary conditions.

DEFINITION 3.1 A nowhere vanishing vector field $n$ along $\tilde{u}_0$ is called an *admissible normal vector field* if it satisfies the following conditions:

(1) $n(s,t) \notin \mathrm{Span}_{\mathbf{R}}\{\partial_s\tilde{u}_0(s,t), \partial_t\tilde{u}_0(s,t)\}$ for all $(s,t) \in S$.

(2) We have

$$\frac{n(s,t)}{|n(s,t)|} \stackrel{s\to\pm\infty}{\longrightarrow} n_{\pm\infty}(t)$$

uniformly in $t$ with all derivatives, where

$$n_{\pm\infty}(t) = \begin{cases} (0, \pm 1, 0, 0) & \text{if } \lambda_\pm = \mp\frac{\pi}{2} \\ (\cos(\pi t/2), -q_\pm(0)\cos(\pi t/2), 0, \mp\sin(\pi t/2)) & \text{if } \lambda_\pm \in \mathbf{Z}\pi \end{cases}$$

in the coordinates (1.3),(1.4) near the endpoints

$$(0, p_\pm) = \lim_{s\to\pm\infty} \tilde{u}_0(s,t) \in \{0\} \times \mathcal{L}$$

of $\tilde{u}_0$,

(3) $n(s,0) \in T_{\tilde{u}_0(s,0)}(\mathbf{R} \times \mathcal{L})$,

(4) There is a path $\gamma = a_1 + ia_2 : \mathbf{R} \to S^1 \subset \mathbf{C}$ so that with $m(s,t) := \tilde{J}(\tilde{u}_0(s,t))\, n(s,t)$ we have

$$a_1(s)n(s,1) + a_2(s)m(s,1) \in T_{\tilde{u}_0(s,1)}(\{0\} \times \mathcal{D}^*)$$

and there is some $R_0 > 0$ such that $\gamma(s) \equiv \pm 1$ for $|s| \geq R_0$.

LEMMA 3.2 *Let $\tilde{u}_0$ be an immersed solution of the boundary value problem* (1.1). *Then $\tilde{u}_0$ has an admissible normal vector field.*

PROOF: Assume that $e_1$ is a nowhere vanishing section in $u_0^*\xi$ which equals

$$\hat{e}_1(s,t) = \big(0, 1, -q'(\bar{\theta})\bar{y} + q(\bar{\theta})(\bar{x} + q(\bar{\theta})\bar{y}), -(\bar{x} + q(\bar{\theta})\bar{y})\big)$$

in coordinates for large $|s|$, where $\tilde{u}_0(s,t) = v(s,t) = (\bar{\tau}, \bar{\theta}, \bar{x}, \bar{y})(s,t)$. Then we define section $e_2$ by $\tilde{J}(u_0(s,t))e_1(s,t) = -e_2$ which equals $-\hat{e}_2 = (0,0,-1,0)$ for large $|s|$ in coordinates. Denote by $(\pi_\lambda\partial_s u_0)_1$ and $(\pi_\lambda\partial_s u_0)_2$ the components of $\pi_\lambda\partial_s u_0$ along $e_1$ and $e_2$ respectively. Define now

$$\bar{n}(s,t) := \big(-(\pi_\lambda\partial_s u_0)_1, \partial_s a_0 \cdot e_1 - (\pi_\lambda\partial_s u_0)_2 X_\lambda(u_0) + (\lambda(u_0)\partial_s u_0)\cdot e_2\big)(s,t)\,,$$

which satisfies conditions 1. and 3. of the lemma. We may now write for large $|s|$ (with $m_{\pm\infty}(t) := \tilde{J}(p_\pm)n_{\pm\infty}(t)$ , $p_\pm := \lim_{s\to\pm\infty}\tilde{u}_0(s,t)$):

$$\begin{aligned} \bar{n}(s,t) &= \alpha_1(s,t)\partial_s\tilde{u}_0(s,t) + \alpha_2(s,t)\partial_t\tilde{u}_0(s,t) \\ &\quad + \beta_1(s,t)n_{\pm\infty}(t) + \beta_2(s,t)m_{\pm\infty}(t) \end{aligned}$$

with suitable functions $\alpha_1, \alpha_2, \beta_1, \beta_2$ so that $\beta_1^2 + \beta_2^2$ never vanishes and $\beta_1(s,0) \neq 0$ , $\beta_2(s,0) \equiv 0$. We modify the coefficient functions $\beta_1, \beta_2$ so that $\beta_1 \equiv 1$ , $\beta_2 \equiv 0$ for large $|s|$ retaining the properties $\beta_1^2 + \beta_2^2 \neq 0$ everywhere and $\beta_1(s,0) \neq 0$ , $\beta_2(s,0) \equiv 0$. This is possible because we did not specify $(\beta_1, \beta_2)$ along $\mathbf{R} \times \{1\}$. We also modify $\alpha_1, \alpha_2$ so that they are zero for large $|s|$. We have now achieved that $\bar{n}$ satisfies condition 2. of the lemma as well. In coordinates, the boundary condition $T_{\tilde{u}_0(s,1)}(\{0\} \times \mathcal{D}^*)$ is represented by $\{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R} \subset \mathbf{R}^4$. For large $|s|$, we have already $\bar{n}(s,1) \in \{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}$. There is up to sign a unique map

$$(b\,; a_1 + ia_2) : \mathbf{R} \to \mathbf{R} \times S^1$$

such that

$$b(s)\partial_t\tilde{u}_0(s,1) + a_1(s)\bar{n}(s,1) + a_2(s)\bar{m}(s,1) \in T_{\tilde{u}_0(s,1)}(\{0\} \times \mathcal{D}^*)\,,$$

where $\bar{m}(s,t) := \tilde{J}(\tilde{u}_0(s,t))\bar{n}(s,t)$. Defining now

$$n(s,t) := tb(s)\big(a_2(s)\partial_s\tilde{u}_0(s,t) + a_1(s)\partial_t\tilde{u}_0(s,t)\big) + \bar{n}(s,t)$$

and $m(s,t) := \tilde{J}(\tilde{u}_0(s,t))n(s,t)$ we see that

$$a_1(s)n(s,1) + a_2(s)m(s,1) \in T_{\tilde{u}_0(s,1)}(\{0\} \times \mathcal{D}^*)$$

as required while conditions 1–3 are still satisfied. □

PROPOSITION 3.3 *Given an embedded solution $\tilde{u}_0$ of the boundary value problem* (1.1). *Let $n$ be an admissible normal vector and let $\gamma = a_1 + ia_2 : \mathbf{R} \to S^1 \subset \mathbf{C}$ be a path as in Definition* 3.1. *Writing $a_1(s) + ia_2(s) = e^{i\phi(s)}$ for a suitable smooth function $\phi : \mathbf{R} \to \mathbf{R}$, the number*

$$\mu(\tilde{u}_0) := \frac{1}{\pi}\Big(\lim_{s\to+\infty}\phi(s) - \lim_{s\to-\infty}\phi(s)\Big) \in \mathbf{Z}$$

*is independent of the choices of $n$ and $\phi$. Moreover, if $\tilde{u}_\tau$ is a smooth family of solutions of* (1.1) *then $\mu(\tilde{u}_\tau)$ is independent of $\tau$.*

DEFINITION 3.4 We call $\mu(\tilde{u}_0)$ above the *Maslov index* of $\tilde{u}_0$.

PROOF OF PROPOSITION 3.3: Assume that $\hat{n}$ is another admissible normal vector. If we write

$$\hat{n}(s,t) = (\alpha_1\partial_s\tilde{u}_0 + \alpha_2\partial_t\tilde{u}_0 + \beta_1 n + \beta_2 m)(s,t)\,,$$

then $\beta_1^2(s,t) + \beta_2^2(s,t) \neq 0$ for all $(s,t) \in S$, and we have for large $|s|$:

$$\alpha_1, \alpha_2, \beta_2 \equiv 0\,, \quad \beta_1 \equiv 1\,.$$

If $\hat{a}_1, \hat{a}_2 : \mathbf{R} \to \mathbf{R}$ are smooth functions so that

$$\hat{a}_1(s)\hat{n}(s,1) + \hat{a}_2(s)\hat{m}(s,1) \in T_{\tilde{u}_0(s,1)}(\{0\} \times \mathcal{D})$$

and

$$\hat{a}_1^2 + \hat{a}_2^2 \equiv 1$$

then we obtain

$$\begin{aligned}\hat{a}_1(s)\hat{n}(s,1) + \hat{a}_2(s)\hat{m}(s,1) = &\big(\hat{a}_1(s)\alpha_1(s,1) - \hat{a}_2(s)\alpha_2(s,1)\big)\partial_s\tilde{u}_0(s,1)\\ &+ \big(\hat{a}_1(s)\alpha_2(s,1) + \hat{a}_2(s)\alpha_1(s,1)\big)\partial_t\tilde{u}_0(s,1)\\ &+ \big(\hat{a}_1(s)\beta_1(s,1) - \hat{a}_2(s)\beta_2(s,1)\big)n(s,1)\\ &+ \big(\hat{a}_1(s)\beta_2(s,1) + \hat{a}_2(s)\beta_1(s,1)\big)m(s,1)\,.\end{aligned}$$

This means that

$$\begin{aligned}&\big(\hat{a}_1(s)\alpha_2(s,1) + \hat{a}_2(s)\alpha_1(s,1)\big)\partial_t\tilde{u}_0(s,1)\\ &\quad + \big(\hat{a}_1(s)\beta_1(s,1) - \hat{a}_2(s)\beta_2(s,1)\big)n(s,1)\\ &\qquad + \big(\hat{a}_1(s)\beta_2(s,1) + \hat{a}_2(s)\beta_1(s,1)\big)m(s,1) \in T_{\tilde{u}_0(s,1)}(\{0\} \times \mathcal{D})\end{aligned}$$

and therefore

$$(\hat{a}_1 + i\hat{a}_2)(s)(\beta_1 + i\beta_2)(s,1) = \pm\rho(s)(a_1 + ia_2)(s)$$

with a suitable positive function $\rho$. Writing $(\hat{a}_1 + i\hat{a}_2)(s) = e^{i\hat{\phi}(s)}$ and $(\beta_1 + i\beta_2)(s,1) = \rho(s)e^{i\psi(s)}$ , $\psi_\pm = \lim_{s\to\pm\infty}\psi(s) \in 2\pi\mathbf{Z}$, we see that $\hat{\phi}_+ - \hat{\phi}_- = \phi_+ - \phi_-$ if $\psi_+ = \psi_-$. Recall that $(\beta_1 + i\beta_2)(s,t) \equiv 1$ if $|s|$ is sufficiently large. Moreover, $\beta_2(s,0) \equiv 0$ since $\partial_s\tilde{u}_0(s,0), n(s,0) \in T_{\tilde{u}_0(s,0)}(\mathbf{R}\times\mathcal{L})$ and the curve

$$\beta_1 + i\beta_2\big|_{\partial([-s_0,s_0]\times[0,1])}, \quad s_0 > 0 \text{ large},$$

has degree zero, hence $\psi_+ = \psi_-$, and the Maslov index does not depend on the choice of the admissible normal vector field $n$ along $\tilde{u}_0$. The function $\phi(s)$ used to compute $\mu(\tilde{u}_0)$ is determined modulo $\mathbf{Z}\pi$, hence $\mu(\tilde{u}_0)$ does not depend on the choice of $\phi(s)$ either. If $\hat{n}_\tau$ is a continuous family of admissible normal vectors, then similar considerations as the one above show that the Maslov index does not depend on the parameter $\tau$. □

We will calculate now the Maslov index of the local solutions $\tilde{u}_\tau$ in (1.2). We need an admissible normal vector field $n$ along $\tilde{u}_\tau$ in the sense of definition 3.1. For the computation of the Maslov index it is sufficient to produce a normal vector field which satisfies $n(s,0) \in T_{\tilde{u}_\tau(s,0)}(\mathbf{R}\times\mathcal{L})$ and $n(s,t) \to (0,\pm1,0,0)$ as $s \to \pm\infty$ (this is less than required by the definition of admissibility).

The following vector field along $\tilde{u}_\tau(s,t)$ does the job:

$$\begin{aligned} n(s,t) &= \frac{d}{d\tau}\tilde{u}_\tau(s,t) \\ &= \Bigg( -\frac{\tau\cos\left(\frac{\pi t}{2}\right)}{\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi s}{2}\right)}, \frac{\sinh\left(\frac{\pi s}{2}\right)}{\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi s}{2}\right)}, \\ &\qquad -\frac{\sin\left(\frac{\pi t}{2}\right)}{\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi s}{2}\right)}, \frac{\tau\sin\left(\frac{\pi t}{2}\right)\sinh\left(\frac{\pi s}{2}\right)}{\left[\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi s}{2}\right)\right]^2} \Bigg). \end{aligned}$$

Indeed, $n(s,1) \in T_{\tilde{u}_\tau(s,1)}(\{0\}\times\mathcal{D})$ since $\tilde{u}_\tau(s,1) \in \{0\}\times\mathcal{D}^*$ for all $s$ and $\tau$. We also have

$$n(s,0) \in T_{\tilde{u}_\tau(s,0)}(\mathbf{R}\times\mathcal{L})$$

and

$$\lim_{s\to\pm\infty} n(s,t) = (0,\pm1,0,0)$$

uniformly in $t$. We define

$$\begin{aligned} m(s,t) &= \tilde{J}_\tau(\tilde{u}_\tau(s,t))n(s,t) \\ &= \Bigg(0, -\frac{\sin\left(\frac{\pi t}{2}\right)}{\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi s}{2}\right)}, \\ &\qquad -\frac{\sinh\left(\frac{\pi s}{2}\right)}{\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi s}{2}\right)}, -\frac{\tau\left[1+\cos\left(\frac{\pi t}{2}\right)\cosh\left(\frac{\pi s}{2}\right)\right]}{\left[\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi s}{2}\right)\right]^2}\Bigg). \end{aligned}$$

Recalling the definition of the Maslov index we have to find functions $a_1(s)$, $a_2(s)$, and $b(s)$ such that

$$b(s)\partial_t\tilde{u}_\tau(s,1) + a_1(s)n(s,1) + a_2(s)m(s,1) \in T_{\tilde{u}_\tau(s,1)}(\{0\}\times\mathcal{D})$$

and $a_1^2 + a_2^2 \equiv 1$. Writing $(a_1 + ia_2)(s) = \rho(s)e^{i\phi(s)}$, we have defined $\mu(\tilde{u}_\tau)$ by

$$\mu(\tilde{u}_\tau) := \frac{1}{\pi}\Big(\lim_{s\to\infty}\phi(s) - \lim_{s\to-\infty}\phi(s)\Big).$$

In our case, $b \equiv a_2 \equiv 0$ and $a_1 \equiv 1$, hence the Maslov index $\mu(\tilde{u}_\tau)$ is zero. We summarize:

PROPOSITION 3.5 *Let* $(\tilde{u}_\tau)$ *be a continuous family of solutions of the boundary value problem* (1.1) *which agrees with the local family in* (1.2) *for small* $\tau$. *Then the Maslov index of each solution* $\tilde{u}_\tau$ *equals zero.*

We choose a metric on $\mathbf{R}\times M$ so that $\mathbf{R}\times\mathcal{L}$ and $\{0\}\times F$ are totally geodesic, where $\{0\}\times F$ is a closed surface obtained by doubling the disk $\{0\}\times\mathcal{D}$. In the coordinates (1.3), where both $\mathbf{R}\times\mathcal{L}$ and $\{0\}\times F$ are given by linear subspaces of $\mathbf{R}^4$, we want this metric to be the standard Euclidean metric. Denote by exp the exponential map with respect to this metric. Let $n(s,t) \in T_{\tilde{u}_0(s,t)}(\mathbf{R}\times M)$ be an admissible normal vector as in Definition 3.1 and

$$m(s,t) = \tilde{J}(\tilde{u}_0(s,t))n(s,t).$$

At this point, we want to be more specific about the behavior of $n(s,t)$ for large $|s|$ (we write $n_+$ for large positive $s$ and $n_-$ for large $|s|$ , $s<0$). If one of the eigenvalues $\lambda_\pm$ obtained by applying the asymptotic formula (Theorem 1.5) to $\tilde{u}_0$ is an integer multiple of $\pi$ then we ask for

$$n_\pm(s,t) = \kappa_\pm\sqrt{\rho_\pm(s)}\left(\cos\left(\frac{\pi t}{2}\right), -q_\pm(0)\cos\left(\frac{\pi t}{2}\right), 0, \mp\sin\left(\frac{\pi t}{2}\right)\right) \tag{3.1}$$

with $\rho_\pm(s) = e^{\int_{s_0}^{s}\alpha_\pm(\tau)d\tau}$. In the cases $\lambda_+ = -\pi/2$ or $\lambda_- = \pi/2$ we ask for

$$n_\pm(s,t) = \sqrt{\rho_\pm(s)}\,(0, \pm 1, 0, 0) \tag{3.2}$$

respectively. If $R_0 > 0$ is the number as in definition 3.1, then pick smooth functions $\beta_\pm : \mathbf{R} \to [0,1]$ so that $\beta_+(s) \equiv 1$ for $s \geq R_0 + 1$ , $\beta_+(s) \equiv 0$ for $s \leq R_0$ and $\beta_-(s) := \beta_+(-s)$. For small $\varepsilon > 0$ and $c_\pm \in \mathbf{R}$ we define the maps

$$\Phi_{c_-,c_+} : S \times \mathbf{R}^2 \supset S \times B_\varepsilon(0) \longrightarrow \mathbf{R} \times M\,, \tag{3.3}$$

$$\Phi_{c_-,c_+}(s,t,x,y) := \exp_{\tilde{u}_0(s,t)}\big(xn(s,t) + ym(s,t) + [c_-\beta_-(s) + c_+\beta_+(s)](0,1,0,0)\big)\,.$$

We are using the parameters $c_\pm$ to move the ends along $\mathcal{L}$. We will sometimes drop the subscript $c_\pm$ and just write $\Phi$. For large $|s|$ the maps $\Phi_{c_-,c_+}$ are then given in coordinates by

$$\Phi_+ : ([R_0,\infty) \times [0,1]) \times \mathbf{R}^2 \longrightarrow \mathbf{R}^4$$

$$\Phi_- : ((-\infty,-R_0] \times [0,1]) \times \mathbf{R}^2 \longrightarrow \mathbf{R}^4$$

$$\begin{aligned}\Phi_\pm(s,t,x,y) = {}& v_\pm(s,t) + xn(s,t)\\ &+ ym(s,t) + [c_-\beta_-(s) + c_+\beta_+(s)](0,1,0,0)\end{aligned} \tag{3.4}$$

which equals

$$\Phi_\pm(s,t,x,y) = v_\pm(s,t) + xn(s,t) + ym(s,t) + c_\pm(0,1,0,0). \tag{3.5}$$

if $|s| \geq R_0 + 1$. In this case we also have

$$D\Phi(s,t,x,y) = \big(\partial_s v_\pm + x\partial_s n + y\partial_s m \mid \partial_t v_\pm + x\partial_t n + y\partial_t m | n | m\big)\,,$$

which does not depend on the parameters $c_\pm$ anymore. If $c_\pm$ are sufficiently small then each map $\Phi_{c_-,c_+}$ is an immersion on some neighborhood of $S \times \{0\}$ whose diameter in the $\mathbf{R}^2$-direction may shrink to zero as $|s| \to \infty$ (recall that $\tilde{u}_0(s,\,.\,)$ converges to points on $\{0\} \times \mathcal{L}$ as $|s| \to \infty$). We obtain almost complex structures $\bar{J}_{c_-,c_+}(z,w) = \bar{J}(z,w)$ on a neighborhood of $S \times \{0\} \subset S \times \mathbf{R}^2$ by

$$\bar{J}_{c_-,c_+}(z,w) := (D\Phi_{c_-,c_+}(z,w))^{-1} \circ \tilde{J}(\Phi_{c_-,c_+}(z,w)) \circ D\Phi_{c_-,c_+}(z,w)\,, \tag{3.6}$$

where $z \in S$ and $w \in \mathbf{R}^2$. We remark at this point that the existence of $\bar{J}$ is only clear on some neighborhood of $S \times \{0\}$ whose diameter shrinks sufficiently fast to zero at the ends. This is because the inverse of $D\Phi(z,w)$ fails to exist in the limit $|z| \to \infty$. We would like to describe solutions $\tilde{u}$ near to $\tilde{u}_0$ which are of the form

$$\tilde{u}(s,t) = \Phi_{c_-,c_+}(s,t,x(s,t),y(s,t))\,.$$

This means we have to make sure that $D\Phi(s,t,x(s,t),y(s,t))$ is invertible in order to get a meaningful PDE for $(x,y)$. Hence, we have to analyze the precise asymptotic behavior of $D\Phi$ and also $\bar{J}$. Before doing so, let us first derive formally the partial differential equation for $(x,y)$ so that $\tilde{u}$ as above is a pseudoholomorphic strip.

If we decompose $\bar{J}_{c_-,c_+}(z,w) = \bar{J}(z,w)$ into blocks of $2 \times 2$-matrices

$$\bar{J}(z,w) := \begin{pmatrix} j_1(z,w) & J_2(z,w) \\ J_1(z,w) & j_2(z,w)\end{pmatrix}, \tag{3.7}$$

then $J_1(z,0) \equiv J_2(z,0) \equiv 0$ and $j_1(z,0) \equiv j_2(z,0) \equiv i$ if $c_- = c_+ = 0$, where $i$ denotes the standard complex structure on $\mathbf{C} \approx \mathbf{R}^2$. We also note that

$$j_1^2 + J_2 J_1 = j_2^2 + J_1 J_2 = -\operatorname{Id}_{\mathbf{R}^2}$$

and

$$j_1 J_2 + J_2 j_2 \equiv J_1 j_1 + j_2 J_1 \equiv 0\,.$$

Searching for $\tilde{J}$-holomorphic curves near $\tilde{u}_0$ then translates into searching for $\bar{J}_{c_-,c_+}$-holomorphic curves in $S \times \mathbf{R}^2$: If $i(z)$ is a complex structure on $S$ then any solution of

$$v = (\zeta, w) : S \longrightarrow S \times \mathbf{R}^2\,,$$

$$Dv(z) \circ i(z) = \bar{J}_{c_-,c_+}(v(z)) \circ Dv(z)\,, \tag{3.8}$$

satisfying the boundary conditions

$$v(s,0) \in (\mathbf{R} \times \{0\}) \times (\mathbf{R} \times \{0\})\,,$$

$$\zeta(s,1) \in \mathbf{R} \times \{1\}\,, \quad w(s,1) \in \mathbf{R} \cdot \big(a_1(\operatorname{Re}\zeta(s,1)) + i a_2(\operatorname{Re}\zeta(s,1))\big)\,,$$

yields a $i(z) - \bar{J}$-holomorphic curve $\tilde{u} = \Phi_{c_-,c_+} \circ v$ in $\mathbf{R} \times M$ with $\tilde{u}(s,0) \in \mathbf{R} \times \mathcal{L}$ and $\tilde{u}(s,1) \in \{0\} \times \mathcal{D}$ provided that $\bar{J}_{c_-,c_+}(v(z))$ exists of course. We remark at this point that every almost complex structure $i(z)$ on $S$ is actually a complex structure. This follows from the result by Newlander and Nirenberg since the Nijenhuis tensor always vanishes in real dimension two, see also [14] for a sketch of an elementary proof or [7]. If we want to find $\tilde{J}$-holomorphic curves near $\tilde{u}_0$ of the form $\tilde{u}(s,t) = \Phi_{c_-,c_+}(s,t,x(s,t),y(s,t))$ then we have to allow the complex structure on $S$ to vary. This will not cause any difficulties because after having shown existence of $\tilde{J}$-holomorphic curves with respect to a different complex structure $i(z)$ on the strip $S$, we will be able to reparametrize the domain and obtain $\tilde{J}$-holomorphic curves with respect to the standard complex structure $i$ on $S$. We will demonstrate in appendix B that all the complex structures $i(z)$ that might occur in our case are in fact equivalent to the standard one.

Inserting $v(z) = (z, w(z))$ into the differential equation (3.8) we obtain

$$\begin{aligned}(i(z),\, Dw(z) \circ i(z)) = \\ \big(j_1(z,w(z)) + J_2(z,w(z)) \circ Dw(z),\; J_1(z,w(z)) + j_2(z,w(z)) \circ Dw(z)\big)\,.\end{aligned} \tag{3.9}$$

Eliminating $i(z)$ we get the equations

$$i(z) = j_1(z,w(z)) + J_2(z,w(z)) \circ Dw(z) \tag{3.10}$$

and

$$\begin{aligned}Dw(z) \circ j_1(z,w(z)) + Dw(z) \circ J_2(z,w(z)) \circ Dw(z) = \\ J_1(z,w(z)) + j_2(z,w(z)) \circ Dw(z)\,.\end{aligned} \tag{3.11}$$

Our aim will be to find a 1-parameter family $(c_-^{(\tau)}, c_+^{(\tau)}, w_\tau : S \to \mathbf{C})$, of solutions of (3.11) near the trivial solution $(c_-^{(0)}, c_+^{(0)}, w_0) \equiv (0,0,0)$ where $w_\tau$ satisfies the boundary conditions $w_\tau(s,0) \in \mathbf{R}$ and $w_\tau(s,1) \in \mathbf{R} \cdot (a_1(s) + ia_2(s))$. In coordinates $s + it$ on $S$ we look at the equivalent equation

$$
\begin{aligned}
F(c_-, c_+, w)(z) := &-Dw(z)\big[j_1^{(c_-,c_+)}(z, w(z))(0,1)\big] \\
&+ j_2^{(c_-,c_+)}(z, w(z))\partial_t w(z) \\
&- Dw(z)\big[J_2^{(c_-,c_+)}(z, w(z))\partial_t w(z)\big] \\
&+ J_1^{(c_-,c_+)}(z, w(z))(0,1) \\
= &\ 0
\end{aligned}
\tag{3.12}
$$

(this time we indicated the parameters $c_\pm$). We have to set up the right function spaces so that $F$ is well-defined and so that it becomes a smooth (nonlinear) Fredholm map. We point out that the space $H^{1,p}(S, \mathbf{R}^2)$ does not work because of the nonlinearity of equation (3.12). It will also turn out that $F$ will not be a Fredholm map unless we consider weighted Sobolev spaces.

We will now compute the almost complex structures $\bar{J}_{c_-,c_+}(z,w)$ for $|z|$ large. Although $D\Phi$ does not depend on the parameters $c_\pm$, the almost complex structure $\bar{J}$ will. Let $\rho$ be a smooth function which agrees with $e^{\int_{s_0}^s \alpha(\tau)d\tau}$ for $|s|$ large (so that $\rho(s) \to 0$ for $|s| \to \infty$, $\alpha(\tau)$ as in the asymptotic formula, Theorem 1.5). We first show that $S \times \{0\}$ has a neighborhood of uniform radius $\varepsilon$ in the $\mathbf{R}^2$-direction where all maps $\Phi_{c_-,c_+}$, $|c_\pm| \le \varepsilon$, are immersions so that $\bar{J}_{c_-,c_+}$ is in fact defined on $S \times B_\varepsilon(0)$ if $|c_\pm| \le \varepsilon$.

LEMMA 3.6 *For every $\varepsilon > 0$ there is $R = R(\varepsilon) > 0$ such that $D\Phi(s,t,x,y)$ is invertible for all $(s,t,x,y) \in S \times \mathbf{R}^2$ with $|s| \ge R$ and $x^2 + y^2 \le \varepsilon^2$.*

*Remark.* The derivative of $\Phi_{0,0}$ along $S \times \{0\}$ is given by

$$
D\Phi_{0,0}(s,t,0,0) = \big(\partial_s \tilde{u}_0(s,t) \mid \partial_t \tilde{u}_0(s,t) \mid n(s,t) \mid m(s,t)\big),
$$

which is nonsingular. Hence $D\Phi_{0,0}$ is also nonsingular on $([-R,R] \times [0,1]) \times B_{\tilde{\varepsilon}}(0)$ for a suitable $\tilde{\varepsilon} > 0$. This is still true for $c_\pm$ nonzero, but small. The above lemma takes care of the ends at infinity, so that $D\Phi_{c_-,c_+}(s,t,x,y)$ are in fact invertible on a set of the form $S \times B_\varepsilon(0) \subset S \times \mathbf{R}^2$ if $|c_\pm|$ are sufficiently small. Therefore, there is $\varepsilon > 0$ such that $\bar{J}_{c_-,c_+}(z,w)$ are defined for all $z \in S$ and $|c_\pm|, |w| \le \varepsilon$.

PROOF OF LEMMA 3.6: We are only concerned with $\Phi(s,t,x,y)$ where $|s|$ is large. Therefore, we pass to coordinates so that $\Phi_\pm$ are given by (3.5) and

$$
\begin{aligned}
D\Phi_\pm(s,t,x,y)(\sigma,\tau,\xi,\eta) = &\ \sigma\big(\partial_s v(s,t) + x\partial_s n(s,t) + y\partial_s m(s,t)\big) \\
&+ \tau\big(\partial_t v(s,t) + x\partial_t n(s,t) + y\partial_t m(s,t)\big) \\
&+ \xi n(s,t) + \eta m(s,t)\,.
\end{aligned}
\tag{3.13}
$$

We first consider the case where $\lambda = \lambda_\pm = \mp\pi/2$. Using (3.2), (1.10), and (1.11) we see that the linear map $D\Phi_\pm(s,t,x,y)$ is given by the matrix $\Xi_\pm + \Delta_\pm$

with

$$\Xi_\pm(s,t,x,y) = \tag{3.14}$$
$$\kappa_\pm\rho(s)\begin{pmatrix} -\alpha_\pm(s)\cos(\lambda t) & \alpha_\pm(s)\sin(\lambda t) & 0 & 0 \\ \alpha_\pm(s)[q(0)\cos(\lambda t)\pm\frac{x}{2}\frac{1}{\sqrt{\rho(s)}}] & -\alpha_\pm(s)q(0)\sin(\lambda t) & \pm\frac{1}{\sqrt{\rho(s)}} & 0 \\ \mp\frac{y}{2}\frac{\alpha_\pm(s)}{\sqrt{\rho(s)}} & 0 & 0 & \mp\frac{1}{\sqrt{\rho(s)}} \\ -\alpha_\pm(s)\sin(\lambda t) & -\alpha_\pm(s)\cos(\lambda t) & 0 & 0 \end{pmatrix}$$

and

$$\Delta_\pm(s,t,x,y) = \rho(s)E_\pm(s,t,x,y)\,,$$

where $E_\pm$ is a $4\times 4$-matrix valued function which converges to 0 uniformly in $t$ with all its derivatives as $s\to\pm\infty$ if $x^2+y^2$ remains bounded. We have used here that

$$m(s,t) = \kappa_\pm\sqrt{\rho(s)}\,(0,0,\mp 1,0) + \rho^{3/2}(s)b(s,t) \tag{3.15}$$

for a suitable bounded vector valued function $b$. The inverse of $\Xi_\pm$ is given by

$$\Xi_\pm(s,t,x,y)^{-1} = \frac{1}{\kappa_\pm\alpha_\pm(s)\rho(s)}\times \tag{3.16}$$
$$\begin{pmatrix} -\cos(\lambda t) & 0 & 0 & -\sin(\lambda t) \\ \sin(\lambda t) & 0 & 0 & -\cos(\lambda t) \\ \frac{1}{2}x\alpha_\pm(s)\cos(\lambda t)\pm\alpha_\pm(s)q(0)\sqrt{\rho(s)} & \pm\alpha_\pm(s)\sqrt{\rho(s)} & 0 & \frac{1}{2}x\alpha_\pm(s)\sin(\lambda t) \\ 0 & 0 & \mp\alpha_\pm(s)\sqrt{\rho(s)} & \frac{1}{2}y\alpha_\pm(s)\sin(\lambda t) \end{pmatrix}.$$

The important fact is the following: The norm of $\Xi_\pm^{-1}(s,t,x,y)$ is bounded by $C\rho(s)^{-1}$ where $C>0$ is a constant not depending on $s,t$ and $x,y$ as long as $x^2+y^2$ remains bounded. On the other hand, we have $\|\Delta_\pm\|\le\varepsilon(s,t,x,y)\rho(s)$ where $\varepsilon(s,t,x,y)$ is a smooth function satisfying

$$\sup_{t,(x,y)\in B_r(0)}\varepsilon(s,t,x,y)\to 0$$

as $s\to\pm\infty$. Therefore,

$$\|\Delta_\pm\| < \frac{1}{\|\Xi_\pm^{-1}\|}$$

if $|s|$ is sufficiently large. This implies that for every $\varepsilon>0$ there is some $R=R(\varepsilon)>0$ such that $(\Xi_\pm+\Delta_\pm)(s,t,x,y)$ is invertible for all $|x|,|y|\le\varepsilon$, $t\in[0,1]$ and $|s|\ge R$ with inverse

$$(\Xi_\pm+\Delta_\pm)^{-1} = \left(\sum_{n=0}^{\infty}(-\Xi_\pm^{-1}\Delta_\pm)^n\right)\Xi_\pm^{-1}\,. \tag{3.17}$$

The case where $\lambda_\pm$ is an integer multiple of $\pi$ is handled similarly. Because

$$\begin{aligned} m(s,t) &= \kappa\sqrt{\rho(s)}\left(\pm\sin\left(\frac{\pi t}{2}\right),\mp q(0)\sin\left(\frac{\pi t}{2}\right),0,\cos\left(\frac{\pi t}{2}\right)\right) \\ &\quad+\rho^{3/2}(s)b(s,t) \end{aligned} \tag{3.18}$$

for a suitable bounded vector valued function $b$, we may also decompose

$$D\Phi_\pm(s,t,x,y) = \Xi_\pm(s,t,x,y) + \rho(s)E_\pm(s,t,x,y)\,,$$

where $E_\pm$ and all its derivatives converge to zero as $|s| \to \infty$ uniformly in $t$ as long as $x^2+y^2$ is confined to a bounded set. Using the asymptotic formula (1.14) and (3.1), the matrix $\Xi_\pm(s,t,x,y)$ is now given by

(3.19)

$$\Xi_\pm(s,t,x,y) = \kappa\sqrt{\rho(s)}\begin{pmatrix} \frac{1}{2}(x\cos(\frac{\pi t}{2}) \pm y\sin(\frac{\pi t}{2}))\alpha_\pm(s) & \frac{\pi}{2}(\pm y\cos(\frac{\pi t}{2}) - x\sin(\frac{\pi t}{2})) & \cos(\frac{\pi t}{2}) & \pm\sin(\frac{\pi t}{2}) \\ (\sqrt{\rho(s)}\cos(t\lambda_\pm) + \frac{1}{2}q(0)[-x\cos(\frac{\pi t}{2}) \mp y\sin(\frac{\pi t}{2})])\alpha_\pm(s) & \frac{\pi}{2}q(0)(x\sin(\frac{\pi t}{2}) \mp y\cos(\frac{\pi t}{2})) - \sqrt{\rho(s)}\alpha_\pm(s)\sin(t\lambda_\pm) & -q(0)\cos(\frac{\pi t}{2}) & \mp q(0)\sin(\frac{\pi t}{2}) \\ -\sqrt{\rho(s)}\alpha_\pm(s)\sin(t\lambda_\pm) & -\sqrt{\rho(s)}\alpha_\pm(s)\cos(t\lambda_\pm) & 0 & 0 \\ \frac{1}{2}\alpha_\pm(s)(y\cos(\frac{\pi t}{2}) \mp x\sin(\frac{\pi t}{2})) & \frac{\pi}{2}(\mp x\cos(\frac{\pi t}{2}) - y\sin(\frac{\pi t}{2})) & \mp\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix}.$$

We calculate

$$\text{(3.20)}\quad \Xi_\pm(s,t,x,y)^{-1} = \frac{1}{\kappa\alpha(s)\rho(s)}$$

$$\cdot\begin{pmatrix} q(0)\cos(t\lambda_\pm) & \cos(t\lambda_\pm) & -\sin(t\lambda_\pm) & 0 \\ -q(0)\sin(t\lambda_\pm) & -\sin(t\lambda_\pm) & -\cos(t\lambda_\pm) & 0 \\ \alpha(s)\sqrt{\rho(s)}\cos\left(\frac{\pi t}{2}\right) - \frac{1}{2}[\mp\pi y q(0)\sin(t\lambda_\pm) + xq(0)\alpha(s)\cos(t\lambda_\pm)] & \frac{1}{2}[\pm\pi y\sin(t\lambda_\pm) - x\alpha(s)\cos(t\lambda_\pm)] & \frac{1}{2}[\pm\pi y\cos(t\lambda_\pm) + x\alpha(s)\sin(t\lambda_\pm)] & \mp\alpha(s)\sqrt{\rho(s)}\sin\left(\frac{\pi t}{2}\right) \\ \pm\alpha(s)\sqrt{\rho(s)}\sin\left(\frac{\pi t}{2}\right) + \frac{1}{2}[\mp\pi xq(0)\sin(t\lambda_\pm) - yq(0)\alpha(s)\cos(t\lambda_\pm)] & \frac{1}{2}[\mp\pi x\sin(t\lambda_\pm) - y\alpha(s)\cos(t\lambda_\pm)] & \frac{1}{2}[\mp\pi x\cos(t\lambda_\pm) + y\alpha(s)\sin(t\lambda_\pm)] & \alpha(s)\sqrt{\rho(s)}\cos\left(\frac{\pi t}{2}\right) \end{pmatrix},$$

and we note that $|\Xi^{-1}| \le c\rho(s)^{-1}$ for a suitable constant $c$, and the argument is the same as before. □

The following two lemmas are vital for the functional analytic framework for equation (3.12). They describe the asymptotic behavior of the almost complex structure $\bar{J}_{c_-,c_+}(z,w)$ for large $|z|$. Recall that $\rho(s)$ is a smooth function which agrees with $e^{\int_{s_0}^{s}\alpha_\pm(\tau)d\tau}$ for large $|s|$, and $\alpha_\pm(s) \to \lambda_\pm$ are the functions appearing in the asymptotic formula for $\tilde{u}_0$ (Theorem 1.5), $\alpha_-$ for negative $s$ and $\alpha_+$ for positive $s$. Although we will deal with the cases $|\lambda_\pm| = \pi/2$ and $\lambda_\pm \in \mathbf{Z}\pi$ separately, the following remarks apply to both.

Working in coordinates for large $|s|$ we have to compute

$$\begin{aligned}\bar{J}_{c_-,c_+}(s,t,x,y) &= D\Phi_\pm(s,t,x,y)^{-1}\circ\hat{J}(\Phi_{c_-,c_+}(s,t,x,y))\circ D\Phi_\pm(s,t,x,y) \\ &= \big(\Xi_\pm(s,t,x,y) + \Delta_\pm(s,t,x,y)\big)^{-1} \\ &\quad\circ\hat{J}\big(v_\pm(s,t) + xn(s,t) + ym(s,t) + (0,c_\pm,0,0)\big) \\ &\quad\circ\big(\Xi_\pm(s,t,x,y) + \Delta_\pm(s,t,x,y)\big)\,,\end{aligned}$$

where $\hat{J}$ is the representative (1.8) of $\tilde{J}$ in local coordinates. We claim that

$$\bar{J}_{c_-,c_+}(s,t,x,y) = (\mathrm{Id} + \varepsilon_1(s,t,x,y))\,\Xi_\pm(s,t,x,y)^{-1} \cdot \hat{J}(\Phi_{c_-,c_+}(s,t,x,y))\,\Xi_\pm(s,t,x,y) + \varepsilon_2(s,t,x,y)\,, \tag{3.21}$$

where $\varepsilon_1, \varepsilon_2$ are matrix-valued functions which converge to the zero matrix uniformly in $t, x, y$ as $|s| \to \infty$ as long as $(x, y)$ is confined to a bounded set in $\mathbf{R}^2$. Multiplying the identity

$$(\Xi_\pm + \Delta_\pm)^{-1} = \left(\sum_{n=0}^{\infty}(-\Xi_\pm^{-1}\Delta_\pm)^n\right)\Xi_\pm^{-1}$$

with $\Xi_\pm + \Delta_\pm$ from the right and using that $\Xi_\pm^{-1}\Delta_\pm \to 0$ as $|s| \to \infty$ we see that

$$\left(\sum_{n=1}^{\infty}(-\Xi_\pm^{-1}\Delta_\pm)^n\right)(s,t,x,y) \longrightarrow 0$$

as $|s| \to \infty$ uniformly in $t, x, y$. Then the claim follows from

$$\begin{aligned} \bar{J}_{c_-,c_+} &= \left(\Xi_\pm^{-1} + \left(\sum_{n=1}^{\infty}(-\Xi_\pm^{-1}\Delta)^n\right)\Xi_\pm^{-1}\right)\hat{J}(\Phi_{c_-,c_+})(\Xi_\pm + \Delta_\pm) \\ &= \Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Xi_\pm + \Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm \\ &\quad + \left(\sum_{n=1}^{\infty}(-\Xi_\pm^{-1}\Delta_\pm)^n\right)\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Xi_\pm \\ &\quad + \left(\sum_{n=1}^{\infty}(-\Xi_\pm^{-1}\Delta_\pm)^n\right)\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm \end{aligned} \tag{3.22}$$

since

$$\left\|\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm\right\| \le \left\|\rho(s)\Xi_\pm^{-1}\right\|\left\|\hat{J}(\Phi_{c_-,c_+})\right\|\|E_\pm\| \overset{|s|\to\infty}{\longrightarrow} 0\,.$$

Hence the following terms contribute to $\bar{J}(s,t,x,y)$ for large $|s|$: First, there is $\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Xi_\pm$, which we decompose into

$$\Xi_\pm^{-1}\hat{J}_{\pm\infty}\Xi_\pm + \Xi_\pm^{-1}(\hat{J}(\Phi_{c_-,c_+}) - \hat{J}_{\pm\infty})\Xi_\pm\,,$$

where

$$\begin{aligned} \hat{J}_{\pm\infty} &:= \lim_{s\to\pm\infty}\hat{J}(\Phi_{c_-,c_+}(s,t,x,y)) = \hat{J}(0, c_\pm, 0, 0) \\ &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & q(c_\pm) \\ -q(c_\pm) & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.23}$$

Comparing (3.21) with (3.22), we see that $\varepsilon_2(s,t,x,y)$ consists essentially of $\Xi_\pm^{-1}(\hat{J}(\Phi_{c_-,c_+}))\Delta_\pm$.

LEMMA 3.7 *Assume that* $\lambda_\pm = \mp\pi/2$. *Moreover, let* $\Omega \subset \mathbf{R}^2$ *be a bounded set and* $(x, y) \in \Omega$. *Denote bounded matrix valued functions whose derivatives are also bounded by* $B_k$ *and matrix valued functions which converge to the zero matrix together with all derivatives uniformly in* $t, x, y$ *by* $\varepsilon_k$. *Then the almost complex structure*

$$\bar{J}(s,t,x,y) := \begin{pmatrix} j_1(s,t,x,y) & J_2(s,t,x,y) \\ J_1(s,t,x,y) & j_2(s,t,x,y) \end{pmatrix}$$

*has the following properties:*

- *For* $|s|$ *large, the matrix valued functions* $j_k(s,t,x,y)$ , $k = 1, 2$, *can be written as*

$$j_k(s,t,x,y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + q_k(s,t,x,y) + \varepsilon_1(s,t,x,y)\,, \tag{3.24}$$

  *where* $q_k$ *are matrix valued functions whose entries are quadratic polynomials in* $x$ *and* $y$.
- *The matrix valued function* $J_2(s,t,x,y)$ *can be written as follows for* $|s|$ *large*

$$J_2(s,t,x,y) = \sqrt{\rho(s)}B_2(s,t,x,y) + yB_3(s,t,x,y)\,. \tag{3.25}$$

- *The matrix valued function* $J_1(s,t,x,y)$ *can be written as follows for* $|s|$ *large*

$$\begin{aligned} J_1(s,t,x,y) = {} & \frac{1}{2}\alpha_\pm(s)\begin{pmatrix} -y & x \\ x & y \end{pmatrix} \\ & + \sqrt{\rho(s)}B_4(s,t,x,y) + x\varepsilon_2(s,t,x,y) \\ & + y\varepsilon_3(s,t,x,y) + Q(s,t,x,y)\,, \end{aligned} \tag{3.26}$$

  *where* $Q$ *is a matrix valued function whose entries are cubic polynomials in* $x$ *and* $y$. *Here, the matrix valued function* $B_4$ *does not vanish for* $x = y = 0$.

PROOF: Using (3.23), (3.14), and (3.16) we compute

$$\Xi_\pm(s,t,x,y)^{-1}\hat{J}_{\pm\infty}\Xi_\pm(s,t,x,y) = \begin{pmatrix} J_0 & 0 \\ X_\pm(s,t,x,y) & J_0 \end{pmatrix},$$

where

$$\begin{aligned} X_\pm(s,t,x,y) := {} & \frac{1}{2}\alpha_\pm(s)\begin{pmatrix} -y & x \\ x & y \end{pmatrix} \\ & \pm \alpha_\pm(s)\sqrt{\rho(s)}(q(0) - q(c_\pm))\begin{pmatrix} \sin\lambda t & \cos\lambda t \\ \cos\lambda t & -\sin\lambda t \end{pmatrix} \end{aligned}$$

and

$$J_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall from (1.8) that

$$\hat{J}(\bar{\tau},\bar{\theta},\bar{x},\bar{y}) = \begin{pmatrix} 0 & -(\bar{x}+q(\bar{\theta})\bar{y}) & 0 & -1 \\ 0 & \bar{y}q'(\bar{\theta}) & 1 & q(\bar{\theta}) \\ -q(\bar{\theta}) & -1+\bar{y}q'(\bar{\theta})((\bar{x}+q(\bar{\theta})\bar{y})q(\bar{\theta})-\bar{y}q'(\bar{\theta})) & (\bar{x}+q(\bar{\theta})\bar{y})q(\bar{\theta})-\bar{y}q'(\bar{\theta}) & q(\bar{\theta})((\bar{x}+q(\bar{\theta})\bar{y})q(\bar{\theta})-\bar{y}q'(\bar{\theta})) \\ 1 & -(\bar{x}+q(\bar{\theta})\bar{y})\bar{y}q'(\bar{\theta}) & -(\bar{x}+q(\bar{\theta})\bar{y}) & -(\bar{x}+q(\bar{\theta})\bar{y})q(\bar{\theta}) \end{pmatrix},$$

where we evaluate at

$$\begin{aligned}(\bar{\tau},\bar{\theta},\bar{x},\bar{y}) &= \Phi_{c_-,c_+}(s,t,x,y)\\ &= v_\pm(s,t)+xn(s,t)+ym(s,t)+(0,c_\pm,0,0)\,.\end{aligned}$$

Using (1.9), (3.2), (3.5) and (3.15) we obtain

$$\bar{\theta} = c_\pm \pm \kappa_\pm x\sqrt{\rho(s)} + \rho(s)\kappa_\pm q(0)\cos(\lambda t) + \rho(s)\varepsilon(s,t)\,, \tag{3.27}$$

$$\bar{x} = \mp\kappa_\pm y\sqrt{\rho(s)} + \rho(s)\varepsilon(s,t)\,, \tag{3.28}$$

and

$$\bar{y} = -\kappa_\pm\rho(s)\sin(\lambda t) + \rho(s)\varepsilon(s,t) \tag{3.29}$$

(the symbol "±" means positive sign if $s > 0$ and negative sign otherwise). We also write

$$q(c_\pm) - q(\bar{\theta}) = x\sqrt{\rho(s)}\,b_1(s,t,x) + \rho(s)b_2(s,t,x)$$

with suitable bounded functions $b_1, b_2$. We now decompose $\hat{J}(\Phi_{c_-,c_+}) - \hat{J}_{\pm\infty}$ now as follows:

$$\hat{J}(\Phi_{c_-,c_+}(s,t,x,y)) - \hat{J}_{\pm\infty} = \sqrt{\rho(s)}\left(\hat{J}_1(s,t,x,y) + \sqrt{\rho(s)}B(s,t,x,y)\right)$$

with some bounded matrix-valued function $B$ and

$$\hat{J}_1(s,t,x,y) = \begin{pmatrix} 0 & \pm y\kappa_\pm & 0 & 0 \\ 0 & 0 & 0 & -xb_1(s,t,x) \\ xb_1(s,t,x) & 0 & \mp\kappa_\pm yq(c_\pm) & \mp\kappa_\pm yq^2(c_\pm) \\ 0 & 0 & \pm\kappa_\pm y & \pm\kappa_\pm yq(c_\pm) \end{pmatrix}. \tag{3.30}$$

We also have

$$\Xi_\pm(s,t,x,y) = \sqrt{\rho(s)}\,B_1(s,t,x,y)\,,\qquad \Xi_\pm(s,t,x,y)^{-1} = \frac{1}{\rho(s)}B_2(s,t,x,y)\,,$$

for suitable bounded matrix valued functions $B_1$, $B_2$ which we will also decompose into sums according to their asymptotic behavior. Using (3.14) we write

$$
\begin{aligned}
&B_1(s,t,x,y)\\
&= \kappa_\pm \begin{pmatrix} -\sqrt{\rho(s)}\alpha_\pm(s)\cos(\lambda t) & \sqrt{\rho(s)}\alpha_\pm(s)\sin(\lambda t) & 0 & 0 \\ \alpha_\pm(s)[\sqrt{\rho(s)}q(0)\cos(\lambda t) \pm \frac{x}{2}] & -\sqrt{\rho(s)}\alpha_\pm(s)q(0)\sin(\lambda t) & \pm 1 & 0 \\ \mp\frac{y}{2}\alpha_\pm(s) & 0 & 0 & \mp 1 \\ -\sqrt{\rho(s)}\alpha_\pm(s)\sin(\lambda t) & -\sqrt{\rho(s)}\alpha_\pm(s)\cos(\lambda t) & 0 & 0 \end{pmatrix}\\
&= \kappa_\pm \begin{pmatrix} 0 & 0 & 0 & 0 \\ \pm\alpha_\pm(s)x/2 & 0 & \pm 1 & 0 \\ \mp\alpha_\pm(s)y/2 & 0 & 0 & \mp 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\\
&\quad + \kappa_\pm\alpha_\pm(s)\sqrt{\rho(s)} \begin{pmatrix} -\cos(\lambda t) & \sin(\lambda t) & 0 & 0 \\ q(0)\cos(\lambda t) & -q(0)\sin(\lambda t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sin(\lambda t) & -\cos(\lambda t) & 0 & 0 \end{pmatrix}\\
&=: B_{1a}(s,x,y) + \sqrt{\rho(s)}\, B_{1b}(s,t)\,.
\end{aligned}
$$

Using (3.16) we write

$$
\begin{aligned}
B_2(s,t,x,y) &= \frac{1}{\kappa_\pm\alpha_\pm(s)} \begin{pmatrix} -\cos(\lambda t) & 0 & 0 & -\sin(\lambda t) \\ \sin(\lambda t) & 0 & 0 & -\cos(\lambda t) \\ \frac{1}{2}x\alpha_\pm(s)\cos(\lambda t) & 0 & 0 & \frac{1}{2}x\alpha_\pm(s)\sin(\lambda t) \\ \frac{1}{2}y\alpha_\pm(s)\cos(\lambda t) & 0 & 0 & \frac{1}{2}y\alpha_\pm(s)\sin(\lambda t) \end{pmatrix}\\
&\quad + \frac{\sqrt{\rho(s)}}{\kappa_\pm} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm q(0) & \pm 1 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 \end{pmatrix}\\
&=: B_{2a}(s,t,x,y) + \sqrt{\rho(s)}\, B_{2b}(s)\,.
\end{aligned}
$$

We compute

$$
\begin{aligned}
&\Xi_\pm^{-1}(\hat{J}(\Phi_{c_-,c_+}) - \hat{J}_{\pm\infty})\Xi_\pm\\
&\quad = \big(\rho^{-1}B_{2a} + \rho^{-1/2}B_{2b}\big)\big(\sqrt{\rho}\hat{J}_1 + \rho B\big)\big(\sqrt{\rho}B_{1a} + \rho B_{1b}\big)\\
&\quad = B_{2a}\hat{J}_1 B_{1a} + \sqrt{\rho}\cdot B\,,
\end{aligned}
$$

where we denote by $B$ any bounded matrix valued function with bounded derivatives, and

(3.31) $$(B_{2a}\hat{J}_1 B_{1a})(s,t,x,y) = \begin{pmatrix} \frac{\kappa_\pm y}{2}(y\sin(\lambda t) - x\cos(\lambda t)) & 0 & -\frac{\kappa_\pm y}{\alpha_\pm(s)}\cos(\lambda t) & \frac{\kappa_\pm y}{\alpha_\pm(s)}\sin(\lambda t) \\ \frac{\kappa_\pm y}{2}(y\cos(\lambda t) + x\sin(\lambda t)) & 0 & \frac{\kappa_\pm y}{\alpha_\pm(s)}\sin(\lambda t) & \frac{\kappa_\pm y}{\alpha_\pm(s)}\cos(\lambda t) \\ \alpha_\pm(s)\frac{\kappa_\pm xy}{4}(x\cos(\lambda t) - y\sin(\lambda t)) & 0 & \frac{\kappa_\pm xy}{2}\cos(\lambda t) & -\frac{\kappa_\pm xy}{2}\sin(\lambda t) \\ \alpha_\pm(s)\frac{\kappa_\pm y^2}{4}(x\cos(\lambda t) - y\sin(\lambda t)) & 0 & \frac{\kappa_\pm y^2}{2}\cos(\lambda t) & -\frac{\kappa_\pm y^2}{2}\sin(\lambda t) \end{pmatrix}.$$

We calculate now $\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm$:

$$\begin{aligned}\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm &= \rho\,\Xi_\pm^{-1}(\hat{J}(\Phi_{c_-,c_+}) - \hat{J}_{\pm\infty})E_\pm + \rho\,\Xi_\pm^{-1}\hat{J}_{\pm\infty}E_\pm \\ &= B_{2a}\hat{J}_{\pm\infty}E_\pm + \sqrt{\rho}\,\varepsilon\,,\end{aligned}$$

where $\varepsilon$ denotes a matrix valued function which converges to zero with all derivatives is $|s| \to \infty$. This shows that the contribution $\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm$ tends to zero, but we need to have a closer look at the term $B_{2a}\hat{J}_{\pm\infty}E_\pm$. The matrix $E_\pm$ can be written as follows:

$$E_\pm(s,t,x,y) = \begin{pmatrix} \varepsilon_{11}(s,t) & \varepsilon_{12}(s,t) & 0 & 0 \\ \varepsilon_{21}(s,t) & \varepsilon_{22}(s,t) & 0 & 0 \\ \varepsilon_{31}(s,t) & \varepsilon_{32}(s,t) & 0 & 0 \\ \varepsilon_{41}(s,t) & \varepsilon_{42}(s,t) & 0 & 0 \end{pmatrix} + \sqrt{\rho(s)}\,B(s,t,x,y)\,,$$

where $\varepsilon_{kl}$ and all its derivatives converge to zero uniformly in $t$ as $|s| \to \infty$ (even at some exponential rate by theorem 1.6), and $B$ is a bounded matrix valued function. The entries $\varepsilon_{kl}$ are the remainder terms in the asymptotic formulas for $\partial_s v_\pm$ and $\partial_t v_\pm$. We obtain

$$\kappa_\pm B_{2a}\hat{J}_{\pm\infty}E_\pm = \begin{pmatrix} \frac{1}{\alpha_\pm}(\varepsilon_{41}\cos(\lambda t) - \varepsilon_{11}\sin(\lambda t)) & \frac{1}{\alpha_\pm}(\varepsilon_{42}\cos(\lambda t) - \varepsilon_{12}\sin(\lambda t)) & 0 & 0 \\ -\frac{1}{\alpha_\pm}(\varepsilon_{11}\cos(\lambda t) + \varepsilon_{41}\sin(\lambda t)) & -\frac{1}{\alpha_\pm}(\varepsilon_{12}\cos(\lambda t) + \varepsilon_{42}\sin(\lambda t)) & 0 & 0 \\ \frac{x}{2}(\varepsilon_{11}\sin(\lambda t) - \varepsilon_{41}\cos(\lambda t)) & \frac{x}{2}(\varepsilon_{12}\sin(\lambda t) - \varepsilon_{42}\cos(\lambda t)) & 0 & 0 \\ \frac{y}{2}(\varepsilon_{11}\sin(\lambda t) - \varepsilon_{41}\cos(\lambda t)) & \frac{y}{2}(\varepsilon_{12}\sin(\lambda t) - \varepsilon_{42}\cos(\lambda t)) & 0 & 0 \end{pmatrix} + \sqrt{\rho}B\,.$$

□

Here is the twin of Lemma 3.7 for the case $\lambda_\pm \in \mathbf{Z}\pi$:

LEMMA 3.8 *Assume that $\lambda_\pm \in \mathbf{Z}\pi$. Moreover, let $\Omega \subset \mathbf{R}^2$ be a bounded set and $(x,y) \in \Omega$. Denote bounded matrix valued functions whose derivatives are also bounded by $B_k$ and matrix valued functions which converge to the zero matrix* *together with all derivatives uniformly in $t, x, y$ by $\varepsilon_k$. Then the almost complex structure*

$$\bar{J}(s,t,x,y) := \begin{pmatrix} j_1(s,t,x,y) & J_2(s,t,x,y) \\ J_1(s,t,x,y) & j_2(s,t,x,y) \end{pmatrix}$$

*has the following properties for large $|s|$:*

- *For $|s|$ large, the matrix valued functions $j_k(s,t,x,y)$ , $k = 1, 2$, can be written as*

$$j_k(s,t,x,y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \sqrt{\rho(s)} B_1(s,t,x,y) + \frac{1}{\sqrt{\rho(s)}} (x B_4(s,t) + y B_5(s,t)) \,, \tag{3.32}$$

$$J_2(s,t,x,y) = \frac{q(0) - q(c_\pm)}{\sqrt{\rho(s)}} B_1(t) + \sqrt{\rho(s)} B_2(s,t,x,y) + x B_3(s,t) + y B_4(s,t) \,. \tag{3.33}$$

- *We have*

$$\begin{aligned} J_1(s,t,x,y) &= x B_1(s,t) + y B_2(s,t) + \rho(s) B_3(s,t,x,y) \\ &\quad + \frac{q(0) - q(c_\pm)}{\sqrt{\rho(s)}} \big(x^2 B_4(s,t) + xy B_5(s,t) + y^2 B_6(s,t)\big) \\ &\quad + \sqrt{\rho(s)} \varepsilon(s,t) \,, \end{aligned} \tag{3.34}$$

*where $\varepsilon(s,t)$ and its derivatives decay like $e^{-\delta|s|}$ for a suitable number $\delta > 0$.*

PROOF: The proof is similar to the one of Lemma 3.7, but a bit more tedious. Starting again from

$$\bar{J}_{c_-,c_+} = (\mathrm{Id} + \varepsilon_1) \big[ \Xi^{-1} \hat{J}_{\pm\infty} \Xi + \Xi^{-1} (\hat{J}(\Phi_\pm) - \hat{J}_{\pm\infty}) \Xi \big] + \varepsilon_2 \tag{3.35}$$

with the same $\hat{J}_{\pm\infty}$ as in (3.23), and using (3.19),(3.20) we obtain

$$\Xi_\pm^{-1}(s,t,x,y) \hat{J}_{\pm\infty} \Xi_\pm(s,t,x,y) = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} (s,t,x,y) \,,$$

where

$$X_{11}(s,t,x,y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{q(0) - q(c_\pm)}{2\alpha(s)\sqrt{\rho(s)}}$$
$$\cdot \begin{pmatrix} -y\alpha(s)\cos(\frac{\pi t}{2} \mp \lambda t) \pm x\alpha(s)\sin(\frac{\pi t}{2} \mp \lambda t) & \pm x\pi\cos(\frac{\pi t}{2} \mp \lambda t) + y\pi\sin(\frac{\pi t}{2} \mp \lambda t) \\ -x\alpha(s)\cos(\frac{\pi t}{2} \mp \lambda t) \mp y\alpha(s)\sin(\frac{\pi t}{2} \mp \lambda t) & \mp y\pi\cos(\frac{\pi t}{2} \mp \lambda t) + x\pi\sin(\frac{\pi t}{2} \mp \lambda t) \end{pmatrix},$$

$$X_{21}(s,t,x,y) = \pm\frac{\pi \pm \alpha(s)}{2}\begin{pmatrix} -y & x \\ x & y\end{pmatrix} + \frac{q(0)-q(c_\pm)}{4\alpha(s)\sqrt{\rho(s)}}$$
$$\cdot \begin{pmatrix} xy\alpha(s)(\pm\pi+\alpha(s))\cos(\frac{\pi t}{2}\mp\lambda t) + \alpha(s)(\pi y^2 \mp \alpha(s)x^2)\sin(\frac{\pi t}{2}\mp\lambda t) \\ \mp xy\alpha(s)(\pm\pi+\alpha(s))\sin(\frac{\pi t}{2}\mp\lambda t) + \alpha(s)(\mp\pi x^2 + \alpha(s)y^2)\cos(\frac{\pi t}{2}\mp\lambda t) \end{pmatrix.$$
$$\begin{matrix} \pi(\pi\pm\alpha(s))xy\sin(\frac{\pi t}{2}\mp\lambda t) + \pi(\pi y^2 \mp \alpha(s)x^2)\cos(\frac{\pi t}{2}\mp\lambda t) \\ -\pi(\pi\pm\alpha(s))xy\cos(\frac{\pi t}{2}\mp\lambda t) + \pi(\pm\pi x^2 - \alpha(s)y^2)\sin(\frac{\pi t}{2}\mp\lambda t)\end{pmatrix},$$

$$X_{22}(s,t,x,y) = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} + \frac{q(0)-q(c_\pm)}{2\alpha(s)\sqrt{\rho(s)}}$$
$$\cdot\begin{pmatrix} \pm\pi y\cos(\frac{\pi t}{2}\mp\lambda t) \mp x\alpha(s)\sin(\frac{\pi t}{2}\mp\lambda t) & \pi y\sin(\frac{\pi t}{2}\mp\lambda t) + x\alpha(s)\cos(\frac{\pi t}{2}\mp\lambda t) \\ \pi x\cos(\frac{\pi t}{2}\mp\lambda t) + \alpha(s)y\sin(\frac{\pi t}{2}\mp\lambda t) & -\pi x\sin(\frac{\pi t}{2}\mp\lambda t) + y\alpha(s)\cos(\frac{\pi t}{2}\mp\lambda t)\end{pmatrix}$$

and

$$X_{12}(s,t,x,y) = \frac{q(0)-q(c_\pm)}{\alpha(s)\sqrt{\rho(s)}}\begin{pmatrix} \pm\sin(\frac{\pi t}{2}\mp\lambda t) & -\cos(\frac{\pi t}{2}\mp\lambda t) \\ -\cos(\frac{\pi t}{2}\mp\lambda t) & \mp\sin(\frac{\pi t}{2}\mp\lambda t)\end{pmatrix}.$$

Using (1.14), (3.1), (3.5) and (3.18) we replace formulae (3.27), (3.28) and (3.29) by

$$\begin{aligned}
\bar\theta &= c_\pm + \kappa q(0)\sqrt{\rho(s)}\left(-x\cos\left(\frac{\pi t}{2}\right) \mp y\sin\left(\frac{\pi t}{2}\right)\right) \\
&\quad + \kappa\rho(s)\cos(\lambda t) + \rho(s)\varepsilon(s,t,x,y)\,, \\
\bar x &= -\kappa\rho(s)\sin(\lambda t) + \rho(s)\varepsilon(s,t,x,y)\,, \\
\bar y &= \kappa\sqrt{\rho(s)}\left(\mp x\sin\left(\frac{\pi t}{2}\right) + y\cos\left(\frac{\pi t}{2}\right)\right) + \rho(s)\varepsilon(s,t,x,y)\,,
\end{aligned}$$

where $\varepsilon$ and all its derivatives converge to zero as $|s|\to\infty$. Moreover,

$$\begin{aligned}
q(c_\pm) - q(\bar\theta) = b_1(s,t,x,y)\Bigg(&\kappa q(0)\sqrt{\rho(s)}\left(-x\cos\left(\frac{\pi t}{2}\right)\mp y\sin\left(\frac{\pi t}{2}\right)\right) \\
&+ \kappa\rho(s)\cos(\lambda t) + \rho(s)\varepsilon(s,t,x,y)\Bigg)
\end{aligned}$$

with some bounded matrix valued function $b_1$. We insert these into equation (1.8) and arrange the terms according to the order of $\rho(s)$. We get

$$\hat J(\Phi_{c_-,c_+}(s,t,x,y)) - \hat J_{\pm\infty} = \sqrt{\rho(s)}\,\hat J_1(s,t,x,y) + \rho(s)B(s,t,x,y)$$

with some bounded matrix-valued function $B$ and

$$\hat J_1(s,t,x,y) =$$
$$\begin{pmatrix} 0 & -q(c_\pm)\eta & 0 & 0 \\ 0 & q'(c_\pm) & 0 & b_1\kappa q(0)\eta' \\ b_1\kappa q(0)\eta' & 0 & \eta(q^2(c_\pm)-q'(c_\pm)) & \eta q(c_\pm)(q^2(c_\pm)-q'(c_\pm)) \\ 0 & 0 & -q(c_\pm)\eta & -q^2(c_\pm)\eta \end{pmatrix}$$

and

$$\eta := \mp x \sin\left(\frac{\pi t}{2}\right) + y \cos\left(\frac{\pi t}{2}\right),$$
$$\eta' := -x \cos\left(\frac{\pi t}{2}\right) \mp y \sin\left(\frac{\pi t}{2}\right).$$

We also have

$$\Xi_\pm(s,t,x,y) = \sqrt{\rho(s)} B_1(s,t,x,y), \quad \Xi_\pm(s,t,x,y)^{-1} = \frac{1}{\rho(s)} B_2(s,t,x,y),$$

for suitable bounded matrix valued functions $B_1$, $B_2$ which we will also decompose into sums according to their asymptotic behavior. Using (3.19) we write

$$\begin{aligned}
&B_1(s,t,x,y)\\
&= \kappa \begin{pmatrix}
\frac{1}{2}(x\cos(\frac{\pi t}{2}) \pm y \sin(\frac{\pi t}{2}))\alpha_\pm(s) & \frac{\pi}{2}(\pm y\cos(\frac{\pi t}{2}) - x\sin(\frac{\pi t}{2})) & \cos(\frac{\pi t}{2}) & \pm\sin(\frac{\pi t}{2})\\
(\sqrt{\rho(s)}\cos(t\lambda) + \frac{1}{2}q(0)[-x\cos(\frac{\pi t}{2}) \mp y\sin(\frac{\pi t}{2})])\alpha_\pm(s) & \frac{\pi}{2}q(0)(x\sin(\frac{\pi t}{2}) \mp y\cos(\frac{\pi t}{2})) - \sqrt{\rho(s)}\alpha_\pm(s)\sin(t\lambda) & -q(0)\cos(\frac{\pi t}{2}) & \mp q(0)\sin(\frac{\pi t}{2})\\
-\sqrt{\rho(s)}\alpha_\pm(s)\sin(t\lambda) & -\sqrt{\rho(s)}\alpha_\pm(s)\cos(t\lambda) & 0 & 0\\
\frac{1}{2}\alpha_\pm(s)(y\cos(\frac{\pi t}{2}) \mp x\sin(\frac{\pi t}{2})) & \frac{\pi}{2}(\mp x\cos(\frac{\pi t}{2}) - y\sin(\frac{\pi t}{2})) & \mp\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2})
\end{pmatrix}\\
&= B_{1a}(s,t,x,y) + \sqrt{\rho(s)} B_{1b}(s,t)
\end{aligned}$$

with

$$\begin{aligned}
&B_{1a}(s,t,x,y)\\
&= \kappa \begin{pmatrix}
\frac{1}{2}(x\cos(\frac{\pi t}{2}) \pm y \sin(\frac{\pi t}{2}))\alpha_\pm(s) & \frac{\pi}{2}(\pm y\cos(\frac{\pi t}{2}) - x\sin(\frac{\pi t}{2})) & \cos(\frac{\pi t}{2}) & \pm\sin(\frac{\pi t}{2})\\
\frac{1}{2}\alpha_\pm(s)q(0)[-x\cos(\frac{\pi t}{2}) \mp y\sin(\frac{\pi t}{2})] & \frac{\pi}{2}q(0)(x\sin(\frac{\pi t}{2}) \mp y\cos(\frac{\pi t}{2})) & -q(0)\cos(\frac{\pi t}{2}) & \mp q(0)\sin(\frac{\pi t}{2})\\
0 & 0 & 0 & 0\\
\frac{1}{2}\alpha_\pm(s)(y\cos(\frac{\pi t}{2}) \mp x\sin(\frac{\pi t}{2})) & \frac{\pi}{2}(\mp x\cos(\frac{\pi t}{2}) - y\sin(\frac{\pi t}{2})) & \mp\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2})
\end{pmatrix}
\end{aligned}$$

and

$$B_{1b}(s,t,x,y) = \kappa\alpha_\pm(s)\begin{pmatrix}
0 & 0 & 0 & 0\\
\cos(\lambda t) & -\sin(\lambda t) & 0 & 0\\
-\sin(\lambda t) & \cos(\lambda t) & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}.$$

Using (3.20) we decompose $B_2$ as follows:

$$B_2(s,t,x,y) = \frac{1}{\kappa\alpha(s)} \cdot \begin{pmatrix} q(0)\cos(t\lambda) \\ -q(0)\sin(t\lambda) \\ \alpha(s)\sqrt{\rho(s)}\cos\left(\frac{\pi t}{2}\right) - \frac{1}{2}[\mp\pi y q(0)\sin(t\lambda) + xq(0)\alpha(s)\cos(t\lambda)] \\ \pm\alpha(s)\sqrt{\rho(s)}\sin\left(\frac{\pi t}{2}\right) + \frac{1}{2}[\mp\pi x q(0)\sin(t\lambda) - yq(0)\alpha(s)\cos(t\lambda)] \end{pmatrix.$$

$$\begin{matrix} \cos(t\lambda) & -\sin(t\lambda) & 0 \\ -\sin(t\lambda) & -\cos(t\lambda) & 0 \\ \frac{1}{2}[\pm\pi y\sin(t\lambda) - x\alpha(s)\cos(t\lambda)] & \frac{1}{2}[\pm\pi y\cos(t\lambda) + x\alpha(s)\sin(t\lambda)] & \mp\alpha(s)\sqrt{\rho(s)}\sin\left(\frac{\pi t}{2}\right) \\ \frac{1}{2}[\mp\pi x\sin(t\lambda) - y\alpha(s)\cos(t\lambda)] & \frac{1}{2}[\mp\pi x\cos(t\lambda) + y\alpha(s)\sin(t\lambda)] & \alpha(s)\sqrt{\rho(s)}\cos\left(\frac{\pi t}{2}\right), \end{matrix}\Bigg)$$

$$= B_{2a}(s,t,x,y) + \sqrt{\rho(s)}B_{2b}(s,t)$$

with

$$B_{2a}(s,t,x,y) = \frac{1}{\kappa\alpha(s)} \cdot \begin{pmatrix} q(0)\cos(t\lambda) & \cos(t\lambda) & -\sin(t\lambda) & 0 \\ -q(0)\sin(t\lambda) & -\sin(t\lambda) & -\cos(t\lambda) & 0 \\ \frac{1}{2}[\pm\pi y q(0)\sin(t\lambda) - xq(0)\alpha(s)\cos(t\lambda)] & \frac{1}{2}[\pm\pi y\sin(t\lambda) - x\alpha(s)\cos(t\lambda)] & \frac{1}{2}[\pm\pi y\cos(t\lambda) + x\alpha(s)\sin(t\lambda)] & 0 \\ \frac{1}{2}[\mp\pi x q(0)\sin(t\lambda) - yq(0)\alpha(s)\cos(t\lambda)] & \frac{1}{2}[\mp\pi x\sin(t\lambda) - y\alpha(s)\cos(t\lambda)] & \frac{1}{2}[\mp\pi x\cos(t\lambda) + y\alpha(s)\sin(t\lambda)] & 0 \end{pmatrix}$$

and

$$B_{2b}(s,t) = \frac{1}{\kappa}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\left(\frac{\pi t}{2}\right) & 0 & 0 & \mp\sin\left(\frac{\pi t}{2}\right) \\ \pm\sin\left(\frac{\pi t}{2}\right) & 0 & 0 & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix}.$$

We compute, denoting bounded matrix valued functions with bounded derivatives by $B'$, $B''$ etc.,

$$\begin{aligned} \Xi_\pm^{-1}(\hat{J}(\Phi_{c_-,c_+}) - \hat{J}_{\pm\infty})\Xi_\pm &= B_{2a}\hat{J}_1 B_{1a} + \sqrt{\rho}(B_{2a}\hat{J}_1 B_{1b} \\ &\quad + B_{2b}\hat{J}_1 B_{1a} + B_{2a}BB_{1a}) + \rho\cdot B' \\ &= xB' + yB'' + \rho B''' + \sqrt{\rho}B_{2a}BB_{1a}\,. \end{aligned}$$

If we write

$$B_{2a}BB_{1a} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

then $Y_{11}$ and $Y_{12}$ are just bounded $2\times 2$-matrix valued functions with bounded derivatives while $Y_{21}$ and $Y_{22}$ are both of the form

$$Y_{21}, Y_{22} = xB' + yB''\,.$$

We are left with the term $\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm$. Denoting by $\bar{\varepsilon}$ a matrix valued function converging to zero with all derivatives and denoting bounded matrix valued functions by $B'$, $B''$ etc., we compute

$$\begin{aligned}\Xi_\pm^{-1}\hat{J}(\Phi_{c_-,c_+})\Delta_\pm &= \rho\,\Xi_\pm^{-1}(\hat{J}(\Phi_{c_-,c_+}) - \hat{J}_{\pm\infty})E_\pm + \rho\,\Xi_\pm^{-1}\hat{J}_{\pm\infty}E_\pm \\ &= (B_{2a} + \sqrt{\rho}B_{2b})(\sqrt{\rho}\hat{J}_1 + \rho B)E_\pm \\ &\quad + B_{2a}\hat{J}_{\pm\infty}E_\pm + \sqrt{\rho}B_{2b}\hat{J}_{\pm\infty}E_{\pm\infty} \\ &= \sqrt{\rho}(xB' + yB'') + \rho\bar{\varepsilon} + B_{2a}\hat{J}_{\pm\infty}E_\pm + \sqrt{\rho}B_{2b}\hat{J}_{\pm\infty}E_\pm\,.\end{aligned}$$

The terms $B_{2a}\hat{J}_{\pm\infty}E_\pm$ and $\sqrt{\rho}B_{2b}\hat{J}_{\pm\infty}E_\pm$ have to be examined further. The matrix $E_\pm$ can be written as follows:

$$E_\pm(s,t,x,y) = \begin{pmatrix}\varepsilon_{11}(s,t) & \varepsilon_{12}(s,t) & 0 & 0\\ \varepsilon_{21}(s,t) & \varepsilon_{22}(s,t) & 0 & 0\\ \varepsilon_{31}(s,t) & \varepsilon_{32}(s,t) & 0 & 0\\ \varepsilon_{41}(s,t) & \varepsilon_{42}(s,t) & 0 & 0\end{pmatrix} + \sqrt{\rho(s)}\tilde{B}(s,t,x,y)\,,$$

where $\varepsilon_{kl}$ and all its derivatives converge to zero uniformly in $t$ as $|s| \to \infty$, and $\tilde{B}$ is a bounded matrix valued function with bounded derivatives. The entries $\varepsilon_{kl}$ are the remainder terms in the asymptotic formulas for $\partial_s v_\pm$ and $\partial_t v_\pm$, and we know that they converge to zero exponentially (Theorem 1.6). Decomposing $\sqrt{\rho}B_{2b}\hat{J}_{\pm\infty}E_\pm$ into blocks of $2\times 2$-matrices we get

$$\sqrt{\rho}B_{2b}\hat{J}_{\pm\infty}E_\pm = \sqrt{\rho}\begin{pmatrix}0 & 0\\ \Upsilon & 0\end{pmatrix},$$

where $\Upsilon = \Upsilon(s,t)$ is a matrix whose entries depend linearly on the $\varepsilon_{kl}$ as above.

Finally, we calculate

$$B_{2a}\hat{J}_{\pm\infty}E_\pm = \begin{pmatrix}\Upsilon_1 & 0\\ x\Upsilon_2 + y\Upsilon_3 & 0\end{pmatrix} + \sqrt{\rho}\begin{pmatrix}B_1 & B_2\\ xB_3 + yB_4 & xB_5 + yB_6\end{pmatrix},$$

where all entries are $2\times 2$-matrices, $\Upsilon_k$ are linear expressions in $\varepsilon_{kl}$, and $B_k$ are bounded with bounded derivatives. Hence this term does not contribute anything new. Collecting all terms together completes the proof of the lemma. □

After having studied the asymptotic behavior of the almost complex structure $\bar{J}$ we may now specify the appropriate function spaces for the nonlinear operator $F$ in (3.12). We define first the following weighted Sobolev spaces

$$\begin{aligned}H_L^{2,p,\gamma}(S,\mathbf{C}) &:= \Big\{u \in H^{2,p}(S,\mathbf{C}) \mid \|u\|_{2,p,\gamma} := \|\rho^\gamma u\|_{2,p} < \infty,\\ &\qquad u(s,0)\in\mathbf{R},\ u(s,1)\in\mathbf{R}\cdot(a_1(s) + ia_2(s))\Big\}\,,\\ H^{1,p,\gamma}(S,\mathbf{C}) &:= \Big\{u \in H^{1,p}(S,\mathbf{C}) \mid \|u\|_{1,p,\gamma} := \|\rho^\gamma u\|_{1,p} < \infty\Big\}\,,\end{aligned}$$

where $\gamma : \mathbf{R} \to \mathbf{R}$ is a smooth function with

$$\gamma(s) \overset{s\to\pm\infty}{\longrightarrow} \gamma_\pm\,,\quad \frac{d^k}{ds^k}\gamma(s) \overset{s\to\pm\infty}{\longrightarrow} 0\,,\quad k\ge 1\,,$$

where the convergence is of exponential nature. Since $p > 2$, the above spaces consist of differentiable and continuous functions respectively. Remember that $\rho(s)$ is a smooth function which agrees with $e^{\int_{s_0}^{s}\alpha_\pm(\tau)d\tau}$ for large $|s|$, and $\alpha_\pm$ are the functions appearing in the asymptotic formula for $\tilde{u}_0$ (Theorem 1.5). Recall also that $\alpha_\pm(s) \to \lambda_\pm$ as $s \to \pm\infty$. Hence, if $\gamma_\pm < 0$ then the above Sobolev spaces consists of functions with a certain exponential decay at infinity. The function $a_1 + ia_2$ (as in Definition 3.1) is smooth with $a_1^2 + a_2^2 \equiv 1$ and $a_2(s) \equiv 0$ for large $|s|$.

PROPOSITION 3.9 *Assume that $p > 2$. Then the map*

$$F : \mathbf{R}^2 \times H_L^{2,p,\gamma}(S, \mathbf{C}) \longrightarrow H^{1,p,\gamma}(S, \mathbf{C}),$$

$$\begin{aligned} F(c_-, c_+, w)(z) := &-Dw(z)\big[j_1^{(c_-,c_+)}(z, w(z))(0, 1)\big] + j_2^{(c_-,c_+)}(z, w(z))\partial_t w(z) \\ &- Dw(z)\big[J_2^{(c_-,c_+)}(z, w(z))\partial_t w(z)\big] + J_1^{(c_-,c_+)}(z, w(z))(0, 1), \end{aligned}$$

*is well-defined and smooth*

- *if $\lambda_\pm = \mp\frac{\pi}{2}$ and $-\frac{1}{2} < \gamma_\pm < 0$, or*
- *if $\lambda_\pm \in \mathbf{Z}\pi$ and $-\frac{1}{2} - \frac{\delta}{|\lambda_\pm|} < \gamma_\pm < -\frac{1}{2}$, where $\delta > 0$ is the exponential decay rate of the remainder terms in the asymptotic formula* (*Theorem* 1.6)*.*

*Its derivative at $(c_-, c_+, w) = (0, 0, 0)$ is given by*

$$DF(0, 0, 0) : \mathbf{R}^2 \times H_L^{2,p,\gamma}(S, \mathbf{C}) \longrightarrow H^{1,p,\gamma}(S, \mathbf{C}), \tag{3.36}$$

$$\begin{aligned} (DF(0, 0, 0)(c_-, c_+, \eta))(z) = &\,\partial_s\eta(z) + i\partial_t\eta(z) + \big(D_2 J_1^{(0,0)}(z, 0)\eta(z)\big)\cdot(0, 1) \\ &+ c_-\frac{d}{d\tau}\bigg|_{\tau=0} J_1^{(\tau,0)}(z, 0)(0, 1) \\ &+ c_+\frac{d}{d\tau}\bigg|_{\tau=0} J_1^{(0,\tau)}(z, 0)(0, 1). \end{aligned}$$

PROOF: Since we assumed $p > 2$ and since $w \in H^{2,p}(S, \mathbf{C})$ we have $F(c_-, c_+, w) \in H^{1,p}_{\mathrm{loc}}(S, \mathbf{C})$. The issue is whether $F(c_-, c_+, w)$ lies in the weighted space $H^{1,p,\gamma}$, i.e., we have to study the decay properties of $F(c_-, c_+, w)(z)$ as $|z| \to \infty$. Most of the terms just require $\gamma_\pm < 0$, but a few impose more restrictions on the admissible weight functions.

Let us first discuss the term

$$Dw(z)\big[j_1^{c_-,c_+}(z, w(z))(0, 1)\big] \tag{3.37}$$

and its derivative

$$\begin{aligned} &D^2w(z)\big[j_1^{c_-,c_+}(z, w(z))(0, 1),\ .\,\big] \\ &\quad + Dw(z)\big[(D_1 j_1^{c_-,c_+}(z, w(z))\ .\,)(0, 1) + D_2 j_1^{c_-,c_+}(z, w(z))Dw(z)\big]. \end{aligned} \tag{3.38}$$

We have to show that their product with $\rho^\gamma$ lies in $L^p(S, \mathbf{C})$. The bounded terms in (3.24) and (3.32) do not cause any problems since $\rho^\gamma w$ and $\rho^\gamma Dw$ are in $L^p$. The same applies to the quadratic term in (3.24). The following expression in (3.32) for the case $\lambda_\pm \in \mathbf{Z}\pi$ imposes a restriction on $\gamma$:

$$\left|\frac{1}{\sqrt{\rho(s)}}\Big[\mathrm{Re}(w(s,t))B_4(s,t) + \mathrm{Im}(w(s,t))B_5(s,t)\Big]\right| \\ \leq c\rho^{-1/2}(s)|w(s,t)| \leq c\rho^{-\gamma_\pm - 1/2}(s)\,,$$

which is bounded for $\gamma_\pm \leq -\frac{1}{2}$, and this holds by assumption.

Let us now consider

$$j_2^{c_-,c_+}(z,w(z))\partial_t w(z)$$

and its derivative

$$j_2^{c_-,c_+}(z,w(z))\partial_t Dw(z) \\ + D_1 j_2^{c_-,c_+}(z,w(z))[\,.\,,\partial_t w(z)] + D_2 j_2^{c_-,c_+}(z,w(z))[Dw(z)\,,\ \partial_t w(z)]\,.$$

We argue exactly as before so that

$$j_2^{c_-,c_+}(z,w(z))\partial_t w(z) \in H^{1,p,\gamma}(S,\mathbf{C})\,.$$

as well.

Consider now

$$Dw(z)\Big[J_2^{c_-,c_+}(z,w(z))\partial_t w(z)\Big] \tag{3.39}$$

and its derivative

$$D^2 w(z)\Big[J_2^{c_-,c_+}(z,w(z))\partial_t w(z)\Big] + Dw(z)\Big[J_2^{c_-,c_+}(z,w(z))D\partial_t w(z)\Big] \\ + Dw(z)\Big[D_1 J_2^{c_-,c_+}(z,w(z))[\,.\,,\partial_t w(z)]\Big] \\ + Dw(z)\Big[D_2 J_2^{c_-,c_+}(z,w(z))[Dw(z),\partial_t w(z)]\Big]\,.$$

The bounded and the linear terms in (3.25) and (3.33) do not require more than $\gamma_\pm < 0$. If $\lambda_\pm \in \mathbf{Z}\pi$, we also have to deal with

$$\frac{q(0)-q(c_\pm)}{\sqrt{\rho(s)}}B_1(t)\,,$$

but the terms containing $J_2$ or its derivatives are at least quadratic in the derivatives of $w$ so that

$$\left|\rho^\gamma(s)\frac{q(0)-q(c_\pm)}{\sqrt{\rho(s)}}B_1(t)\right|\,|Dw(s,t)|^2 \leq c\,\rho^{\gamma_\pm - \frac{1}{2} - 2\gamma_\pm}(s)\,,$$

which is bounded if $\gamma_\pm \leq -\frac{1}{2}$.

We are left with the term

$$J_1^{c_-,c_+}(z,w(z))\,(0,1)$$

and its derivative

$$D_1 J_1^{c_-,c_+}(z,w(z))(0,1) + \Big[D_2 J_1^{c_-,c_+}(z,w(z))Dw(z)\Big](0,1)\,.$$

The expressions in (3.26) which are linear or cubic in $w$ can be multiplied with $\rho^\gamma$ for any $\gamma_\pm < 0$, and the product is still in $L^p$. The product of the term $\sqrt{\rho(s)}B_4(s,t;w(s,t))$ in (3.26) with $\rho^\gamma$ is also in $L^p$ since we assumed $\gamma_\pm > -\frac{1}{2}$ in the case $|\lambda_\pm| = \pi/2$. The case $\lambda_\pm \in \mathbf{Z}\pi$ is more delicate because the asymptotic behavior of $J_1(z, w(z))$ is worse. The term $\rho(s)B_3$ in (3.34) requires $\gamma_\pm > -1$. We also have to make sure that

$$\rho^\gamma \sqrt{\rho(s)}\varepsilon(s,t) \in L^p(S, \mathbf{C}),$$

with $\varepsilon$ as in (3.34). This holds if $\gamma_\pm > -\frac{1}{2} - \frac{\delta}{|\lambda_\pm|}$. This term competes with the expression

$$\frac{q(0) - q(c_\pm)}{\sqrt{\rho(s)}}\big(x^2 B_4(s,t) + xyB_5(s,t) + y^2 B_6(s,t)\big), \quad x + iy = w,$$

in (3.34). We estimate

$$\left|\rho^\gamma(s)\frac{q(0) - q(c_\pm)}{\sqrt{\rho(s)}}\big(x^2 B_4(s,t) + xyB_5(s,t) + y^2 B_6(s,t)\big)\right| \le$$
$$c\rho^{-\frac{1}{2}+\gamma}(s)|w(s,t)|^2 \le \rho^{-\frac{1}{2}+\gamma-2\gamma},$$

which is in $L^p$ if $\gamma_\pm < -\frac{1}{2}$.

Hence we have shown that the map $F$ is well-defined. The formula for the derivative and the verification of smoothness is straightforward, and we omit the proof. □

Our aim is to use the implicit function theorem and to prove the following theorem:

THEOREM 3.10 *Let $\tilde{u}_0$ be an embedded solution of* (1.1) *so that its Maslov index* (*see Definition* 3.4) *$\mu(\tilde{u}_0)$ vanishes and its exponential decay rates $\lambda_\pm$* (*as in Theorem* 1.5) *satisfy $|\lambda_\pm| \in \{\frac{\pi}{2}, \pi\}$. Consider the map*

$$F : \mathbf{R}^2 \times H_L^{2,p,\gamma}(S, \mathbf{C}) \longrightarrow H^{1,p,\gamma}(S, \mathbf{C})$$

*as in Proposition* 3.9, *where the limits $\gamma_\pm = \lim_{s\to\pm\infty}\gamma(s)$ of the weight function $\gamma$ satisfy*

- $-\frac{1}{2} < \gamma_\pm < 0$ *if $\lambda_\pm = \mp\frac{\pi}{2}$ and*
- $-\frac{1}{2} - \frac{\delta}{|\lambda_\pm|} < \gamma_\pm < -\frac{1}{2}$ *if $\lambda_\pm = \mp\pi$, where $\delta > 0$ is the exponential decay rate of the remainder terms in the asymptotic formula* (*Theorem* 1.6).

*Then there is $\tilde{\delta} > 0$ and a unique smooth map*

$$(c_-, c_+, \psi) : (-\tilde{\delta}, \tilde{\delta}) \to \mathbf{R}^2 \times H_L^{2,p,\gamma}(S, \mathbf{C})$$

*with the following properties*:

- $(c_-(0), c_+(0), \psi(0)) = (0, 0, 0)$,
- $F(c_-(\tau), c_+(\tau), \psi(\tau)) \equiv 0$ *for all* $-\tilde{\delta} < \tau < \tilde{\delta}$,
- $c_-(\tau), c_+(\tau) \neq 0$ *for all* $\tau \neq 0, \tau \in (-\tilde{\delta}, \tilde{\delta})$.

The proof requires results about the linearization of the map $F$ at the point $(0,0,0)$. These are the subject of the following section. We will postpone the proof of Theorem 3.10 until the end of the following section.

## 4 The Linearized Problem: Fredholm theory and Transversality

In this section we will study the linearization of the map $F$ in Proposition 3.9 at the solution $\tilde{u}_0$ which corresponds to $(c_-, c_+, w) = (0,0,0)$. It is given by (3.36) which is

$$DF(0,0,0) : \mathbf{R}^2 \times H_L^{2,p,\gamma}(S,\mathbf{C}) \longrightarrow H^{1,p,\gamma}(S,\mathbf{C})$$

$$\begin{aligned}(DF(0,0,0)(c_-,c_+,\eta))(z) &= \partial_s\eta(z) + i\,\partial_t\eta(z) + \big(D_2 J_1^{(0,0)}(z,0)\eta(z)\big)\cdot(0,1)\\ &\quad + c_-\frac{d}{d\tau}\bigg|_{\tau=0} J_1^{(\tau,0)}(z,0)(0,1)\\ &\quad + c_+\frac{d}{d\tau}\bigg|_{\tau=0} J_1^{(0,\tau)}(z,0)(0,1)\,.\end{aligned}$$

We recall that $\gamma = \gamma(s)$ is a smooth function defined on the real line which converges to $\gamma_\pm$ as $s \to \pm\infty$, and all derivatives of $\gamma$ tend to zero as $s \to \pm\infty$ (exponentially fast will be convenient later on). If $\lambda_\pm$ are the exponential decay rates of $\tilde{u}_0$ as in Theorem 1.5 then the nonlinear map $F$ is well-defined if

- $-\frac{1}{2} < \gamma_+ < 0$ in the case $|\lambda_+| = \frac{\pi}{2}$,
- $-\frac{1}{2} - \frac{\delta}{|\lambda_+|} < \gamma_+ < -\frac{1}{2}$ in the case where $\lambda_+$ is an integer multiple of $\pi$

and if the same conditions are satisfied for the pair $(\gamma_-, \lambda_-)$. The linear operator above is of course defined for all real values of $\gamma_\pm$. We have to study $DF(0,0,0)$ for negative and for positive weights $\gamma_\pm$ in order to prove surjectivity of $DF(0,0,0)$.

PROPOSITION 4.1 *Let $p \geq 2$ and $\gamma_\pm$ such that $\gamma_\pm\lambda_\pm$ are not integer multiples of $\pi$. Denote the bounded interval with end points $\gamma_-\lambda_-$ and $\gamma_+\lambda_+$ by $I$. Then the map $DF(0,0,0)$ above is a Fredholm operator. In the case $\mu(\tilde{u}_0) = 0$ and $\gamma_-\gamma_+ > 0$ its index is given by*

$$\mathrm{ind}(DF(0,0,0)) = 2 + \mathrm{sign}(\gamma_\pm)\cdot\#(\pi\mathbf{Z} \cap I)\,.$$

*In particular, if $|\lambda_\pm| \in \{\frac{\pi}{2}, \pi\}$ and*

- $-2 < \gamma_\pm < 0$ *in the case* $\lambda_\pm = \mp\frac{\pi}{2}$,
- $-1 < \gamma_\pm < 0$ *in the case* $\lambda_\pm = \mp\pi$

*then the Fredholm index equals* $+1$.

PROOF: The map $DF(0,0,0)$ is a Fredholm operator of index $N$ if and only if the operator

(4.1) $$T_0 : H_L^{2,p,\gamma}(S,\mathbf{C}) \longrightarrow H^{1,p,\gamma}(S,\mathbf{C})$$

$$T_0(\eta)(z) := \partial_s\eta(z) + i\,\partial_t\eta(z) + \big(D_2 J_1^{(0,0)}(z,0)\eta(z)\big)\cdot(0,1)\,,\quad z=(s,t)\in S\,,$$

is a Fredholm operator of index $N-2$. We will successively simplify the operator $T_0$ preserving the Fredholm property and the index. The first step is to get rid of the weighted spaces. The assignment $\eta \longmapsto \rho^\gamma \eta$ induces isomorphisms

$$H_L^{2,p,\gamma}(S,\mathbf{C}) \xrightarrow{\sim} H_L^{2,p}(S,\mathbf{C})$$

and

$$H^{1,p,\gamma}(S,\mathbf{C}) \xrightarrow{\sim} H^{1,p}(S,\mathbf{C})\,.$$

Recall that the subscript "$L$" refers to the boundary conditions $\eta(s,0) \in \mathbf{R}$ and $\eta(s,1) \in \mathbf{R}\cdot e^{i\phi(s)}$, where the limits $\phi_\pm = \lim_{s\to\pm\infty}\phi(s) \in \mathbf{Z}\pi$ satisfy $\mu(\tilde u_0) = \frac{1}{\pi}(\phi_+ - \phi_-)$ with $\mu(\tilde u_0)$ being the Maslov index of the solution $\tilde u_0$. We may also assume that $\phi'(s) \to 0$ as $|s| \to \infty$ and $\phi_- = 0$. We obtain an operator $T_1$ between the unweighted spaces whose kernel and cokernel are isomorphic to those of $T_0$:

$$T_1 : H_L^{2,p}(S,\mathbf{C}) \longrightarrow H^{1,p}(S,\mathbf{C})\,, \tag{4.2}$$

$$\begin{aligned} T_1(\eta)(s,t) &:= \rho^\gamma(s)\big(T_0(\rho^{-\gamma}\eta)(s,t)\big) \\ &= \partial_s\eta(s,t) + i\partial_t\eta(s,t) - \gamma(s)\rho'(s)\rho^{-1}(s)\eta(s,t) \\ &\quad + \big[D_2J_1^{(0,0)}(s,t;0)\eta(s,t)\big](0,1) - \gamma'(s)\log(\rho(s))\eta(s,t)\,. \end{aligned}$$

For large $|s|$ we have $\rho(s) = e^{\int_{s_0}^{s}\alpha(\tau)d\tau}$ so that

$$-\gamma\rho'(s)\rho^{-1}(s)\eta(s,t) = -\gamma\alpha(s)\eta(s,t) \tag{4.3}$$

and

$$\big|\gamma'(s)\log(\rho(s))\big| \le |\gamma'(s)|\big(\|\alpha\|_{L^1([s_0,\infty))} + |\lambda_\pm|\,|s-s_0|\big)\,, \tag{4.4}$$

which tends to zero if $s \to \pm\infty$. We now simplify the boundary condition. The assignment $\eta(s,t) \longmapsto e^{-it\phi(s)}\eta(s,t)$ defines isomorphisms

$$H_L^{2,p}(S,\mathbf{C}) \xrightarrow{\sim} H_{\mathbf{R}}^{2,p}(S,\mathbf{C})$$

and

$$H^{1,p}(S,\mathbf{C}) \xrightarrow{\sim} H^{1,p}(S,\mathbf{C})\,,$$

where

$$H_{\mathbf{R}}^{2,p}(S,\mathbf{C}) := \big\{u \in H_{\mathbf{R}}^{2,p}(S,\mathbf{C}) \mid u(s,0), u(s,1) \in \mathbf{R}\big\}\,.$$

The induced operator $T_2$ is given by

$$T_2 : H_{\mathbf{R}}^{2,p}(S,\mathbf{C}) \longrightarrow H^{1,p}(S,\mathbf{C})\,, \tag{4.5}$$

$$\begin{aligned} T_2(\eta)(s,t) &:= e^{-it\phi(s)}\big(T_1(e^{it\phi}\eta)(s,t)\big) \\ &= \partial_s\eta(s,t) + i\partial_t\eta(s,t) - \gamma(s)\rho'(s)\rho^{-1}(s)\eta(s,t) \\ &\quad + (it\phi'(s) - \phi(s))\eta(s,t) \\ &\quad + e^{-it\phi(s)}\big[D_2J_1(s,t;0)(e^{it\phi(s)}\eta(s,t))\big](0,1) \\ &\quad - \gamma'(s)\log(\rho(s))\eta(s,t)\,. \end{aligned}$$

We note that the kernel and cokernel of $T_2$ are isomorphic to those of $T_1$. The zero order operator

$$\eta(s,t) \longmapsto it\phi'(s)\eta(s,t) + e^{-it\phi(s)}\big[D_2 J_1(s,t;0)(e^{it\phi(s)}\eta(s,t))\big](0,1) \\ - \gamma'(s)\log(\rho(s))\eta(s,t)$$

is a compact perturbation because its operator norm converges to zero as $|s| \to \infty$ (these are lemma 3.16 in [15] and (3.26) in Lemma 3.7). Hence its presence is irrelevant for the Fredholm property and the Fredholm index. For the same reason (using (4.3)) we may replace the expression $-\gamma\rho'(s)\rho^{-1}(s)\,\eta(s,t)$ by $f(s)\eta(s,t)$ where $f(s)$ is a smooth function converging to $-\gamma_\pm\lambda_\pm$ as $s \to \pm\infty$, for example

$$f(s) = -\frac{\gamma_-\lambda_- + \gamma_+\lambda_+}{2} + \frac{\gamma_-\lambda_-\gamma_+\lambda_+}{2}\tanh(s)\,, \tag{4.6}$$

and we may replace the function $\phi(s)$ with

$$\phi(s) = \frac{\pi\mu(\tilde{u}_0)}{2}(1+\tanh(s))\,. \tag{4.7}$$

Summarizing, the operator $DF(0,0,0)$ is Fredholm of index $N$ if and only if the operator

$$T_3 : H^{2,p}_{\mathbf{R}}(S,\mathbf{C}) \longrightarrow H^{1,p}(S,\mathbf{C})\,, \tag{4.8}$$
$$T_3(\eta)(s,t) = \partial_s\eta(s,t) + i\,\partial_t\eta(s,t) + (f(s)-\phi(s))\eta(s,t)$$

is Fredholm of index $N-2$. The Fredholm property and the index of $T_3$ actually do not depend on the particular choice of $p \geq 2$ (this follows from a result by Mazja and Plamenevski, see [9], [13] or [16]).

We will choose $p = 2$ for this purpose and use the spectral flow of the operator family $A(s) = -i\frac{d}{dt} + (\phi(s) - f(s))$ for the computation of the index. By regularity we may use also the spaces $H^{k,p}$, $H^{k-1,p}$ , $k \geq 1$, instead of $H^{2,p}$, $H^{1,p}$. We introduce some notation before we quote a theorem which will be our main tool: Let $W, H$ be separable real Hilbert spaces with $W \subset H$ so that the inclusion $W \hookrightarrow H$ is compact with dense range. Then we define the Hilbert spaces $\mathcal{H}$ and $\mathcal{W}$ by

$$\mathcal{H} := L^2(\mathbf{R}, H)$$

and

$$\mathcal{W} := L^2(\mathbf{R}, W) \cap H^{1,2}(\mathbf{R}, H)$$

with inner products

$$(\xi_1,\xi_2)_{\mathcal{H}} := \int_{\mathbf{R}} (\xi_1(s),\xi_2(s))_H\,ds$$

and

$$(\xi_1,\xi_2)_{\mathcal{W}} := \int_{\mathbf{R}} \big[(\xi_1(s),\xi_2(s))_W + (\xi_1'(s),\xi_2'(s))_H\big]ds\,.$$

We would like to use the following theorem ([9], [15] theorems 3.12. and 4.21):

THEOREM 4.2 *Let $W$, $H$ be separable real Hilbert spaces as above and let $A(s) : W \to H$ be a family of bounded linear operators satisfying the following conditions:*

(i) *The map $A : \mathbf{R} \to \mathcal{L}(W, H)$ is continuously differentiable in the weak operator topology and there exists a constant $c > 0$ such that*

$$\|A(s)\xi\|_H + \|\partial_s A(s)\xi\|_H \leq c\|\xi\|_W$$

*for every $s \in \mathbf{R}$ and $\xi \in W$.*

(ii) *The operators $A(s)$ are self-adjoint and there is a constant $c > 0$ so that*

$$\|\xi\|_W^2 \leq c\big(\|A(s)\|_H^2 + \|\xi\|_H^2\big)$$

*for all $s \in \mathbf{R}$ and $\xi \in W$.*

(iii) *There are invertible operators $A_\pm \in \mathcal{L}(W, H)$ such that*

$$\lim_{s\to\pm\infty} \|A(s) - A_\pm\|_{\mathcal{L}(W,H)} .$$

*Given a differentiable curve $\xi : \mathbf{R} \to W$, we define an operator*

$$D_A : \mathcal{W} \longrightarrow \mathcal{H}$$

*by*

$$(D_A\xi)(s) := \partial_s\xi(s) - A(s)\xi(s) .$$

*Then $D_A$ is a Fredholm operator and its index is given by minus the spectral flow of the operator family $A(s)$.*

The Hilbert spaces $\mathcal{W}$ and $\mathcal{H}$ where we would like to apply Theorem 4.2 are

$$\mathcal{H} := L^2(\mathbf{R}, L^2([0,1], \mathbf{C}))$$

and

$$\mathcal{W} := L^2\big(\mathbf{R}, H^{1,2}_{\mathbf{R}}([0,1], \mathbf{C})\big) \cap H^{1,2}\big(\mathbf{R}, L^2([0,1], \mathbf{C})\big) ,$$

where

$$H^{1,2}_{\mathbf{R}}([0,1], \mathbf{C}) := \big\{h \in H^{1,2}([0,1], \mathbf{C}) \mid h(0), h(1) \in \mathbf{R}\big\}$$

with norms

$$\|\xi\|_{\mathcal{H}}^2 := \int_{\mathbf{R}} \|\xi(s)\|^2_{L^2([0,1],\mathbf{C})} ds ,$$

$$\|\xi\|_{\mathcal{W}}^2 := \int_{\mathbf{R}} \big(\|\xi(s)\|^2_{H^{1,2}([0,1],\mathbf{C})} + \|\partial_s\xi(s)\|^2_{L^2([0,1],\mathbf{C})}\big) ds$$

which actually equal $\|\xi\|^2_{L^2(S,\mathbf{C})}$ and $\|\xi\|^2_{H^{1,2}(S,\mathbf{C})}$. The operator family $A(s)$ in Theorem 4.2 will be

$$A(s) := -i\frac{d}{dt} + (\phi(s) - f(s))\,\mathrm{Id} . \tag{4.9}$$

Condition (i) of Theorem 4.2 is obviously satisfied since $|f'(s)|, |\phi'(s)|$ remain bounded. The estimate in condition(ii) is also obvious. The operators $A(s)$ are self-adjoint because the operator $-i\frac{d}{dt}$ is self-adjoint and the matrix $(\phi(s) - f(s))\,\mathrm{Id}$ is symmetric. If we define operators $A_\pm \in \mathcal{L}(W, H)$ by

$$A_+ := -i\frac{d}{dt} + (\pi\mu(\tilde{u}_0) + \gamma_+\lambda_+)\,\mathrm{Id}\,, \quad A_- := -i\frac{d}{dt} + \gamma_-\lambda_-\,\mathrm{Id}\,, \tag{4.10}$$

then condition (iii) is satisfied as well since both $\gamma_\pm\lambda_\pm$ are not integer multiples of $\pi$ by assumption. In order to show this and for the computation of the spectral flow of the family $A(s)$ we start with the eigenvalues of the operators $A(s)$. Since $A(s)$ acts on paths with real boundary condition the eigenvalues of $A(s)$ are given by

$$\lambda_n(s) = n\pi + (\phi(s) - f(s))\,, \quad n \in \mathbf{Z}\,. \tag{4.11}$$

We will show now that the operators $A(s)$ have a compact resolvent, hence the spectrum $\sigma(A(s))$ consists of the eigenvalues (4.11) only. Assume that $\nu$ is a real number with $\nu \neq \lambda_n(s)$ for all $n \in \mathbf{Z}$ ($s$ is fixed at the moment). Then

$$\kappa := \phi(s) - f(s) - \nu \notin \mathbf{Z}\pi\,.$$

Define now the following bounded linear operator

$$R(\nu) : L^2([0,1], \mathbf{C}) \longrightarrow L^2([0,1], \mathbf{C})\,,$$
$$(R(\nu)v)(t) := \left(\int_0^t i\,v(\tau)e^{i\kappa\tau}d\tau + c\right)e^{-i\kappa t}\,,$$

where

$$c = \cot\kappa \int_0^1 \big(v_1(\tau)\cos(\kappa\tau) - v_2(\tau)\sin(\kappa\tau)\big)d\tau + \int_0^1 (v_1(\tau)\sin(\kappa\tau) + v_2(\tau)\cos(\kappa\tau))d\tau$$

with $v = v_1 + iv_2$. With the above choice of $c$ we have $(R(\nu)v)(0), (R(\nu)v)(1) \in \mathbf{R}$. Moreover, $R(\nu) = (A(s) - \nu\,\mathrm{Id})^{-1}$ and actually $R(\nu)v \in H^{1,2}_{\mathbf{R}}([0,1], \mathbf{C})$, which is compactly embedded into $L^2([0,1], \mathbf{C})$, hence the resolvent is compact and the spectrum consists of the eigenvalues $\lambda_n(s) = n\pi + (\phi(s) - f(s))\,, \quad n \in \mathbf{Z}$ only. Note also that $0 \notin \sigma(A_\pm)$, hence $A_\pm$ are invertible.

Therefore, we have shown that the operator $T_3$ in (4.8) is Fredholm. Let us compute now the spectral flow of the operator family $A(s)$. Let us review its definition. Following [15], a number $s_0 \in \mathbf{R}$ is called a *crossing* if the operator $A(s_0)$ is not injective, i.e. there is some $n_0 \in \mathbf{Z}$ so that $\lambda_{n_0}(s_0) = 0$. In our case the kernel of $A(s_0)$ is always one-dimensional. Recall that $\mu(\tilde{u}_0) = 0$ and $\mathrm{sign}(\gamma_-) = \mathrm{sign}(\gamma_+) \neq 0$ by assumption. Using $\lambda_n(s) = n\pi + (\phi(s) - f(s))$ with $f$ as in (4.6) and $\phi$ as in (4.7) we see that

$$\lambda_n'(s) = -\,\mathrm{sign}(\gamma_\pm) \neq 0\,.$$

In the language of [15], all crossings in our situation are regular. Then the spectral flow is defined as the number

$$\sum_{s_0 \text{ crossing}} \operatorname{sign} \lambda'_{n_0}(s_0) \, .$$

If we let $s$ increase then we count the eigenvalues $\lambda_n$ which become zero. Decreasing eigenvalues are counted with negative sign, and increasing eigenvalues are counted with positive sign. Using theorem 4.2, the Fredholm index of $T_3$ is then given by

$$\begin{aligned}
\operatorname{ind} T_3 &= \operatorname{ind} \left( \frac{\partial}{\partial s} - A(s) \right) \\
&= - \sum_{s_0 \text{ crossing}} \operatorname{sign} \lambda'_{n_0}(s_0) \\
&= \operatorname{sign}(\gamma_\pm) \, \#\{n \in \mathbf{Z} \mid \phi(s) - f(s) = n\pi \text{ for some } s \in \mathbf{R}\} \\
&= \operatorname{sign}(\gamma_\pm) \, \#(\pi \mathbf{Z} \cap I) \, .
\end{aligned}$$

This completes the proof of Proposition 4.1. □

Figure 4.1 shows the case where $|\lambda_\pm| \in \{\frac{\pi}{2}, \pi\}$ and

- $-2 < \gamma_\pm < 0$ in the case $\lambda_\pm = \mp\frac{\pi}{2}$,
- $-1 < \gamma_\pm < 0$ in the case $\lambda_\pm = \mp\pi$.

In these cases there is exactly one positive crossing, hence the Fredholm indices of $T_3$ and the operator $T_0$ in (4.1) equal $-1$ as desired.

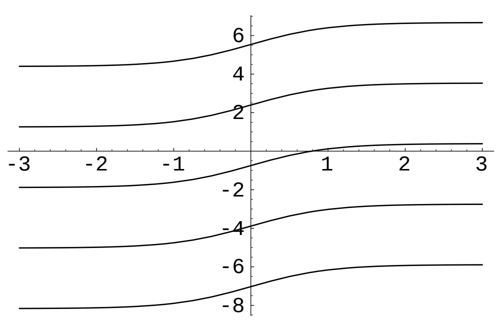


FIGURE 4.1. The figure shows the eigenvalues $\lambda_n(s)$ of the operators $A(s)$ for $\gamma_+ = -1/4$, $\gamma_- = -2/3$, $\lambda_- = \pi$, $\lambda_+ = -\pi/2$.

Our aim is to show next that in this case, the operator $T_0$ is as good as it can be, namely the kernel is trivial and the cokernel is exactly one-dimensional. It suffices to consider the somewhat simpler operator $T_2$ in (4.5) since its kernel and cokernel are isomorphic to those of $T_0$. The operator $T_2$ is of the following form:

$$T_2 : H^{2,p}_{\mathbf{R}}(S,\mathbf{C}) \longrightarrow H^{1,p}(S,\mathbf{C})\,,$$
$$(T_2\eta)(s,t) = \partial_s\eta(s,t) + i\,\partial_t\eta(s,t) + \Gamma(s,t)\eta(s,t)\,, \tag{4.12}$$

where $\Gamma : S \to \mathbf{R}^{2\times 2}$ is a smooth function with the following properties:

$$\partial_s\Gamma(s,t) \longrightarrow 0$$

uniformly in $t$ as $|s| \to \infty$,

$$\Gamma(s,t) \longrightarrow -\gamma_\pm\lambda_\pm \cdot \mathrm{Id}$$

as $s \to \pm\infty$ uniformly in $t$ since we will focus on situations with zero Maslov index. If we wish to emphasize the weight function $\gamma$, we will sometimes denote the operator by $T_2(\gamma)$. We have proved in proposition 4.1 that $T_2$ is Fredholm and its index is given by

$$\operatorname{ind} T_2(\gamma) = \operatorname{sign}(\gamma_\pm)\,\#(\pi\mathbf{Z}\cap I)\,, \tag{4.13}$$

where $I$ is the bounded interval with end points $\gamma_-\lambda_-$ and $\gamma_+\lambda_+$. In order to understand how large the kernel of $T_2$ is we first have to investigate the behavior of elements in the kernel. We note that all elements in the kernel of $T_2$ are smooth by elliptic regularity.

LEMMA 4.3 *Let $\eta \in H^{1,p}_{\mathbf{R}}(S,\mathbf{C})$, $p \geq 2$, be a solution of the linear differential equation $T_2\eta = 0$ with $T_2 = T_2(\gamma)$ as in* (4.12) *and $\lambda_\pm\gamma_\pm \notin \mathbf{Z}\pi$. Then there are constants $s_0, c > 0$ such that*

$$\sup_{0\leq t\leq 1} |\eta(s,t)| \leq ce^{-1/2\kappa_\pm|s|}$$

*for all $|s| \geq s_0$, where*

$$\kappa_\pm = \min_{n\in\mathbf{Z}} |n\pi + \gamma_\pm\lambda_\pm| > 0\,.$$

PROOF: We define the following unbounded operators

$$A_\pm : L^2([0,1],\mathbf{C}) \supset H^{1,2}_{\mathbf{R}}([0,1],\mathbf{C}) \longrightarrow L^2([0,1],\mathbf{C})\,,$$
$$A_\pm := -i\frac{d}{dt} + \gamma_\pm\lambda_\pm\,\mathrm{Id}\;.$$

We also define the smooth functions

$$\Delta_\pm(s,t) := -\gamma_\pm\lambda_\pm\,\mathrm{Id} - \Gamma(s,t)$$

so that

$$|\Delta_\pm(s,t)|\,,\quad |\partial_s\Delta_\pm(s,t)| \longrightarrow 0 \quad \text{as } s \to \pm\infty\,.$$

The differential equation $T_2\eta = 0$ is then the same as

$$\partial_s\eta(s,t) = (A_\pm\eta(s))(t) + \Delta_\pm(s,t)\eta(s,t)\,. \tag{4.14}$$

The operators $A_\pm$ are self-adjoint. Moreover, their resolvents are compact operators so that the spectra $\sigma(A_\pm)$ consist of eigenvalues only. We have

$$\sigma(A_\pm) = \{n\pi + \gamma_\pm \lambda_\pm \mid n \in \mathbf{Z}\},$$

hence $0 \notin \sigma(A_\pm)$ and the inverses $A_\pm^{-1} : L^2([0,1],\mathbf{C}) \to H^{1,2}_{\mathbf{R}}([0,1],\mathbf{C}) \subset L^2([0,1],\mathbf{C})$ exist. Moreover, they are bounded operators with norms

$$\left\| A_\pm^{-1} \right\|_{\mathcal{L}(L^2,L^2)} = \frac{1}{\kappa_\pm}.$$

This implies that for all $\delta \in H^{1,2}_{\mathbf{R}}([0,1],\mathbf{C})$

$$\|A_\pm \delta\|_{L^2([0,1],\mathbf{C})} \geq \kappa \|\delta\|_{L^2([0,1],\mathbf{C})}. \tag{4.15}$$

We define now the smooth function

$$g : \mathbf{R} \longrightarrow (0,\infty), \quad g(s) = \frac{1}{2}\|\eta(s)\|^2_{L^2([0,1],\mathbf{C})},$$

and we claim that there are sequences $(s_k^-)$ and $(s_k^+)$ with $s_k^\pm \to \pm\infty$ as $s \to \pm\infty$ such that $g(s_k^\pm) \to 0$. If the number $p$ in the lemma equals 2 then this is trivial. If $p > 2$ then $\eta \in L^p(S,\mathbf{C})$, so there must be sequences with $\|\eta(s_k^\pm)\|_{L^p([0,1],\mathbf{C})} \to 0$. The claim follows from Hölder's inequality:

$$\int_0^1 \left|\eta(s_k^\pm,t)\right|^2 dt \leq \left(\int_0^1 \left|\eta(s_k^\pm,t)\right|^p dt\right)^{2/p}, \quad \frac{p}{2} > 1.$$

Denoting functions which converge to zero as $s \to \pm\infty$ by $\varepsilon(s)$, we estimate with (4.14)

$$\begin{aligned}
g''(s) &= \|\partial_s \eta(s)\|^2_{L^2([0,1],\mathbf{C})} + (\partial_{ss}\eta(s),\eta(s))_{L^2([0,1],\mathbf{C})} \\
&\geq (\partial_{ss}\eta(s),\eta(s))_{L^2([0,1],\mathbf{C})} \\
&= (A_\pm \partial_s \eta(s),\eta(s))_{L^2([0,1],\mathbf{C})} + (\partial_s \Delta_\pm(s)\eta(s),\eta(s))_{L^2([0,1],\mathbf{C})} \\
&\quad + (\Delta_\pm(s)\partial_s\eta(s),\eta(s))_{L^2([0,1],\mathbf{C})} \\
&= (\partial_s\eta(s), A_\pm\eta(s))_{L^2([0,1],\mathbf{C})} + (\partial_s \Delta_\pm(s)\eta(s),\eta(s))_{L^2([0,1],\mathbf{C})} \\
&\quad + (A_\pm\eta(s) + \Delta_\pm(s)\eta(s), \Delta_\pm(s)^T\eta(s))_{L^2([0,1],\mathbf{C})} \\
&\geq \|A_\pm\eta(s)\|^2_{L^2([0,1],\mathbf{C})} - \varepsilon(s)\|A_\pm\eta(s)\|_{L^2([0,1],\mathbf{C})}\|\eta(s)\|_{L^2([0,1],\mathbf{C})} \\
&\quad - \varepsilon(s)\|\eta(s)\|^2_{L^2([0,1],\mathbf{C})} \\
&= \|A_\pm\eta(s)\|_{L^2([0,1],\mathbf{C})}\big(\|A_\pm\eta(s)\|_{L^2([0,1],\mathbf{C})} - \varepsilon(s)\|\eta(s)\|_{L^2([0,1],\mathbf{C})}\big) \\
&\quad - \varepsilon(s)\|\eta(s)\|^2_{L^2([0,1],\mathbf{C})}.
\end{aligned}$$

Using (4.15) we obtain

$$g''(s) \geq [\kappa_\pm(\kappa_\pm - \varepsilon(s)) - \varepsilon(s)]\|\eta(s)\|^2_{L^2([0,1],\mathbf{C})} \geq \kappa_\pm^2\, g(s)$$

if $|s|$ is sufficiently large. Together with the existence of sequences $s_k^{\pm}$ where $g(s_k^{\pm}) \to 0$ this implies $g(s) \le c\, e^{-\kappa_{\pm}|s|}$, i.e.,

$$\|\eta(s)\|_{L^2([0,1],\mathbf{C})} \le ce^{-1/2\kappa_{\pm}|s|} .$$

Taking now $\tilde{g}(s) := \frac{1}{2}\|\partial_s\eta(s)\|_{L^2([0,1],\mathbf{C})}$ and observing that $\partial_s\eta$ satisfies a similar differential equation as $\eta$, we obtain $\tilde{g}''(s) \ge \kappa_{\pm}^2\tilde{g}(s)$ as well, which implies $\|\partial_s\eta(s)\|_{L^2([0,1],\mathbf{C})} \le ce^{-1/2\kappa|s|}$. Since $\partial_t\eta(s,t) = i\,\partial_s\eta(s,t) + i\,\Gamma(s,t)\eta(s,t)$ we obtain the same exponential decay estimate for $\|\partial_t\eta(s)\|_{L^2([0,1],\mathbf{C})}$. The Sobolev embedding theorem yields

$$\sup_{0\le t\le 1} |\eta(s,t)| \le c\|\eta(s)\|_{H^{1,2}([0,1],\mathbf{C})} \le ce^{-1/2\kappa_{\pm}|s|} ,$$

which is the desired exponential decay estimate. □

LEMMA 4.4 *Let $\eta \in H^{1,p}_{\mathbf{R}}(S,\mathbf{C})$, $p \ge 2$, be a nontrivial solution of the linear differential equation $T_2\eta = 0$ with $T_2 = T_2(\gamma)$ as in* (4.12)*. Then we have the following asymptotic formulas for large $|s|$:*

$$\eta(s,t) = e^{\int_{s_0}^{s} a(\tau)d\tau}\,(e_{\pm}(t) + r_{\pm}(s,t)) ,$$

*where $a$ is a smooth real-valued function satisfying $a(\tau) \to \nu_{\pm}$ as $s \to \pm\infty$ and $\nu_+ < 0$ , $\nu_- > 0$ being eigenvalues of the operators*

$$A_{\pm} : L^2([0,1],\mathbf{C}) \supset H^{1,2}_{\mathbf{R}}([0,1],\mathbf{C}) \longrightarrow L^2([0,1],\mathbf{C}) , \tag{4.16}$$

$$A_{\pm} := -i\frac{d}{dt} + \gamma_{\pm}\lambda_{\pm}\,\mathrm{Id} .$$

*Moreover, $e_{\pm}(t)$ are eigenvectors of $A_{\pm}$ belonging to the eigenvalues $\nu_{\pm}$ with $e(t) \ne 0$ for all $0 \le t \le 1$, and $r_{\pm}$ are smooth functions so that $r_{\pm}$ and all their derivatives converge to zero uniformly in $t$ as $s \to \pm\infty$.*

PROOF: We will only sketch the proof since it is very similar to the proof of Theorem 1.5, but much simpler. As in [2], we define

$$a(s) := \frac{\frac{d}{ds}\|\eta(s)\|^2_{L^2([0,1],\mathbf{C})}}{2\|\eta(s)\|^2_{L^2([0,1],\mathbf{C})}}$$

and

$$\xi(s,t) := \frac{\eta(s,t)}{\|\eta(s)\|_{L^2([0,1],\mathbf{C})}} .$$

As in the paper [2], the function $\xi(s,t)$ satisfies a perturbed Cauchy Riemann type equation:

$$\partial_s\xi(s) = A_{\pm}\xi(s) + \Delta_{\pm}(s)\xi(s) - a(s)\xi(s) .$$

Using $\|\xi(s)\|_{L^2([0,1],\mathbf{C})} = 1$, we obtain the following formula for $\alpha(s)$:

$$a(s) = (A_{\pm}\xi(s),\xi(s))_{L^2([0,1],\mathbf{C})} + (\Delta_{\pm}(s)\xi(s),\xi(s))_{L^2([0,1],\mathbf{C})} ,$$

and the estimate

$$a'(s) \geq 2\|\partial_s \xi(s)\|^2_{L^2([0,1],\mathbf{C})} - \varepsilon(s)\|\partial_s \xi(s)\|_{L^2([0,1],\mathbf{C})} - \varepsilon(s)\,.$$

If $a(s) \notin \sigma(A_-)$ or $a(s) \notin \sigma(A_+)$ we conclude

$$\begin{aligned} 1 &= \|\xi(s)\|_{L^2([0,1],\mathbf{C})} \\ &= \|(A_\pm - a(s)\,\mathrm{Id})^{-1}(\partial_s \xi(s) - \Delta_\pm(s)\xi(s))\|_{L^2([0,1],\mathbf{C})} \\ &\leq \frac{1}{\operatorname{dist}(a(s), \sigma(A_\pm))}\,\|\partial_s \xi(s)\|_{L^2([0,1],\mathbf{C})} - \varepsilon(s)\,. \end{aligned}$$

With these key estimates in place and the decay property from Lemma 4.3, the arguments are the same as in the paper [2], but much simpler. Eigenvectors $e_\pm$ belonging to eigenvalues $\nu_n^\pm = n\pi + \gamma_\pm \lambda_\pm$ of $A_\pm$ are given by

$$e_\pm(t) = ce^{i\pi n t}\,, \tag{4.17}$$

so $e_\pm$ never vanishes. □

LEMMA 4.5 *Let $\eta \in H^{1,p}_{\mathbf{R}}(S, \mathbf{C})$, $p \geq 2$ be a non constant solution of the linear differential equation $T_2\eta = 0$ with $T_2 = T_2(\gamma)$ as in* (4.12). *Then*

$$\#\{z \in S \mid \eta(z) = 0\} < \infty\,.$$

PROOF: Since $\eta$ satisfies a Cauchy Riemann type equation $0 = \partial_s \eta + i\,\partial_t \eta + \Gamma\eta$ the similarity principle (Theorems A.1 and A.2) implies that the set $\{z \in S \mid \eta(z) = 0\}$ is discrete. We have to show that it is also bounded. Arguing indirectly, we assume that there are $(s_k, t_k) \in S$ with $|s_k| \to \infty$ and $\eta(s_k, t_k) = 0$. We may assume without loss of generality that $t_k \to t_0 \in [0, 1]$ and $s_k \to +\infty$. Then

$$0 = \eta(s_k, t_k) = e^{\int_{s_0}^{s_k} a(\tau)d\tau}(e_+(t_k) + r_+(s_k, t_k))$$

and

$$0 = e_+(t_k) + r_+(s_k, t_k)\,.$$

Passing to the limit yields $e_+(t_0) = 0$ contradicting the fact that $e_+$ never vanishes. □

Let $z_0 \in S$ be a zero of $\eta$, where we assume that $\eta$ does not vanish identically. If $z_0 \in \overset{\circ}{S}$ we have $\eta(z) = \Phi(z)\sigma(z)$ with

$$\sigma(z) = \sum_{k=k_0}^{\infty} a_k (z - z_0)^k$$

where $a_{k_0} \neq 0$ using the similarity principle. We note that the number $k_0 \geq 1$ does not depend on the representation $\Phi\sigma$ for $\eta$ because it equals the winding number $w(\eta|_{\partial B_\rho(z_0)}, 0)$ of $\eta|_{\partial B_\rho(z_0)}$ around 0, where $\rho > 0$ is some small number. We define the local order $o(z_0, \eta)$ by

$$o(z_0, \eta) := k_0\,. \tag{4.18}$$

If $z_0 \in \partial S$ then we can apply Theorem A.2 and we define similarly the local order $o(z_0, \eta)$ for a boundary point.

The following proposition is our main tool for determining the size of the kernel of the operator $T_2(\gamma)$. We define the sets

$$N_{int} := \{z \in \overset{\circ}{S} \mid \eta(z) = 0\}\,, \quad N_{bd} := \{z \in \partial S \mid \eta(z) = 0\}\,,$$

which are finite sets by lemma 4.5.

PROPOSITION 4.6 *Let $\eta \in H^{1,p}_{\mathbf{R}}(S, \mathbf{C})$, $p \geq 2$ be a nontrivial solution of the linear differential equation $T_2\eta = 0$ with $T_2 = T_2(\gamma)$ as in* (4.12)*. Let $n_\pm$ be integers given by*

$$a(s) \stackrel{s\to\infty}{\longrightarrow} \nu_+ = n_+\pi + \gamma_+\lambda_+ < 0\,,$$
$$a(s) \stackrel{s\to-\infty}{\longrightarrow} \nu_- = n_-\pi + \gamma_-\lambda_- > 0\,,$$

*where $a$ is the function in the asymptotic formula in Lemma* 4.4*. Then*

$$0 \leq 2\sum_{z\in N_{int}} o(z,\eta) + \sum_{z\in N_{bd}} o(z,\eta) = n_+ - n_-\,.$$

PROOF: Using lemma 4.4 we have

$$\frac{\eta(s,t)}{|\eta(s,t)|} \longrightarrow e_\pm(t) = c_\pm\, e^{in_\pm\pi t}$$

as $s \to \pm\infty$ for suitable nonzero constants $c_\pm$. Compactifying $S$ at the ends and identifying it with $Q = [-1,1]\times[0,1]$ we see that there is some $0 < \delta < 1$ such that $\eta(s,t) \neq 0$ for all $0 \leq t \leq 1$ and $|s| \geq 1-\delta$. Denote by $r$ the distance between the finite set of interior zeros of $\eta$ and the set $([-1,1]\times\{1\}) \cup ([-1,1]\times\{0\})$ (see Figure 4.2). Moreover, let $\varepsilon_0$ be the minimal distance between two boundary zeros.

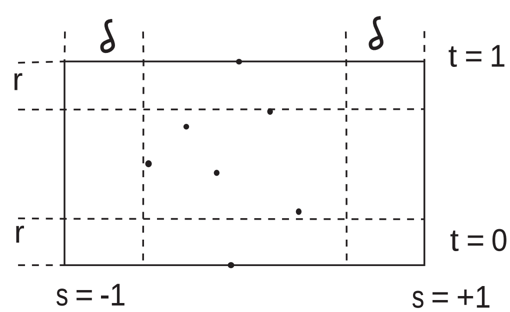


FIGURE 4.2. The dots represent zeros of $\eta$.

Let $0 < \varepsilon < \min\{\varepsilon_0, r, \delta\}$ and define the domain

$$S_\varepsilon := S \setminus \bigcup_{z\in N_{bd}} B_\varepsilon(z)$$

(see Figure 4.3). Let $\Omega \subset \mathring{S}_\varepsilon \subset S_\varepsilon$ be a connected set which contains all the interior

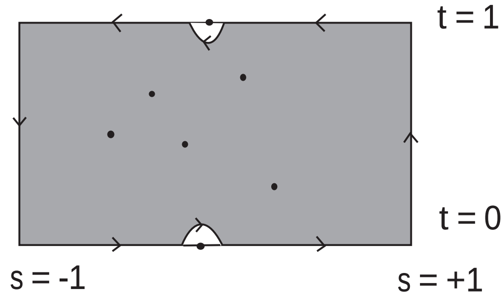


FIGURE 4.3. The domain $S_\varepsilon$.

zeros of $\eta$. The winding numbers $w(\eta|_{\partial S_\varepsilon}, 0)$ and $w(\eta|_{\partial\Omega}, 0)$ are well-defined since $0 \notin \eta(\partial S_\varepsilon) \cup \eta(\partial\Omega)$. Since $\eta$ has no zeros in $\bar{S}_\varepsilon \backslash \Omega$ we have

$$w(\eta|_{\partial S_\varepsilon}, 0) = w(\eta|_{\partial\Omega}, 0) = \sum_{z \in N_{int}} o(z, \eta)\,.$$

Consider the loop

$$L_\varepsilon(z) := \left.\frac{\eta(z)}{|\eta(z)|}\right|_{z \in \partial S_\varepsilon} \cdot \mathbf{R} \subset \mathbf{C}$$

of Lagrangian subspaces of $\mathbf{C}$. Its Maslov index is then given by

$$\mu(L_\varepsilon) = 2w\big(\eta\big|_{\partial S_\varepsilon}, 0\big)\,.$$

We will now relate the Maslov index $\mu(L_\varepsilon)$ to the boundary zeros of $\eta$ and the eigenvectors $e_\pm$. Let $z_0$ be a zero of $\eta$ on the boundary. Applying the Similarity Principle (boundary version) we may write $\eta(z) = \Phi(z)\sigma(z)$ near $z_0$, where $\sigma$ is a holomorphic function with real boundary values. We compute the Maslov index of the loop

$$L_{z_0}(z) = \left.\frac{\eta(z)}{|\eta(z)|}\right|_{z \in \partial D^+}$$

as shown in Figure 4.4. We may replace the loop $L_{z_0}$ by the loop $\frac{\sigma(z)}{|\sigma(z)|}|_{z \in \partial D^+}$ since

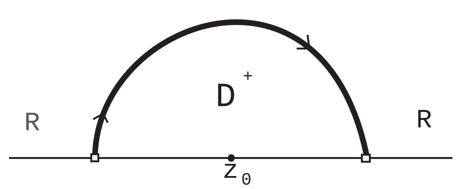


FIGURE 4.4. The situation near a boundary zero $z_0$ of $\eta$. Bypass the boundary zero $z_0$ and evaluate $\eta/|\eta|$ along the arc.

the two are homotopic via $t \longmapsto \Phi(tz)^{-1}L_{z_0}(z)$, $0 \leq t \leq 1$. Expanding the holomorphic function $\sigma$ in a power series

$$\sum_{k=k_0}^{\infty} a_k z^k$$

we may also remove the terms $\sum_{k=k_0+1}^{\infty} a_k z^k$ by a homotopy argument, if we replace the domain $D^+$ by a smaller half-disk. Then the Maslov index of $L_{z_0}$ is given by

$$\mu(L_{z_0}) = -o(z_0, \eta)\,.$$

At the ends we obtain the loops

$$L_+(t) = c_+ e^{in_+\pi t}\,, \quad L_-(t) = c_- e^{in_-\pi(1-t)}\,, \quad 0 \leq t \leq 1\,,$$

of Lagrangian subspaces in $\mathbf{C}$ which contribute

$$\mu(L_+) = n_+ \quad \text{and} \quad \mu(L_-) = -n_-\,.$$

The Maslov index of $L_\varepsilon$ is given by

$$\mu(L_\varepsilon) = \mu(L_-) + \mu(L_+) + \sum_{z_0 \in N_{bd}} \mu(L_{z_0}) = -n_- + n_+ - \sum_{z \in N_{bd}} o(z, \eta)\,.$$

This completes the proof. □

The previous proposition permits us to determine the size of the kernel of the operator $T_2$.

PROPOSITION 4.7 *Assume that $\gamma_\pm\lambda_\pm$ are not integer multiples of $\pi$, $\mu(\tilde{u}_0) = 0$ and $p \geq 2$. Let $T_2 = T_2(\gamma)$ be the operator in* (4.12).

- *If $\gamma_-, \gamma_+ < 0$ then $T_2$ has trivial kernel.*
- *If $0 < \gamma_\pm < \frac{\pi}{2|\lambda_\pm|}$ then the kernel of $T_2$ is one-dimensional generated by an element $\eta_0$ which does not vanish anywhere.*

PROOF: Consider first the case where $\gamma_\pm$ are negative. Arguing indirectly, we assume that $\eta$ is a nontrivial element in the kernel of $T_2$. By Lemma 4.4 we have the following asymptotic formulas for $|s|$ large:

$$\eta(s,t) = e^{\int_{s_0}^{s} a(\tau)d\tau}(e_\pm(t) + r_\pm(s,t))\,,$$

where $a$ is a smooth function satisfying $a(s) \to \nu_\pm$ as $s \to \pm\infty$, where $\nu_+ < 0$, $\nu_- > 0$ are eigenvalues of the operators $A_\pm$ in (4.16). We have

$$\nu_+ = \pi n_+ + \gamma_+\lambda_+\,, \quad \nu_- = \pi n_- + \gamma_-\lambda_-$$

for suitable integers $n_\pm \in \mathbf{Z}$. We also recall that $\lambda_- > 0$ and $\lambda_+ < 0$. The inequalities $\nu_+ < 0$ and $\nu_- > 0$ imply

$$n_+ < \frac{-\gamma_+\lambda_+}{\pi} < 0 \quad \text{and} \quad n_- > \frac{-\gamma_-\lambda_-}{\pi} > 0\,,$$

hence $n_+ - n_- < 0$ contradicting Proposition 4.6.

Consider now the case where $0 < \gamma_\pm < \frac{\pi}{2|\lambda_\pm|}$. By Proposition 4.1 (or rather the proof), the operator $T_2$ has index $+1$, hence the kernel of $T_2$ is nontrivial. If $\eta$ is a nontrivial element in the kernel, then we have the same asymptotic formulas as before, and the inequalities $\nu_+ < 0$ , $\nu_- > 0$ imply

$$n_+ - n_- < \frac{1}{\pi}(\gamma_-\lambda_- - \gamma_+\lambda_+) < 1\,.$$

On the other hand we have $n_+ - n_- \geq 0$ by Proposition 4.6 so that $n_+ - n_- = 0$. Invoking Proposition 4.6 once again we see that $\eta$ does not vanish anywhere. If the kernel of $T_2$ were more than one-dimensional then we could take two linear independent elements in the kernel, say $\eta_1$ and $\eta_2$ and a point $z_0 \in S$ where both $\eta_1$ and $\eta_2$ are nonzero. Then

$$\eta(z) := \eta_2(z) - \frac{\eta_2(z_0)}{\eta_1(z_0)}\eta_1(z)$$

would be an element in the kernel of $T_2$ which cannot be trivial since $\eta_1$ , $\eta_2$ were linear independent. On the other hand, $\eta(z_0) = 0$ contradicting Proposition 4.6. □

We summarize what we have shown in the following proposition.

PROPOSITION 4.8 *Assume that* $\gamma_\pm\lambda_\pm \notin \mathbf{Z}\pi$ *and* $p \geq 2$. *Then the operator*

$$T_0 : H_L^{2,p,\gamma}(S, \mathbf{C}) \longrightarrow H^{1,p,\gamma}(S, \mathbf{C})$$

*in* (4.1) *is Fredholm. If in addition* $\mu(\tilde{u}_0) = 0$ *then we have the following*:

- *If* $\gamma_+, \gamma_- < 0$ *then its index is given by* $-1$ *and its kernel is trivial.*
- *If* $0 < \gamma_\pm < \frac{\pi}{2|\lambda_\pm|}$ *then its index is* $+1$ *and its kernel is one-dimensional generated by some* $\eta_0 \in H^{2,p,\gamma}(S, \mathbf{C})$ *with* $\eta_0(s, t) \neq 0$ *for all* $(s, t) \in S$.

The main theorem about the linearization of $F$ is the following:

THEOREM 4.9 *Consider the linearization* $DF(0, 0, 0)$ *of the map* $F$ *in Proposition* 3.9. *Assume further that* $|\lambda_\pm| \in \{\frac{\pi}{2}, \pi\}$ *and that*

- $-\frac{1}{2} < \gamma_\pm < 0$ *if* $\lambda_\pm = \mp\frac{\pi}{2}$ *and*
- $-\frac{1}{2} - \frac{\delta}{|\lambda_\pm|} < \gamma_\pm < -\frac{1}{2}$, *where* $\delta > 0$ *is the exponential decay rate of the remainder terms in the asymptotic formula* (*Theorem* 1.6) *if* $\lambda_\pm = \mp\pi$.

*Then* $DF(0, 0, 0)$ *is a surjective Fredholm operator of index* $+1$. *The kernel of* $DF(0, 0, 0)$ *is generated by some element* $(c_-, c_+, \eta) \in \mathbf{R}^2 \times H_L^{2,p,\gamma}(S, \mathbf{C})$ *where* $c_- \neq 0$ *and* $c_+ \neq 0$.

PROOF: The Fredholm property and the computation of the index was done in Proposition 4.1. Recall that

$$DF(0,0,0) : \mathbf{R}^2 \times H_L^{2,p,\gamma}(S,\mathbf{C}) \longrightarrow H^{1,p,\gamma}(S,\mathbf{C}) ,$$

$$\begin{aligned}(DF(0,0,0)(c_-,c_+,\eta))(z) &= \partial_s \eta(z) + i\partial_t \eta(z) + \big(D_2 J_1^{(0,0)}(z,0)\eta(z)\big)\cdot(0,1)\\ &\quad + c_- \frac{d}{d\tau}\Big|_{\tau=0} J_1^{(\tau,0)}(z,0)(0,1)\\ &\quad + c_+ \frac{d}{d\tau}\Big|_{\tau=0} J_1^{(0,\tau)}(z,0)(0,1) .\end{aligned}$$

The kernel of $DF(0,0,0)$ is at least one-dimensional, and we have to show that it is one-dimensional. Pick a nontrivial element $(c_-,c_+,\eta) \in \ker DF(0,0,0)$. We cannot have $c_- = c_+ = 0$ since then $T_0(\gamma)\eta = 0$ with $T_0$ as in (4.1), but we have shown in proposition 4.8 that $\ker T_0(\gamma) = \{0\}$. Let now $\bar{\gamma}$ be a weight function with

$$0 < \bar{\gamma}_\pm := \lim_{s\to\pm\infty} \bar{\gamma}(s) < \frac{\pi}{2|\lambda_\pm|} . \tag{4.19}$$

Using Proposition 4.8 again, the operator

$$T_0(\bar{\gamma}) : H_L^{2,p,\bar{\gamma}}(S,\mathbf{C}) \longrightarrow H^{1,p,\bar{\gamma}}(S,\mathbf{C})$$

is a surjective Fredholm operator with one-dimensional kernel generated by some $\eta_0 \in H_L^{2,p,\bar{\gamma}}(S,\mathbf{C})$ which does not vanish anywhere. Let $\eta_\pm \in H_L^{2,p,\bar{\gamma}}(S,\mathbf{C})$ be solutions to the equations

$$\begin{aligned} T_0(\bar{\gamma})\eta_+ &= \frac{d}{d\tau}\Big|_{\tau=0} J_1^{(0,\tau)}(z,0)(0,1) ,\\ T_0(\bar{\gamma})\eta_- &= \frac{d}{d\tau}\Big|_{\tau=0} J_1^{(\tau,0)}(z,0)(0,1) .\end{aligned}$$

Recalling that $H_L^{2,p,\gamma}(S,\mathbf{C}) \subset H_L^{2,p,\bar{\gamma}}(S,\mathbf{C})$ we compute

$$0 = DF(0,0,0)(c_-,c_+,\eta) = T_0(\bar{\gamma})(c_-\eta_- + c_+\eta_+ + \eta) ,$$

this means

$$\hat{\eta} = c_-\eta_- + c_+\eta_+ + \eta \in \ker T_0(\bar{\gamma}) = \mathbf{R}\eta_0 .$$

We note that $\eta_\pm \notin H_L^{2,p,\gamma}(S,\mathbf{C})$ and that $\eta_\pm$ are not collinear so that indeed $\dim(\ker(DF(0,0,0))) = 1$ and $DF(0,0,0)$ is surjective.

We would like to show next that both $c_-$ and $c_+$ have to be nonzero if $(c_-,c_+,\eta) \in \ker(DF(0,0,0))$. We have to look at the asymptotic behavior of $\hat{\eta} = c_-\eta_- + c_+\eta_+ + \eta$, a generator of the kernel of $T_0(\bar{\gamma})$. The linear version of the asymptotic formula, Lemma 4.4, yields the following formula for the absolute value of $\hat{\eta} \in \ker T_0(\bar{\gamma})$ for large $|s|$:

$$|\hat{\eta}(s,t)| = \rho^{-\bar{\gamma}(s)}(s) e^{\int_{s_0}^{s} a(\tau)d\tau} |e_\pm(t) + r_\pm(s,t)| .$$

The factor $\rho^{\bar{\gamma}}$ is present because lemma 4.4 originally refers to the operator $T_2(\bar{\gamma})$, not to $T_0(\bar{\gamma})$. Revisiting the proof of Proposition 4.7, we have shown that $n_- = n_+$, where $n_\pm$ are the integers given by $a(s) \to \nu_\pm = \pi n_\pm + \bar{\gamma}_\pm\lambda_\pm$. Using $\nu_+ < 0$, $\nu_- > 0$ and the assumption $0 < \bar{\gamma}_\pm < \frac{\pi}{2|\lambda_\pm|}$ we estimate

$$n_+ < -\frac{1}{\pi}\bar{\gamma}_+\lambda_+ = \frac{\bar{\gamma}_+|\lambda_+|}{\pi} < \frac{1}{2}$$

and

$$n_- > -\frac{\bar{\gamma}_-\lambda_-}{\pi} > -\frac{1}{2},$$

so that $n_- = n_+ = 0$, i.e. $a(s) \to \bar{\gamma}_\pm\lambda_\pm$ as $s \to \pm\infty$ and $e_\pm$ are nonzero constants. We point out that Theorem 1.6 is also valid for the function $a(s) - \bar{\gamma}_\pm\lambda_\pm$ (the proof is actually much simpler since we are dealing here with a linear differential equation for $\eta$). The function $\rho$ equals $e^{\int_{s_0}^{s}\alpha(\tau)d\tau}$, but here the function $\alpha$ comes from the asymptotic formula for the nonlinear Cauchy Riemann equation, Theorem 1.5. The function $\alpha(s)$ converges to $\lambda_\pm$. Then

$$|\hat{\eta}(s,t)| = e^{-\bar{\gamma}(s)\int_{s_0}^{s}\alpha(\tau)d\tau + \int_{s_0}^{s}a(\tau)d\tau}|e_\pm + r_\pm(s,t)|$$

and

$$\begin{aligned} &-\bar{\gamma}(s)\int_{s_0}^{s}\alpha(\tau)d\tau + \int_{s_0}^{s}a(\tau)d\tau = \\ &\quad -\bar{\gamma}(s)\int_{s_0}^{s}(\alpha(\tau)-\lambda_\pm)d\tau + \int_{s_0}^{s}(a(\tau)-\bar{\gamma}_\pm\lambda_\pm)d\tau + (\bar{\gamma}_\pm - \bar{\gamma}(s))\lambda_\pm(s-s_0) \\ &\quad \geq -C > -\infty \end{aligned}$$

for a suitable positive constant $C$ because of theorem 1.6 and because $|\bar{\gamma}_\pm - \bar{\gamma}(s)|$ decays exponentially fast. Hence $|\hat{\eta}(s,t)|$ cannot converge to zero for $|s| \to \infty$. This shows that actually both $c_-$ and $c_+$ have to be nonzero. □

PROOF OF THEOREM 3.10: The proof is a simple application of the implicit function theorem. By theorem 4.9 the linearization $DF(0,0,0)$ is surjective and the kernel is one-dimensional generated by some element $x_0 = (c_-, c_+, \eta)$. Moreover, both $c_-$ and $c_+$ are nonzero. Because the kernel is finite-dimensional, we have a splitting

$$\mathbf{R}^2 \times H_L^{2,p,\gamma}(S,\mathbf{C}) = \ker DF(0,0,0) \oplus X,$$

where $X$ is some closed subspace of $\mathbf{R}^2 \times H_L^{2,p,\gamma}(S,\mathbf{C})$. The implicit function theorem then yields a unique smooth map

$$\phi : \ker DF(0,0,0) \supset U \longrightarrow X,$$

where $U$ is a suitable open neighborhood of $(0,0,0)$ in $\ker DF(0,0,0)$, such that

$$\phi(0) = 0 \quad \text{and} \quad F(u + \phi(u)) \equiv 0$$

for all $u \in U$. The map

$$(-\delta, \delta) \longrightarrow \mathbf{R}^2 \times H_L^{2,p,\gamma}(S, \mathbf{C})$$

in the theorem is then given by

$$(c_-(\tau), c_+(\tau), \psi(\tau)) = \tau x_0 + \phi(\tau x_0) \tag{4.20}$$

so that $(c_-(0), c_+(0), \psi(0)) = (0, 0, 0)$. We have

$$0 = \frac{d}{d\tau}\bigg|_{\tau=0} F(c_-(\tau), c_+(\tau), \psi(\tau)) = DF(0,0,0)D\phi(0)x_0 \,.$$

Since $\phi$ maps into the linear space $X$ which is a complement of $\ker DF(0,0,0)$, we conclude that $D\phi(0) = 0$. Then

$$\frac{d}{d\tau}\bigg|_{\tau=0} (c_-(\tau), c_+(\tau), \psi(\tau)) = x_0 = (c_-, c_+, \eta) \,.$$

This shows that $c_-(\tau)$ and $c_+(\tau)$ are nonzero if $0 \neq |\tau|$ is sufficiently small. □

We now state and prove Theorem 1.2:

THEOREM 4.10 *Let $\tilde{u}_0$ be an embedded solution of* (1.1) *so that its Maslov index $\mu(\tilde{u}_0)$ vanishes. Assume moreover, that $|\tilde{u}_0(s,t) - p_\pm|$ decays either like $e^{-\pi|s|}$ or like $e^{-\pi/2|s|}$ for large $|s|$ in local coordinates near the points $p_\pm := \lim_{s\to\pm\infty} \tilde{u}_0(s,t)$ and that $p_- \neq p_+$. Then there is a smooth family $(\tilde{v}_\tau)_{-1<\tau<1}$ of embedded solutions of (*1.1*) with the following properties*:

- $\tilde{v}_0 = \tilde{u}_0$,
- *The solutions $\tilde{v}_\tau$ have the same Maslov index and the same decay rates as $\tilde{u}_0$,*
- *The sets*
$$U_\pm := \bigcup_{-1<\tau<1} \left\{ \lim_{s\to\pm\infty} \tilde{v}_\tau(s,t) \right\}$$
*are open neighborhoods of the points $p_\pm$ in $\mathcal{L}$.*

*If $|\tilde{u}_0(s,t) - p_\pm|$ decays like $e^{-\pi/2|s|}$ for both $s \to +\infty$ and $s \to -\infty$ then we have in addition*

- $\tilde{v}_\tau(S) \cap \tilde{v}_{\tau'}(S) = \varnothing$ *if* $\tau \neq \tau'$.

PROOF: Let $(c_-(\tau), c_+(\tau), \psi(\tau))$ be the map guaranteed by Theorem 3.10. We obtain pseudoholomorphic curves in $\mathbf{R} \times M$ by composing with the map $\Phi$ in (3.3), i.e.,

$$\tilde{v}_\tau(s,t) = \Phi_{c_-(\tau),c_+(\tau)}(s, t, x_\tau(s,t), y_\tau(s,t)) \,,$$

where we split $\psi(\tau)(s,t) = x_\tau(s,t) + iy_\tau(s,t)$ into its real and imaginary parts. The maps $\tilde{v}_\tau$ are actually pseudoholomorphic curves with respect to some complex structure $i_\tau(z)$ on $S$ as in (3.10) (with $w$ replaced by $\psi(\tau)$), but all the complex structures $i_\tau(z)$ are conformally equivalent to the standard one (see Appendix B), so that we may assume $i_\tau(z) = i$ after reparametrization. The statement that $U_\pm$ are open neighborhoods of the points $p_\pm$ follows from $c_\pm(\tau) \neq 0$ for $\tau \neq 0$. As for the decay rates of $|\tilde{v}_\tau(s,t)-(0,c_\pm(\tau),0,0)|$ we consult equation (3.5) for large $|s|$ which is

$$\tilde{v}_\tau(s,t)-(0,c_\pm(\tau),0,0)=\tilde{u}_0(s,t)+x_\tau(s,t)n(s,t)+y_\tau(s,t)m(s,t)\,.$$

For simplicity let us only discuss the case $s\to+\infty$. If $\lambda_+=-\pi$, i.e., $|\tilde{u}_0(s,t)|\le ce^{-\pi s}$, then we obtain from (3.1) and (3.18)

$$|n(s,t)|,\,|m(s,t)|\le ce^{-\frac{\pi}{2}s}$$

and

$$|x_\tau(s,t)|,\,|y_\tau(s,t)|\le ce^{\pi\gamma_+ s}$$

since $x_\tau+iy_\tau\in H^{2,p,\gamma}_L(S,\mathbf{C})$. Now $\gamma_+<-\frac{1}{2}$ implies that $|x_\tau n+y_\tau m|$ decays exponentially at a faster rate than $|\tilde{u}_0|$, hence $|\tilde{v}_\tau-(0,c_+(\tau),0,0)|$ has the same rate of decay as $|\tilde{u}_0|$. In the case $\lambda_+=-\pi/2$ we have $-1/2<\gamma_+<0$. We compute using the asymptotic formula (1.9) for $\tilde{u}_0$, (3.2) and (3.15):

$$\begin{aligned}\tilde{v}_\tau(s,t)&=\kappa e^{\int_{s_0}^s\alpha(\zeta)d\zeta}\left(-\cos\left(\frac{\pi t}{2}\right),q(0)\cos\left(\frac{\pi t}{2}\right),0,\sin\left(\frac{\pi t}{2}\right)\right)\\&\quad+\kappa e^{1/2\int_{s_0}^s\alpha(\zeta)d\zeta}(0,x_\tau(s,t),-y_\tau(s,t),0)\\&\quad+(0,c_+(\tau),0,0)+e^{\int_{s_0}^s\alpha(\zeta)d\zeta}\varepsilon(s,t)\,,\quad \alpha(s)\overset{s\to\infty}{\to}-\frac{\pi}{2}\,.\end{aligned}\tag{4.21}$$

On the other hand, we may also apply Theorem 1.5 to the solution $\tilde{v}_\tau$, and we obtain an asymptotic formula of the form (1.9) or (1.14). Comparing (4.21) with those two asymptotic formulas, we see that (1.14) is not possible, hence

$$\begin{aligned}\tilde{v}_\tau(s,t)&=\kappa_\tau e^{\int_{s_0}^s\alpha_\tau(\zeta)d\zeta}\left(-\cos\left(\frac{\pi t}{2}\right),q(c_+(\tau))\cos\left(\frac{\pi t}{2}\right),0,\sin\left(\frac{\pi t}{2}\right)\right)\\&\quad+(0,c_+(\tau),0,0)+e^{\int_{s_0}^s\alpha_\tau(\zeta)d\zeta}\varepsilon_\tau(s,t)\,,\end{aligned}\tag{4.22}$$

where $\kappa_0=\kappa$, $\alpha_0\equiv\alpha$ and $\alpha_\tau(s)\to-\frac{\pi}{2}$ for each $\tau$. Since $\psi(\tau)$ and its derivatives converge uniformly to zero as $\tau\to0$, we also have $\alpha_\tau\to\alpha$ uniformly as $\tau\to0$. We have so far shown that the solutions $\tilde{v}_\tau$ exhibit the same decay behavior as $\tilde{u}_0$. The maps $\tilde{v}_\tau$ are all immersions for small $|\tau|$ because of Lemma 3.6 and the remark following the lemma. If the decay rates of $|\tilde{v}_\tau-(0,c_+(\tau),0,0)|$ do not depend on $\tau$ then we also have $\mu(\tilde{v}_\tau)=0$ because of Proposition 3.3. In order to establish the embedding property it is sufficient to show that the maps $\tilde{v}_\tau$ are injective for small $|\tau|$ since the end points $\lim_{s\to\pm\infty}\tilde{v}_\tau(s,t)$ are not in the range of $\tilde{v}_\tau$ (see Proposition 2.1). Arguing indirectly, we assume that there are sequences $\tau_k\to0$ and $z'_k=(s'_k,t'_k)\ne z_k=(s_k,t_k)$ such that

$$\tilde{v}_{\tau_k}(z'_k)=\tilde{v}_{\tau_k}(z_k)\quad\forall k\,.$$

We may assume without loss of generality that the sequences $t_k$ and $t'_k$ converge, say to $t$ and $t'$ respectively. We note that none of the sequences $(s_k)$ or $(s'_k)$ can be bounded. Indeed, if $(s_k)$ were bounded we could assume that $s_k\to s$ after passing to some subsequence. The corresponding subsequence of $(s'_k)$ can not be unbounded since this would imply that either $p_-$ or $p_+$ lies in the range of $\tilde{u}_0$. If $(s'_k)$ were bounded then we would obtain $s'_k \to s'$ after passing to another subsequence and $s = s'$, $t = t'$ since $\tilde{u}_0$ is injective. This is also impossible because for every $R > 0$ there is some $\varepsilon > 0$ so that the map $\Phi$ as in (3.3) restricted to $[-R, R] \times [0, 1] \times B_\varepsilon(0) \subset S \times \mathbf{R}^2$ is injective. We have assumed $p_- \neq p_+$ which rules out the case $|s'_k|, |s_k| \to \infty$, $\operatorname{sign}(s'_k) \neq \operatorname{sign}(s_k)$ as well. The only remaining case to consider is where both sequences $(s_k)$ and $(s'_k)$ tend to infinity in absolute value and have the same sign. Hence assume that $s_k, s'_k \to +\infty$ and $s_k < s'_k$. Using the first and the last component of the asymptotic formula (4.21) we obtain

$$\sin\left(\frac{\pi t_k}{2}\right) = e^{\int_{s_k}^{s'_k} \alpha(\zeta)d\zeta}\left[\sin\left(\frac{\pi t'_k}{2}\right) + \varepsilon_1(s'_k, t'_k)\right] + \varepsilon_2(s_k, t_k) \tag{4.23}$$

and

$$\cos\left(\frac{\pi t_k}{2}\right) = e^{\int_{s_k}^{s'_k} \alpha(\zeta)d\zeta}\left[\cos\left(\frac{\pi t'_k}{2}\right) + \varepsilon_3(s'_k, t'_k)\right] + \varepsilon_4(s_k, t_k)\,, \tag{4.24}$$

where $\varepsilon_k(s, t)$ denote functions which converge uniformly to zero with all derivatives as $s \to \infty$. If the sequence $(s'_k - s_k)$ has an unbounded subsequence then we may assume that $s'_k - s_k \to \infty$ after passing to a suitable subsequence, so that

$$e^{\int_{s_k}^{s'_k} \alpha(\zeta)d\zeta} \longrightarrow 0\,.$$

We then obtain in the limit $\sin(\frac{\pi t}{2}) = 0 = \cos(\frac{\pi t}{2})$ which is impossible. Hence we may assume that the sequence $s'_k - s_k$ converges to some limit $C \geq 0$. If $C > 0$ then

$$e^{\int_{s_k}^{s'_k} \alpha(\zeta)d\zeta} \longrightarrow e^{-\frac{\pi}{2}C} < 1\,.$$

Then equations (4.23) and (4.24) imply

$$\sin\left(\frac{\pi t}{2}\right) = e^{-\frac{\pi}{2}C}\sin\left(\frac{\pi t'}{2}\right),$$
$$\cos\left(\frac{\pi t}{2}\right) = e^{-\frac{\pi}{2}C}\cos\left(\frac{\pi t'}{2}\right),$$

which is also impossible. If $s'_k - s_k \to 0$ then the above equations lead to $t = t'$. This means that the points $z_k$ and $z'_k$ come closer to each other,

$$0 \neq |z'_k - z_k| \longrightarrow 0\,.$$

Using Taylor's formula we get

$$D\tilde{v}_{\tau_k}(z_k)(z'_k - z_k) = -\left(\int_0^1 (1-\sigma)D^2\tilde{v}_{\tau_k}(z_k + \sigma(z'_k - z_k))d\sigma\right)(z'_k - z_k)^2\,. \tag{4.25}$$

We multiply this equation with

$$\frac{1}{|z_k' - z_k|} e^{-\int_{s_0}^{s_k} \alpha_{\tau_k}(\zeta)d\zeta} ,$$

with $\alpha_{\tau_k}$ as in the asymptotic formula (4.22). Recalling that $\alpha_{\tau_k} \to \alpha$ uniformly and consulting (4.22), we see that the right hand side of equation (4.25) converges to zero as $k \to \infty$ while the absolute value of the left hand side remains larger than some positive number. This contradiction concludes the proof that $\tilde{v}_\tau$ is embedded for sufficiently small $|\tau|$.

Let us now focus on the case where $\tilde{u}_0$ and $\tilde{v}_\tau$ converge at both ends at the rate $e^{-\frac{\pi}{2}|s|}$. In order to show $\tilde{v}_\tau(S) \cap \tilde{v}_{\tau'}(S) = \varnothing$ if $\tau \neq \tau'$ and $\tau, \tau'$ small, it suffices to show that $\tilde{v}_\tau(S) \cap \tilde{u}_0(S) = \varnothing$ for $0 \neq \tau$ small. This is because we can apply Theorem 3.10 to $\tilde{v}_{\tau'}$ instead of $\tilde{u}_0$. Because of (3.4) and (3.2) we have to show that the function

$$S \ni (s,t) \longmapsto \big(x_\tau(s,t) - c_-(\tau)\beta_-(s) + c_+(\tau)\beta_+(s), y_\tau(s,t)\big) \in \mathbf{C}$$

never vanishes whenever $|\tau|$ is sufficiently small. We have shown in the proof of theorem 3.10 that

$$(c_-(\tau), c_+(\tau), \psi(\tau)) = \tau x_0 + \phi(\tau x_0) ,$$

where $x_0 = (c_-, c_+, \eta) \in \ker DF(0,0,0)$ with $c_\pm \neq 0$ and $\phi$ is some map with $\phi(0) = 0$, $D\phi(0) = 0$ (see (4.20)). Hence we have to show that the function

$$\begin{aligned} \tilde{\eta} : S &\longrightarrow \mathbf{C} , \\ (s,t) &\longmapsto \eta(s,t) - c_-\beta_-(s) + c_+\beta_+(s) \end{aligned}$$

has no zeros. We will accomplish this by showing that $\tilde{\eta} \in \ker T_0(\bar{\gamma})$ since $\tilde{\eta} \not\equiv 0$ and $\ker T_0(\bar{\gamma})$ is one-dimensional generated by an element which has no zeros. The operator $T_0$ is given by (4.1), which is

$$T_0 : H_L^{2,p,\gamma}(S, \mathbf{C}) \longrightarrow H^{1,p,\gamma}(S, \mathbf{C}) ,$$

$$T_0(\eta)(z) := \partial_s \eta(z) + i\partial_t \eta(z) + \big(D_2 J_1^{(0,0)}(z,0)\eta(z)\big)\cdot(0,1) , \quad z = (s,t) \in S ,$$

and $\bar{\gamma}$ refers to a weight function as in (4.19). We will show that the expressions

$$\left.\frac{d}{d\tau}\right|_{\tau=0} J_1^{(\tau,0)}(z,0)(0,1) , \quad z = (s,t) ,$$

and

$$\left.\frac{d}{d\tau}\right|_{\tau=0} J_1^{(0,\tau)}(z,0)(0,1)$$

are equal to

$$\begin{aligned} -[T_0(\bar{\gamma})\beta_-](s,t) &= -\beta_-'(s) - \big(D_2 J_1^{(0,0)}(z,0)\beta_-(s)\big)\cdot(0,1) \\ &= -\beta_-'(s) + \left.\frac{d}{d\tau}\right|_{\tau=0} J_1^{(0,0)}(z, -\tau\beta_-(s))(0,1) \end{aligned}$$

and

$$\begin{aligned}[T_0(\bar\gamma)\beta_+](s,t) &= \beta_+'(s) + \big(D_2 J_1^{(0,0)}(z,0)\beta_+(s)\big)\cdot(0,1)\\ &= \beta_+'(s) + \frac{d}{d\tau}\Big|_{\tau=0} J_1^{(0,0)}(z,\tau\beta_+(s))(0,1)\,,\end{aligned}$$

respectively so that $(c_-,c_+,\eta) \in \ker DF(0,0,0)$ indeed implies $\eta - c_-\beta_- + c_+\beta_+ \in \ker T_0(\bar\gamma)$ by Proposition 3.9. Starting with

$$\Phi_{\tau_-,\tau_+}(z,0) = \Phi_{0,0}(z,\tau_+\beta_+(s) - \tau_-\beta_-(s))$$

and $(h,k) = (h_1 + ih_2, k_1 + ik_2) \in \mathbf{C}^2$ we obtain

$$\begin{aligned}&D\Phi_{\tau_-,\tau_+}(z,0)(h,k)\\ &= D_1\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))h + D_2\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))k\\ &\quad + (\tau_+\beta_+'(s) - \tau_-\beta_-'(s))D_2\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))h_1\\ &= D\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))\big(h, k + h_1(\tau_+\beta_+'(s) - \tau_-\beta_-'(s))\big)\,.\end{aligned}$$

Returning to (3.6) and lemma 3.7, we compute with

$$\pi_1,\pi_2 : \mathbf{C}^2 \to \mathbf{C}$$

being the projections onto the first and second factor, respectively,

$$\begin{aligned}&\tilde J(\Phi_{\tau_-,\tau_+}(z,0))D\Phi_{\tau_-,\tau_+}(z,0)(i,0)\\ &= D\Phi_{\tau_-,\tau_+}(z,0)\bar J_{\tau_-,\tau_+}(z,0)(i,0)\\ &= D\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))\big(\pi_1\bar J_{\tau_-,\tau_+}(z,0)(i,0),\\ &\qquad \pi_2\bar J_{\tau_-,\tau_+}(z,0)(i,0)\\ &\qquad + [\tau_+\beta_+'(s) - \tau_-\beta_-'(s)]\\ &\qquad \mathrm{Re}(\pi_1\bar J_{\tau_-,\tau_+}(z,0)(i,0))\big)\,.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}&\tilde J(\Phi_{\tau_-,\tau_+}(z,0))D\Phi_{\tau_-,\tau_+}(z,0)(i,0)\\ &= \tilde J\big(\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))\big)D\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))(i,0)\\ &= D\Phi_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))\bar J_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))(i,0)\,.\end{aligned}$$

A remark about the notation: It is $(i,0)\in\mathbf{C}^2$ and $(0,1)\approx i\in\mathbf{C}$ after identifying $\mathbf{C}$ with $\mathbf{R}^2$. We conclude

$$\begin{aligned}&\bar J_{0,0}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))(i,0)\\ &= \big(j_1^{(0,0)}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))(0,1),\ J_1^{(0,0)}(z,\tau_+\beta_+(s)-\tau_-\beta_-(s))(0,1)\big)\\ &= \big(*,\ J_1^{(\tau_-,\tau_+)}(z,0)(0,1) + j_2^{(\tau_-,\tau_+)}(z,0)\cdot 0\\ &\qquad + [\tau_+\beta_+'(s) - \tau_-\beta_-'(s)]\,\mathrm{Re}[j_1^{(\tau_-,\tau_+)}(z,0)(0,1)]\big)\,,\end{aligned}$$

and therefore

$$J_1^{(0,0)}(z, \tau_+\beta_+(s) - \tau_-\beta_-(s))(0,1) = \\ J_1^{(\tau_-,\tau_+)}(z,0)(0,1) + [\tau_+\beta_+'(s) - \tau_-\beta_-'(s)]\,\mathrm{Re}[j_1^{(\tau_-,\tau_+)}(z,0)(0,1)]\,.$$

We note that

$$\mathrm{Re}[j_1^{(0,0)}(z,0)(0,1)] = \mathrm{Re}\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right] = \mathrm{Re}(i^2) = -1\,,$$

so that

$$\left.\frac{d}{d\tau_\pm}\right|_{\tau_\pm=0} J_1^{(0,0)}(z, \tau_+\beta_+(s) - \tau_-\beta_-(s))(0,1) = \\ \mp\,\beta_\pm'(s) + \left.\frac{d}{d\tau_\pm}\right|_{\tau_\pm=0} J_1^{(\tau_-,\tau_+)}(z,0)(0,1)\,.$$

This is what we wanted to show, hence the proof of the theorem is complete. □

## Appendix A: The Similarity Principle

In this appendix, we review the similarity principle in its original version and also a version near boundary points. For $2 < p < \infty$ we denote by $V^p$ the Banach space consisting of all $u \in W^{1,p}(D, \mathbf{C}^n)$ satisfying $u(\partial D) \subset \mathbf{R}^n$, where $D \subset \mathbf{C}$ is the unit disk and let

$$\bar\partial : V^p \to L^p$$

be the standard Cauchy Riemann operator

$$u \longmapsto \frac{\partial u}{\partial s} + i\frac{\partial u}{\partial t}\,.$$

The operator $\bar\partial : V^p \to L^p(D, \mathbf{C}^n)$ is surjective and Fredholm of index $n$ with kernel being the constants in $\mathbf{R}^n$.

THEOREM A.1 *Assume* $A \in L^\infty(D, \mathcal{L}_{\mathbf{R}}(\mathbf{C}^n))$, $2 < p < \infty$ *and* $w \in W^{1,p}_{\mathrm{loc}}(\mathring{D}, \mathbf{C}^n)$. *Let* $w$ *be a solution of*

$$\bar\partial w + Aw = 0 \quad in\ \mathring{D}\,, \\ w(0) = 0\,.$$

*Then there exists*

$$\Phi \in \bigcap_{2<q<\infty} W^{1,q}(D, \mathcal{L}_{\mathbf{C}}(\mathbf{C}^n))$$

*with*

$$\Phi(0) = \mathrm{Id}\,, \quad \Phi(z) \in \mathrm{GL}(\mathbf{C}^n)\,,$$

*and a map* $f : D \to \mathbf{C}^n$ *with* $f(0) = 0$ *such that for* $z \in D$

$$w(z) = \Phi(z)f(z)\,,$$

*and for some* $0 < \varepsilon \le 1$, *a sufficiently small number and* $D_\varepsilon \subset \mathbf{C}$ *the disk of radius* $\varepsilon$, $f$ *is holomorphic on* $D_\varepsilon$.

PROOF: See [5] or [11]. □

Next we consider a boundary version of the similarity principle. Let

$$D^+ := \{z \in D \mid \mathrm{Im}(z) \geq 0\}\,.$$

THEOREM A.2 *Assume* $A \in L^\infty(D^+, \mathcal{L}_{\mathbf{R}}(\mathbf{C}^n))$ *and* $w \in W^{1,p}_{\mathrm{loc}}(D^+, \mathbf{C}^n)$, $2 < p < \infty$*, satisfying*

$$\bar{\partial} w + Aw = 0 \quad \textit{on} \;\; \overset{\circ}{D}{}^+\,,$$
$$w((-1,1)) \subset \mathbf{R}^n\,, \quad w(0) = 0\,.$$

*Then there exists* $\Phi \in \bigcap_{2<q<\infty} W^{1,q}(D^+, \mathcal{L}_{\mathbf{C}}(\mathbf{C}^n))$ *with*

$$\Phi(z) \in \mathrm{GL}(\mathbf{C}^n)\,, \quad \Phi(0) = \mathrm{Id}\,,$$
$$\Phi(z) \in \mathrm{GL}(\mathbf{R}^n) \subset \mathcal{L}(\mathbf{R}^n) \quad \textit{for } z \in (-1,1)\,,$$

*and a map* $f : D^+ \to \mathbf{C}^n$ *with*

$$f(z) \in \mathbf{R}^n \;\; \textit{for } z \in (-1,1)\,, \quad f(0) = 0\,,$$

*holomorphic on some smaller half-disk* $D^+_\varepsilon$*, such that*

$$w(z) = \Phi(z) f(z)\,.$$

PROOF: This result can be reduced to Theorem A.1. Extend $A$ to a map in $L^\infty(D, \mathcal{L}_{\mathbf{R}}(\mathbf{C}^n))$ by

$$A(z) = \overline{A(\bar{z})} \quad \text{if } \mathrm{Im}(z) < 0\,,$$

where "$\overline{\phantom{xx}}$" means replacing all coefficients by the complex conjugate ones. Extend $w$ similarly by

$$w(z) = \overline{w(\bar{z})}\,.$$

Then $w \in W^{1,p}(D, \mathbf{C}^n)$ as one verifies easily. Now apply Theorem A.1 and find

$$w(z) = \Phi(z) f(z),$$

where $z$ lies in some disk $D_\varepsilon$, and it turns out that

$$\Phi((-1,1)) \subset \mathcal{L}(\mathbf{R}^n)$$

and consequently

$$f((-\varepsilon, \varepsilon)) \subset \mathbf{R}^n\,.$$

□

*Remark.* There is also a parameterized version of the similarity principles: If $A_\tau$ is a continuous path in $L^\infty(D, \mathcal{L}_{\mathbf{R}}(\mathbf{C}^n))$ with $A_0 = 0$ and if $w_\tau$ is a continuous family of solutions of $\bar{\partial} w_\tau + A_\tau w_\tau = 0$ then $w_\tau = \Phi_\tau \sigma_\tau$ with $\Phi_\tau, \sigma_\tau \in C^0(D)$ depending continuously on $\tau$ as well. The maps $\Phi_\tau$ converge in $C^0(D)$ to the identity matrix. The important fact is that the path of operators $\Phi \mapsto (\bar{\partial}\Phi + A_\tau \Phi, \Phi(1))$ and the corresponding path of the inverses are continuous in $\tau$ with respect to the operator norm.

## Appendix B: Uniformization of $(S, i(z))$

Using theorem 3.10 we obtain a 1-parameter family of pseudoholomorphic curves

$$\tilde{v}_\tau = (b_\tau, v_\tau) : S \to \mathbf{R} \times M\,,$$
$$\tilde{v}_\tau(z) = \Phi_{c_-(\tau),c_+(\tau)}(z, \psi(\tau)(z))\,,$$

near a given embedded one $\tilde{u}_0$ with respect to complex structures $i_\tau(z)$ on $S$ given by (3.10), replace $w(z)$ there by $\psi(\tau)(z)$. We conclude from equation (3.10), Lemma 3.7, Lemma 3.8 and the decay behavior of $w(z)$ in (3.10) that $i(z) \to i$ whenever $|z| \to \infty$. We would like to show that there is a conformal isomorphism $\phi : (\overset{\circ}{S}, i(z)) \to (\overset{\circ}{S}, i)$ with $i$ being the standard complex structure on the strip $S$, which extends analytically to the boundary of $S$. The surface $(\overset{\circ}{S}, i(z))$ is a simply connected Riemannian surface. By the uniformization theorem [6, 8] those are conformally equivalent either to the open unit disk or the complex plane. Working in conformal coordinates, we see that the **R**-parts $b_\tau$ of $\tilde{v}_\tau$ are subharmonic and not constant. Arguing similarly as in Proposition 2.1 (the maximum principle for subharmonic functions also holds on arbitrary Riemann surfaces) we obtain that $b_\tau < 0$ in the interior of $S$. Simply connected Riemann surfaces which admit a non constant negative subharmonic function are called 'hyperbolic' and cannot be conformally equivalent to the whole complex plane. Hence for all complex structures $i_\tau(z)$ on $S$ obtained by our construction, the interior of $S$ is conformally isomorphic to the unit disk. Since $\partial S$ is an analytic curve the conformal isomorphism extends analytically to the boundary. This is essentially Schwarz reflection (see [12, theorem 2.5, chapter IX.2]).

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