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A Note on V. I. Arnold's Chord Conjecture

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1 Introduction

This paper makes a contribution to a conjecture of V. I. Arnold in contact geometry, which he stated in his 1986 paper [4]. A [4] on a closed, oriented three manifold M is a 1-form τ so that $\tau \wedge d\tau$ is a volume form. There is a distinguished vectorfield X_τ, called the Reeb vectorfield of τ, which is defined by $i_{X_\tau} d\tau \equiv 0$ and $i_{X_\tau}\tau \equiv 1$. The standard example on S^3 is the following: Consider the 1-form λ on $\mathbf{R}^4$ defined by

$$\lambda = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2).$$

This induces a contact form on the unit three sphere S^3. Observe that all the orbits of the Reeb vectorfield are periodic; they are the fibres of the Hopf fibration. Note that the dynamics of the Reeb vectorfield changes drastically in general if we replace λ by the contact form $f\lambda$ where f is a nowhere vanishing function on S^3 (see the example in [3]; see also [1], [2]). A *Legendrian knot* in a contact manifold (M, τ) is a knot which is everywhere tangent to $\ker \tau$. Then V. I. Arnold stated the following conjecture.

Conjecture ([4]). Let λ be the above contact form on the three sphere. If $f\colon S^3 \to (0, \infty)$ is a smooth function and $\mathcal{L}$ is a Legendrian knot in S^3, then there is a *characteristic chord* for $(f\lambda, \mathcal{L})$, i.e., a trajectory x of the Reeb vectorfield and some number $T \neq 0$ so that $x(0) \in \mathcal{L}$ and $x(T) \in \mathcal{L}$.

There is almost nothing known about this problem. V. I. Arnold only mentioned the case where $f \equiv 1$: Projecting the Legendrian knot onto S^2 using the Hopf fibration, we observe that the area enclosed by this curve is a multiple of 4π. Hence, it must have

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a self-intersection, which in turn shows that an orbit of the Reeb vectorfield intersects the Legendrian knot at two different times.

Definition 1.1. A three-dimensional submanifold of $\mathbf{R}^4$ is called a *strictly convex hypersurface in* $\mathbf{R}^4$ if it is closed and if it bounds a compact, strictly convex set containing the origin.

We provide an existence proof for characteristic chords in the case where the contact manifold is a strictly convex hypersurface $S \subset \mathbf{R}^4$ with contact form $\lambda|_S$, and where we confine ourselves to special Legendrian knots: We only consider Legendrian knots which are not linked with a certain periodic orbit of the Reeb vectorfield. Our result is based on the following crucial theorem by H. Hofer, K. Wysocki, and E. Zehnder [10], [8].

Theorem 1.2. Let S be a strictly convex hypersurface in $\mathbf{R}^4$ and let J be a complex structure on the symplectic vectorbundle $\ker\lambda \to S$ compatible with $d\lambda$; i.e., $d\lambda \circ (\mathrm{Id} \times J)$ should be a bundle metric on $\ker\lambda$ (such J always exist; see [3]). Then there exist: an unknotted periodic orbit P_0 of the Reeb vectorfield X_λ with self-linking number -1 and generalized Conley-Zehnder index 3 (*binding orbit*); and a diffeomorphism

$$\Phi\colon\ S^1 \times \mathbf{C} \longrightarrow S\backslash P_0$$

with the following properties.

For every $\tau \in S^1$, the map $u_\tau := \Phi(\tau, .)$ is an embedding and has the following properties:

- $u_\tau(\mathbf{C})$ is transversal to X_λ.
- $u_\tau(Re^{2\pi i t}) \to P_0(T_0 t)$ as $R \to +\infty$ with convergence in C^∞, where T_0 is the minimal period of P_0.
- The maps u_τ satisfy the following partial differential equation:

$$\pi_\lambda \partial_s u_\tau + J(u_\tau)\pi_\lambda \partial_t u_\tau = 0,\ \text{ where } \pi_\lambda\colon\ TS \to \ker\lambda \text{ is the projection along } \chi_\lambda,$$

$$d(u_\tau^* \lambda \circ i) = 0.$$

- $d\lambda|_{u_\tau(\mathbf{C})}$ is nondegenerate and $\int_{\mathbf{C}} u_\tau^* d\lambda = T_0 \in (0, \infty)$.
- Each $u_\tau(\mathbf{C})$ gives rise to a global surface of section; i.e. the closure of $u_\tau(\mathbf{C})$ is an embedded disk D_τ with $\partial D_\tau = P_0$, and for each point $p \in S\backslash P_0$, there are real numbers $t^-(p) < 0$ and $t^+(p) > 0$ so that

$$\varphi_{t^-(p)}(p),\ \varphi_{t^+(p)}(p) \in \overset{\circ}{D}_\tau = u_\tau(\mathbf{C}),$$

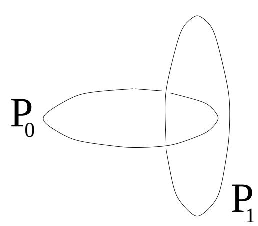
where φ denotes the flow of X_λ. □

H. Hofer, K. Wysocki, and E. Zehnder have actually shown that any periodic orbit P_0 of X_λ, which is unknotted with self-linking number -1 and generalized Conley-Zehnder index 3, is a binding orbit as described in Theorem 1.2 (see Theorem 1.6 in [9] and Section 7 of [10]). Our result is the following theorem.

Theorem 1.3. Let S be a strictly convex hypersurface in $\mathbf{R}^4$ equipped with the contact form $\lambda|_S$. Let $\mathcal{L} \subset S$ be a Legendrian knot with the following additional property: Assume there is an unknotted periodic orbit P_0 of X_λ which has self-linking number -1 and generalized Conley-Zehnder index 3, so that $\mathcal{L}$ and P_0 are not linked. Then there exists a characteristic chord for the knot $\mathcal{L}$. $\square$

We note that such an orbit P_0 always exists in view of Theorem 1.2, so there are many knots $\mathcal{L}$ for which the assumptions are satisfied. The obvious drawback is, of course, that the hypothesis is in general difficult to verify.

Remark. The assumption of P_0 having generalized Conley-Zehnder index 3 can be dropped since Theorem 1.2 also holds without this assumption (see [7]). In fact, it follows from Theorem 1.2 that there always exist at least two unknotted periodic orbits P_0 and P_1 which have self-linking number -1 and are linked with $\text{lk}(P_0, P_1) = 1$. In view of this remark, the following is true.



Theorem 1.4. Let S be a strictly convex hypersurface in $\mathbf{R}^4$ equipped with the contact form $\lambda|_S$. Let $\mathcal{L} \subset S$ be a Legendrian knot with the following additional property: Assume there is an unknotted periodic orbit P_0 of X_λ which has self-linking number -1, so that $\mathcal{L}$ and P_0 are not linked. Then there exists a characteristic chord for the knot $\mathcal{L}$. $\square$

The relation between Theorem 1.3 and V. I. Arnold's conjecture is the following: The projection along the rays issuing from the origin in $\mathbf{R}^4$ defines a diffeomorphism $\phi\colon S^3 \to S$ by $z \longmapsto h(z)z$ with a suitable smooth function $h\colon S^3 \to (0, \infty)$. Then $\phi^*\lambda|_S = h^2\lambda|_{S^3}$. Hence Theorem 1.3 gives an affirmative answer to V. I. Arnold's conjecture if we restrict the class of admissible Legendrian knots and the class of contact forms $f\lambda$.

2 Proof of Theorem 1.3

We assume, of course, that $P_0 \cap \mathcal{L} = \emptyset$ since we are done otherwise. Let us choose $\tau_0 \in S^1$ so that our given knot $\mathcal{L}$ and $u_{\tau_0}(\mathbf{C})$ intersect transversally.

We define the map

$$\Psi\colon\ Z = D \times \mathbf{R} \longrightarrow S \backslash P_0,$$

$$(x, y, z) \longmapsto \varphi_z(u_0(x, y)),$$

where φ denotes the flow of X_λ, D is the open unit disk in $\mathbf{R}^2$, and $u_0 = \Phi(\tau_0, \phi(.))$ with ϕ being the diffeomorphism $\phi\colon\ D \to \mathbf{C}$, $re^{i\varphi} \longmapsto (1 - r^2)^{-1} re^{i\varphi}$. Then $\mathcal{L}$ and $\Sigma := u_0(D)$ intersect transversally.

Denote by

$$T^{\pm}\colon\ \Sigma \longrightarrow \mathbf{R}^{\pm}$$

the positive (resp., negative) return time; i.e., $T^+(p)$ (resp., $T^-(p)$) is the smallest positive (resp., largest negative) time t where we have $\varphi_t(p) \in \Sigma$. Let us denote by $P_\Sigma\colon\ \Sigma \to \Sigma$ the Poincaré map; i.e., $P_\Sigma(p) = \varphi_{T^+(p)}(p)$. The inverse P_Σ^{-1} is given by $p \longmapsto \varphi_{T^-(p)}(p)$. We observe that

$$\inf_{p \in \Sigma} |T^{\pm}(p)| > 0 \tag{1}$$

and that the derivative of Ψ is an isomorphism at every point $(x, y, z) \in Z$. Assume now

$$\Psi(x, y, z) = \Psi(x', y'.z')$$

for some points $(x, y, z), (x', y', z') \in Z$. Then

$$\varphi_{z-z'}(u_0(x, y)) = u_0(x', y')$$

and there is some $k \in \mathbf{Z}$ so that

$$u_0(x', y') = P_\Sigma^k(u_0(x, y))$$

and

$$z' = z - \sum_{l=0}^{|k|-1} (T^{\pm} \circ P_\Sigma^{\pm l} \circ u_0)(x, y),$$

where $\pm = \operatorname{sign} k$ and $P_\Sigma^0 := \mathrm{Id}_\Sigma$. Using this and (1), we see that $\Psi\colon\ Z \to S \backslash P_0$ is the universal cover of $S \backslash P_0$. A straightforward calculation shows that the 1-form $\nu = \Psi^* \lambda$ on Z is given by

$$\nu = dz + u_0^* \lambda.$$

Moreover, it is a contact form with Reeb vectorfield

$$X_\nu(x,y,z) = \frac{\partial}{\partial z} = (0,0,1).$$

Let $\gamma\colon [0,1] \to S\backslash P_0$ be a parametrization of our Legendrian knot $\mathcal{L}$; i.e., $\gamma(0) = \gamma(1)$, $\gamma([0,1]) = \mathcal{L}$, and γ is an embedding. Take any lift $\tilde{\gamma}\colon [0,1] \to Z$ of γ. Note that $\tilde{\gamma}$ is a smoothly embedded curve since Ψ is a local diffeomorphism. Moreover, $\tilde{\gamma}$ is Legendrian with respect to the contact structure $\ker\nu$ on Z.

The homology groups $H_1(\mathcal{L},\mathbf{Z})$ and $H_1(S\backslash P_0,\mathbf{Z})$ are isomorphic to $\mathbf{Z}$; denote generators by $[\mathcal{L}]$ and $[S\backslash P_0]$. Then the inclusion $\mathcal{L}\hookrightarrow S\backslash P_0$ induces a homomorphism $i_*\colon H_1(\mathcal{L},\mathbf{Z}) \to H_1(S\backslash P_0,\mathbf{Z})$, so there exists $k \in \mathbf{Z}$ so that $i_*[\mathcal{L}] = k\,[S\backslash P_0]$. Then $\mathrm{lk}(\mathcal{L}, S\backslash P_0) = k$ is the linking number of $\mathcal{L}$ and P_0 in S. By our assumption that $\mathcal{L}$ and P_0 are not linked, we have $k = 0$.

If $\{\mathcal{L}\}$ is a generator of $\pi_1(\mathcal{L}) \approx \mathbf{Z}$, and if $i_\#\colon \pi_1(\mathcal{L}) \to \pi_1(S\backslash P_0)$ is the homomorphism induced by the inclusion i, then the following hold.

- $\mathcal{L}$ and P_0 are not linked if and only if $i_\#\{\mathcal{L}\} \in \pi_1(S\backslash P_0)$ is the trivial class.
- A lift $\tilde{\gamma}\colon [0,1] \to Z$ of γ is a closed curve if and only if $\mathcal{L}$ and P_0 are not linked.

Hence, any lift $\tilde{\gamma}\colon [0,1] \to Z = D \to \mathbf{R}$ of γ is a closed curve. Writing $\tilde{\gamma}(t) = (\zeta(t), z(t)) \in D \times \mathbf{R}$, we obtain

$$0 = \int_{S^1} \tilde{\gamma}^*\nu = \int_{S^1} \zeta^*(u_0^*\lambda).$$

We use the following lemma (Proposition 5.4. of [10]).

Lemma 2.1. Consider on D the 2-form $f(x,y)dx \wedge dy$ with $f > 0$ and integrable on D. Then there exists a diffeomorphism $\tau\colon D \to D$ satisfying

$$\tau^*(f\,dx \wedge dy) = c\,dx \wedge dy$$

for a suitable constant $c > 0$. □

Applying this to our situation, we get

$$\tau^* d(u_0^*\lambda) = c\,dx \wedge dy.$$

If $\overline{\zeta}\colon \overline{D} \to D$ is any smooth extension of $\zeta\colon S^1 \to D$, and if we write $\overline{\xi} := \tau^{-1} \circ \overline{\zeta}$, $\xi := \tau^{-1} \circ \zeta$, then

$$\begin{aligned} 0 &= \int_{S^1} \zeta^*(u_0^*\lambda) \\ &= \int_{\overline{D}} \overline{\zeta}^* d(u_0^*\lambda) \\ &= c\int_{\overline{D}} \overline{\xi}^*(dx \wedge dy) \\ &= \frac{c}{2}\int_{S^1} \xi^*(x\,dy - y\,dx). \end{aligned}$$

This means that the area of the set enclosed by the curve ξ equals zero, so ξ must have a self-intersection. This implies that ζ must also have a self-intersection; hence there is a characteristic chord for the Legendrian knot $\tilde{\gamma}$. Because $\tilde{\gamma}$ was a lift of the curve γ with respect to Ψ and because Ψ maps orbits of X_γ to orbits of X_λ, we also have obtained a characteristic chord for the knot $\mathcal{L}$. ■

References

[1] C. Abbas, *Asymptotic behaviour of pseudoholomorphic half-planes in symplectisations*, PhD thesis, Eidgenössische Technische Hochschule, Zürich, 1997.

[2] ———, *Finite energy surfaces and the chord problem*, Duke Math. J. **96** (1999), 241-316.

[3] C. Abbas and H. Hofer, *Holomorphic curves and global questions in contact geometry*, to appear in Birkhäuser.

[4] V. I. Arnold, *First steps in symplectic topology*, Russ. Math. Surv. **41** (1986), 1–21.

[5] Y. Eliashberg, to appear in IAS/Park City Math. Ser.

[6] Y. Eliashberg and H. Hofer, *Contact homology*, in preparation.

[7] H. Hofer, private communication.

[8] H. Hofer, K. Wysocki, and E. Zehnder, *A characterization of the tight three sphere*, Duke Math. J. **81** (1995), 159–226.

[9] ———, *A characterization of the tight three sphere, II*, to appear in Comm. Pure Appl. Math.

[10] ———, *The dynamics on a strictly convex energy surface in* $\mathbf{R}^4$, to appear in Ann. of Math., 1998.

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