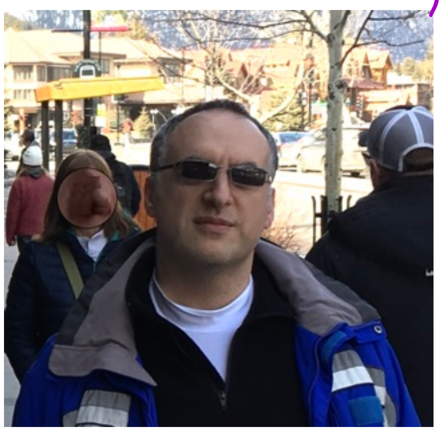
Plethysm and the algebra of uniform block permutations

joint work with Rosa Orellana, Franco Saliola, Anne Schilling



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Context: Algebras/groups

$B \subseteq A$

$$\mathrm{Res}^A_B V^\lambda_A \cong \bigoplus_\mu \left(V^\mu_B\right)^{\oplus a_{\lambda\mu}}$$

Exception to notation: $\mathfrak{S}^\lambda$

$$\begin{array}{|cccc} 9 & 12 & & \\ 6 & 10 & 11 & \\ 3 & 7 & 8 & \\ 1 & 2 & 4 & 5 \\ \hline \end{array} \in \mathfrak{S}^{(4332)}$$

$$\dim V_A^\lambda = \sum_\mu a_{\lambda\mu} \cdot \dim V_B^\mu$$

Goal find/compute $a_{\lambda\mu}$

Context: Algebras/groups

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Open problem:

Restriction problem

Find the decomposition of

$$\mathrm{Res}^{GL_n}_{S_n} V^{\lambda}_{GL_n} = \bigoplus_{\mu \vdash n} \left(S^{\mu} \right)^{\oplus r_{\lambda\mu}}$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 4 & 5 \\ \cline{1-2} 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline \end{array} \in V^{(6422)}_{GL_5}$$

$V^{\lambda}_{GL_n}$ = span semistandard Young tableaux shape λ in $\{1, 2, \ldots, n\}$

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Find the decomposition of

Restriction problem

$$\mathrm{Res}^{Gl_n}_{S_n} V^{\lambda}_{Gl_n} = \bigoplus_{\mu \vdash n} \left(\mathbb{S}^{\mu}\right)^{\oplus r_{\lambda\mu}}$$

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$V^{\lambda}_{Gl_n}$ = span semistandard Young tableaux shape λ in $\{1,2,\dots,n\}$

Example:

$$V^{(2)}_{Gl_4} \cong \mathcal{L}\left\{ \boxed{1\ 1}\,.\ \boxed{1\ 2},\ \boxed{1\ 3},\ \boxed{1\ 4},\ \boxed{2\ 2},\ \boxed{2\ 3},\ \boxed{2\ 4},\ \boxed{3\ 3},\ \boxed{3\ 4},\ \boxed{4\ 4} \right\}$$

$$\cong \left(\mathbb{S}^{(4)}\right)^{\oplus 2} \oplus \left(\mathbb{S}^{(31)}\right)^{\oplus 2} \oplus \mathbb{S}^{(22)}$$

$$\boxed{1\ 2\ 3\ 4}$$

$$\boxed{\begin{array}{lll} 4 & & \\ 1 & 2 & 3 \end{array}} \quad \boxed{\begin{array}{lll} 3 & & \\ 1 & 2 & 4 \end{array}} \quad \boxed{\begin{array}{lll} 2 & & \\ 1 & 3 & 4 \end{array}}$$

$$\boxed{\begin{array}{ll} 3 & 4 \\ 1 & 2 \end{array}} \quad \boxed{\begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array}}$$

symmetric function approach (from 1900's)

Characters chararaterize!

$A \in Gl_n$ has eigenvalues $x_1, x_2, \ldots, x_n$

$$A \sim \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & x_n \end{bmatrix}$$

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$$S_n \subseteq Gl_n$$

$$\mathrm{Char}_A(V^{\lambda}_{Gl_n}) = S_\lambda(x_1, x_2, \ldots, x_n) = \sum_T \mathrm{wt}(T)$$

character of permutation matrix: s_λ(eigenvalues of permutation matrix)

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Frobenius image: general technique for obtaining decomposition from S_n-character

$$\mathcal{F}_{S_n}(V) = \sum_{\mu \vdash n} \text{char } V(\mu) \frac{p_\mu}{z_\mu} \xrightarrow[\text{expansion}]{\text{Schur}} \text{decomposition}$$

symmetric function approach (from 1900's)

Characters chararaterize!

problem: changing bases + evaluating Schur functions at roots of unity is slow/hard

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$$\mathcal{F}_{S_4}\left(\text{Res}^{Gl_n}_{S_n} V^\lambda_{Gl_n}\right) = s_2(1,1,1,1)\frac{P_{1^4}}{24} + S_2(1,-1,1,1)\frac{P_{211}}{4} + S_2(1,-1,1,-1)\frac{P_{22}}{8} + S_2(1,\zeta,\zeta^2,1)\frac{P_{31}}{3} + s_2(1,-1,i,-i)\frac{P_4}{4}$$

$$= 2s_4 + 2s_{31} + s_{22}$$

Approach Littlewood 50's reformulated Scharf-Thibon 90's:

$$\mathrm{Res}^{Gl_n}_{S_n} V^{\lambda}_{Gl_n} = \bigoplus_{\mu \vdash n} \left(S^{\mu}\right)^{\oplus r_{\lambda\mu}}$$

Theorem:

$$r_{\lambda\mu} = \text{coefficient } s_\lambda(x_1, x_2, \ldots\ldots) \text{ in } s_\mu(1, x_1, x_2, \ldots, x_1^2, x_1x_2, x_2^2, \ldots, x_1^3, x_1^2x_2, x_1x_2x_3, x_1x_2^2, x_2^3, \ldots, \ldots)$$

$$= \langle s_\lambda, s_\mu[1 + s_1 + s_2 + s_3 + \cdots] \rangle$$

$f[g]$ is the operation of plethysm

g = char of Gl_n rep $\phi: Gl_n \to Gl_m$

f = char of Gl_m rep $\psi: Gl_m \to Gl_d$

$f[g]$ = char of $\psi \circ \phi$

problem: computing plethysm is slow/hard

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problem: computing plethysm is slow/hard

Example: $\langle s_2, s_4[1 + s_1 + s_2 + s_3 + s_4 + \cdots] \rangle = 2$

$$s_k[X_1 + X_2 + X_3 + \cdots] = \sum_{\alpha \vDash_w k} s_{\alpha_1}[X_1]\, s_{\alpha_2}[X_2]\, s_{\alpha_3}[X_3] \cdots$$

Approach 70's (Butler & King):

$$\begin{tikzcd} Gl_n \arrow[d, dash, "\cup|"] \\ O_n \arrow[d, dash, "\cup|"] \\ S_n \end{tikzcd}$$

$$\mathrm{Res}^{Gl_n}_{O_n} V^{(2)}_{Gl_n} \cong V^{(1)}_{O_n} \oplus V^{(2)}_{O_n}$$

General formula from Weyl character formula

$$\mathrm{Res}^{Gl_n}_{O_n} V^{\lambda}_{Gl_n} = \bigoplus_{\mu} \left(V^{\mu}_{O_n}\right)^{\oplus d_{\lambda\mu}}$$

$$d_{\lambda\mu} = \sum_{\gamma \text{ even}} c^{\lambda}_{\gamma\mu} \qquad c^{\lambda}_{\gamma\mu} = \langle s_\lambda, s_\gamma s_\mu \rangle$$

Littlewood-Richardson rule = # of tableaux satisfying "lattice" condition

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Littlewood-Richardson rule = # of tableaux satisfying "lattice" condition

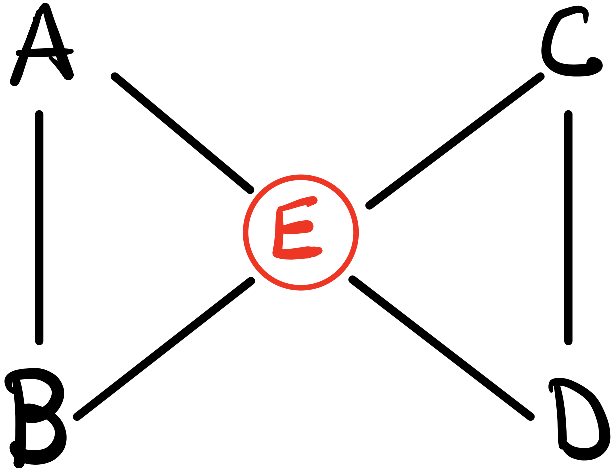
problem: no clear way of decomposing $\mathrm{Res}^{O_n}_{S_n} V^{\lambda}_{O_n}$ into symmetric group irreducibles

Example:

$$\mathrm{Res}^{O_n}_{S_n} V^{()}_{O_n} \cong \mathbb{S}^{(n)}$$

$$\mathrm{Res}^{O_n}_{S_n} V^{(2)}_{O_n} \cong \mathbb{S}^{(n)} \oplus \left(\mathbb{S}^{(n-1,1)}\right)^{\oplus 2} \oplus \mathbb{S}^{(n-2,2)}$$

See-saw pairs general setup:



A, B, C, D algebras that act on $\textcircled{E}$

if $C \cong \mathrm{End}_B(\textcircled{E})$

$D \cong \mathrm{End}_A(\textcircled{E})$

$$\mathrm{Res}^A_B V^\lambda_A \cong \bigoplus_\mu \left(V^\mu_B\right)^{\oplus a_{\lambda\mu}}$$

$$\mathrm{Res}^C_D V^\lambda_C \cong \bigoplus_\lambda \left(V^\mu_D\right)^{\oplus a_{\mu\lambda}}$$

Schur-Weyl duality

early 1900's

$$V_n = \text{span}\{v_1, v_2, \dots, v_n\}$$

$$A \in Gl_n \quad A(v_i) = \sum_j a_{ij} v_j$$

$V_n^{\otimes k}$ span

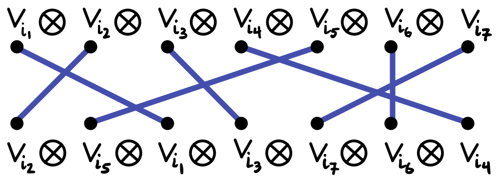
$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$

Gl_n —— $V_n^{\otimes k}$ —— S_k

semi-standard tableaux

words

standard tableaux



$$S_k \cong \text{End}_{Gl_n}(V_n^{\otimes k})$$

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$$\begin{tikzcd} Gl_n \arrow[r, dash] & V_n^{\otimes k} \arrow[r, dash] & S_k \end{tikzcd}$$

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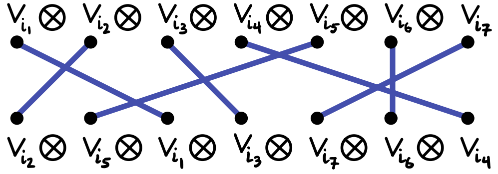
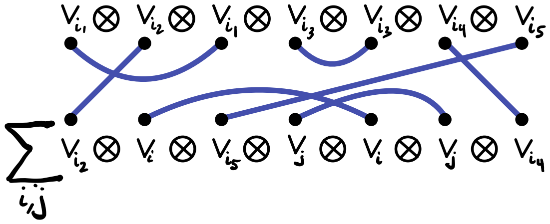


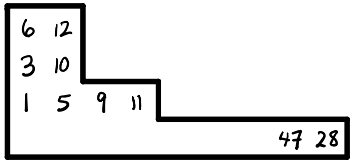
$$S_k \cong \text{End}_{Gl_n}(V_n^{\otimes k})$$

$$\begin{tikzcd} O_n \arrow[r, dash] & V_n^{\otimes k} \arrow[r, dash] & B_k(n) \end{tikzcd}$$

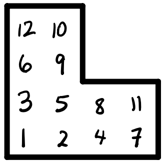
Brauer 1937



$$\begin{tikzcd}
Gl_n \arrow[dd, dash] \arrow[dr, dash] & & B_k(n) \arrow[dd, dash] \arrow[dl, dash] \\
& V_n^{\otimes k} & \\
O_n \arrow[ur, dash] & & S_k \arrow[ul, dash]
\end{tikzcd}$$



$\in V^{(4,2,2)}_{B_{12}(n)}$



$\in S^{(4,4,2,2)}$

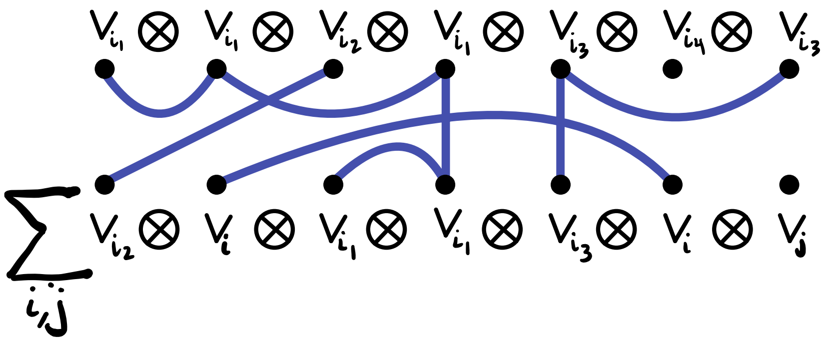
$$\mathrm{Res}^{B_k(n)}_{S_k} V^{\lambda}_{B_k(n)} \cong \bigoplus_{\mu} \left(S^{\mu}\right)^{\oplus d_{\mu\lambda}}$$

$$d_{\mu\lambda} = \sum_{\gamma \text{ even}} c^{\mu}_{\gamma\lambda} \qquad c^{\lambda}_{\gamma\mu} = \langle s_{\mu}, s_{\gamma} s_{\lambda} \rangle$$

"Schur-Weyl" duality in the early 1990's

Martin
Jones

$$S_n \text{ —— } V_n^{\otimes k} \text{ —— } P_k(n)$$



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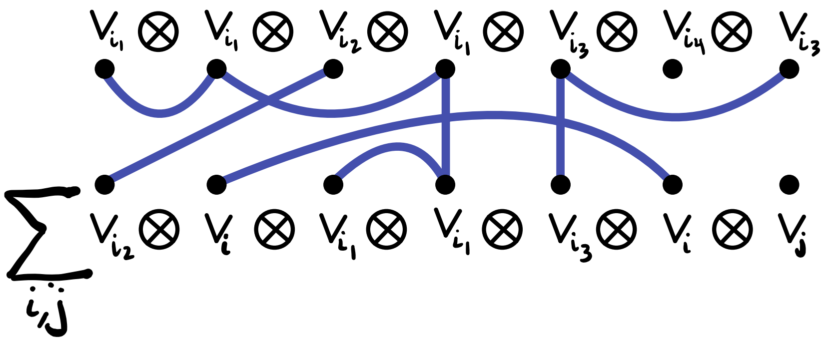
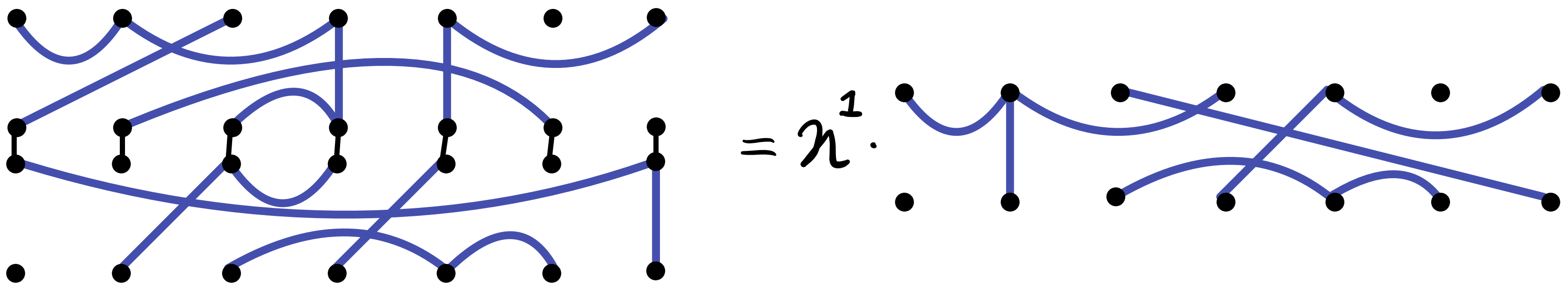
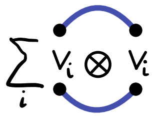
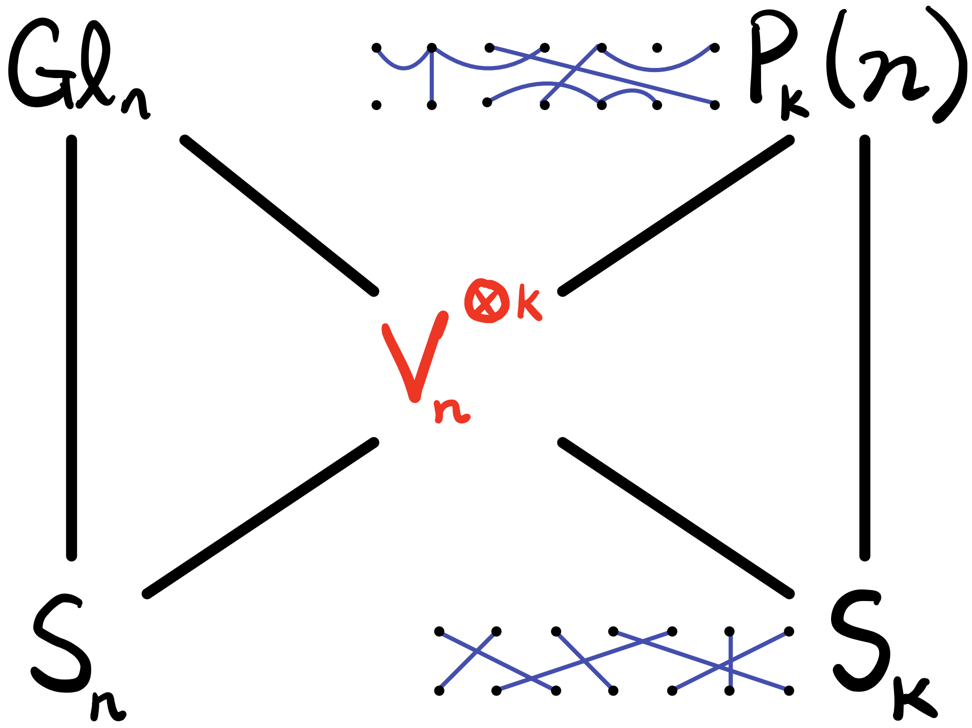




Note: closed loops
in composition contribute
a factor of n





$\in V_{P_q(n)}^{(n-3,2,1)}$

$$\mathrm{Res}^{P_k(n)}_{S_k} V^{\lambda}_{P_k(n)} = \bigoplus_{\mu \vdash k} \left(\mathbb{S}^{\mu}\right)^{\oplus r_{\mu\lambda}}$$

$$r_{\mu\lambda} = \langle s_\mu , s_\lambda[1+s_1+s_2+s_3+\cdots]\rangle$$

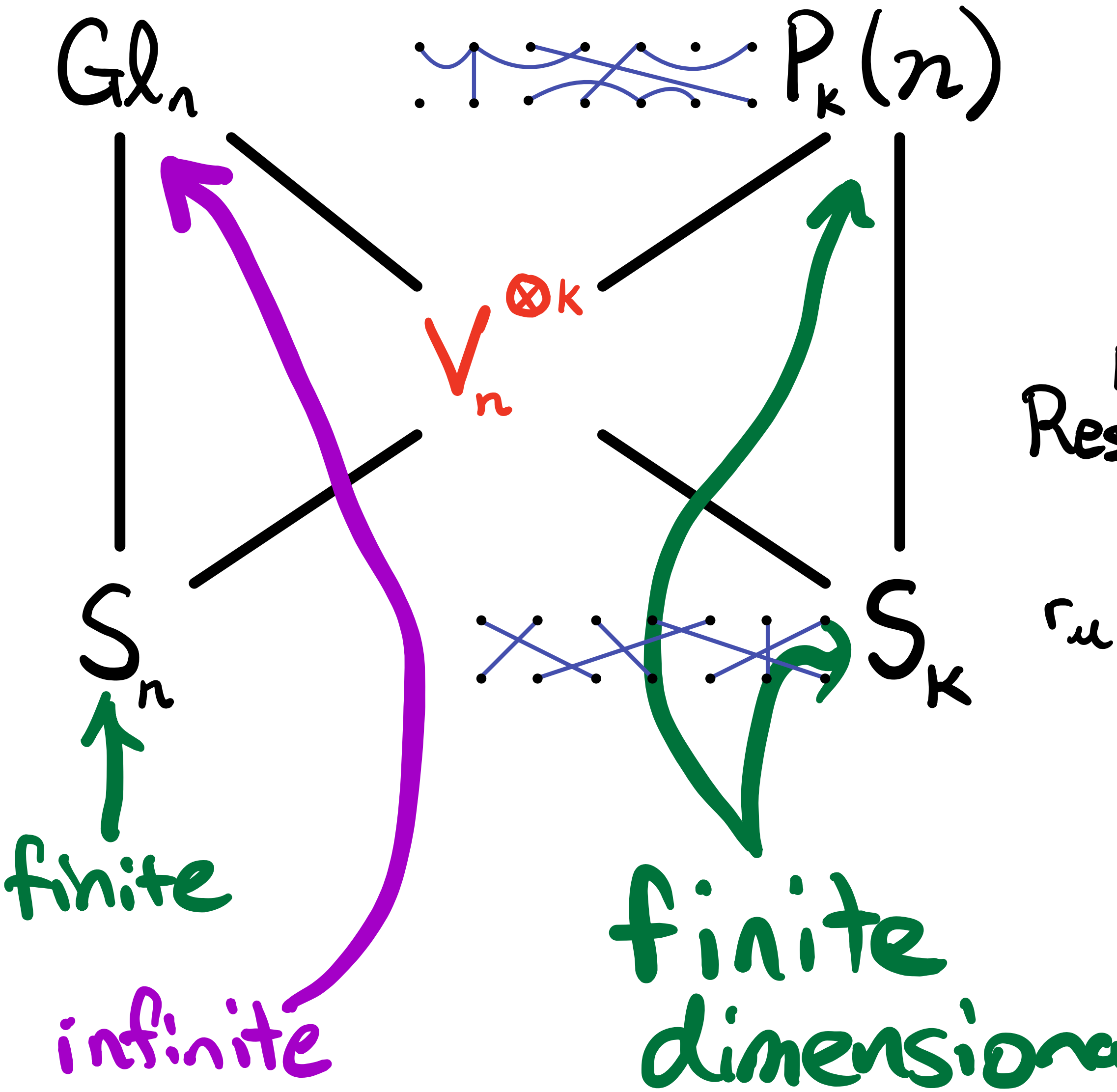
$P_k(n)$ has non-propagating blocks

S_k is a group with monoid product

$$\dim P_k(n) = B_{2k} = \#\text{ set partitions of } [k]\cup[\bar{k}]$$

Bell number

0	1	2	3	4	$\cdots$
1	2	15	877	21147	$\cdots$



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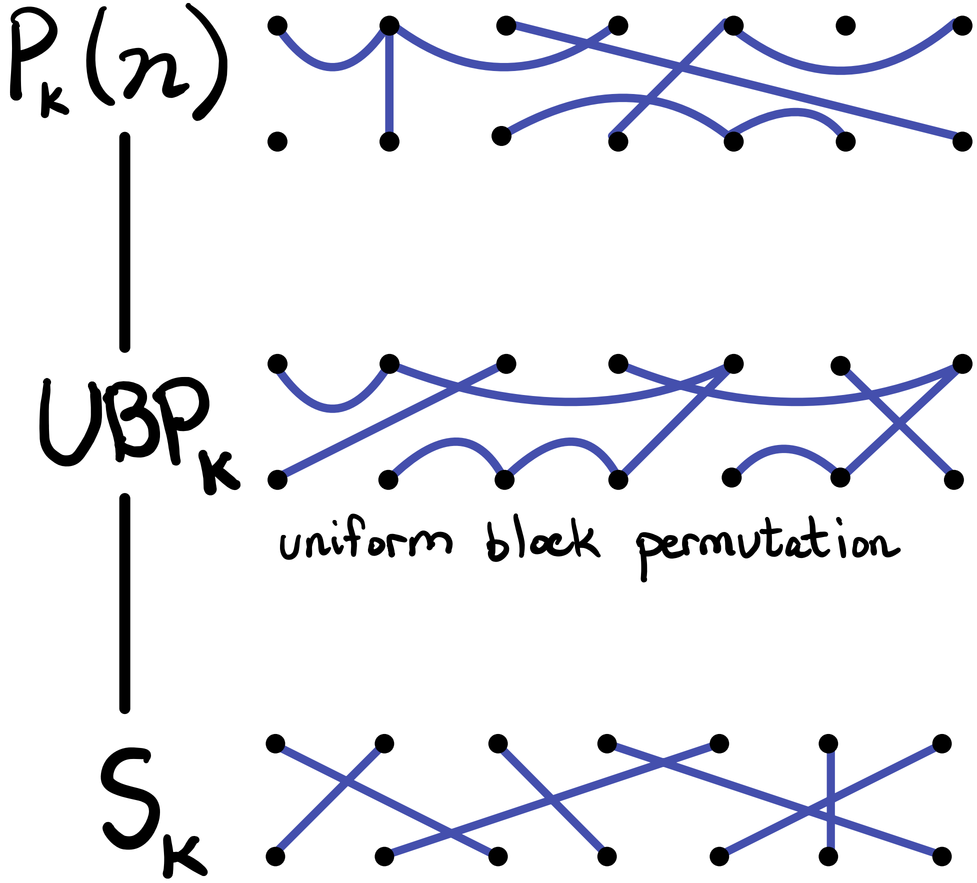


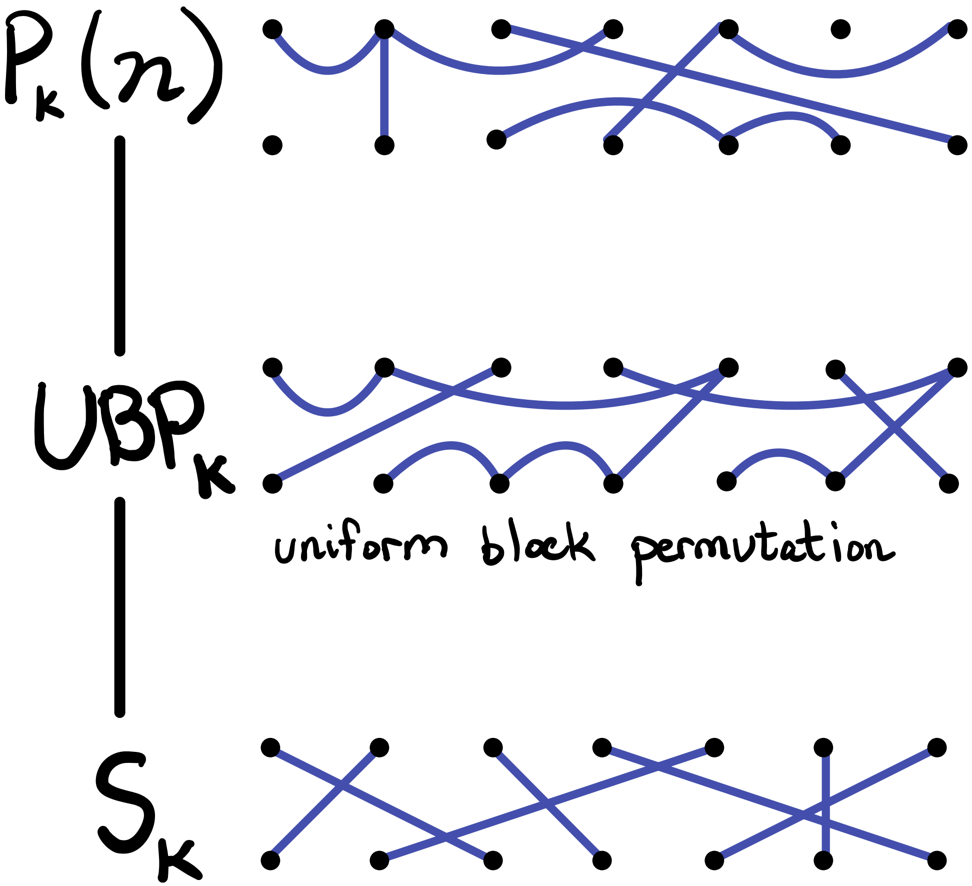
uniform block permutation

	0	1	2	3	4	5	6
$\dim P_k(n) =$	1	2	15	877	21147	678570	27644437
$\dim UBP_k =$	1	1	3	16	131	1496	22482

Summary of results from OSSZ'22

- UBP_k is a monoid (factorizable inverse semigroup) – no parameter n, with S_k maximal subgroup

$$\left(\begin{array}{|c|c|}\cline{1-1} 14 & \multicolumn{1}{c}{} \\ \hline 6 & 10 \\ \hline \end{array}, \begin{array}{|c|}\hline 9,15 \\ \hline 2,13 \\ \hline \end{array}, \phi, \begin{array}{|c|c|}\hline 1357 & 48,11,12 \\ \hline \end{array} \right) \in V_{UBP_{15}}^{(21,11,\bullet,2)}$$



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- irreducibles are indexed by sequences of partitions
  $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ $k = |\lambda^{(1)}| + 2|\lambda^{(2)}| + \dots + k|\lambda^{(k)}|$

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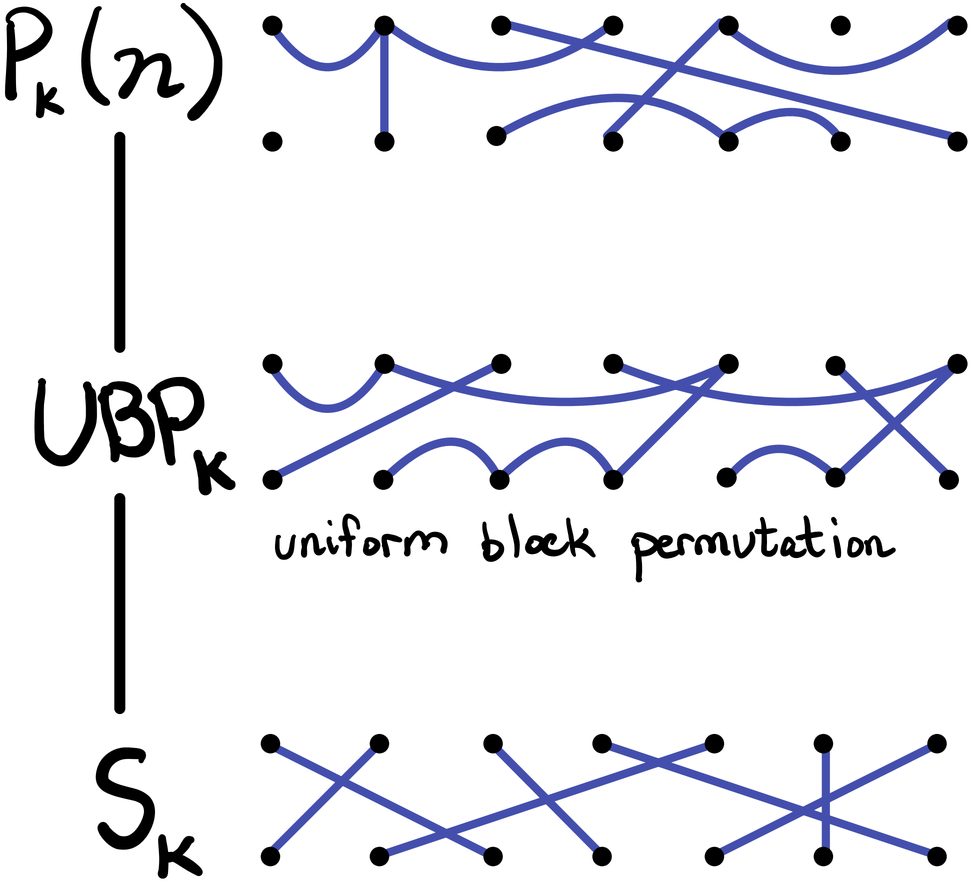

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- dimension of irreducibles is equal to number of tuples of set valued tableaux

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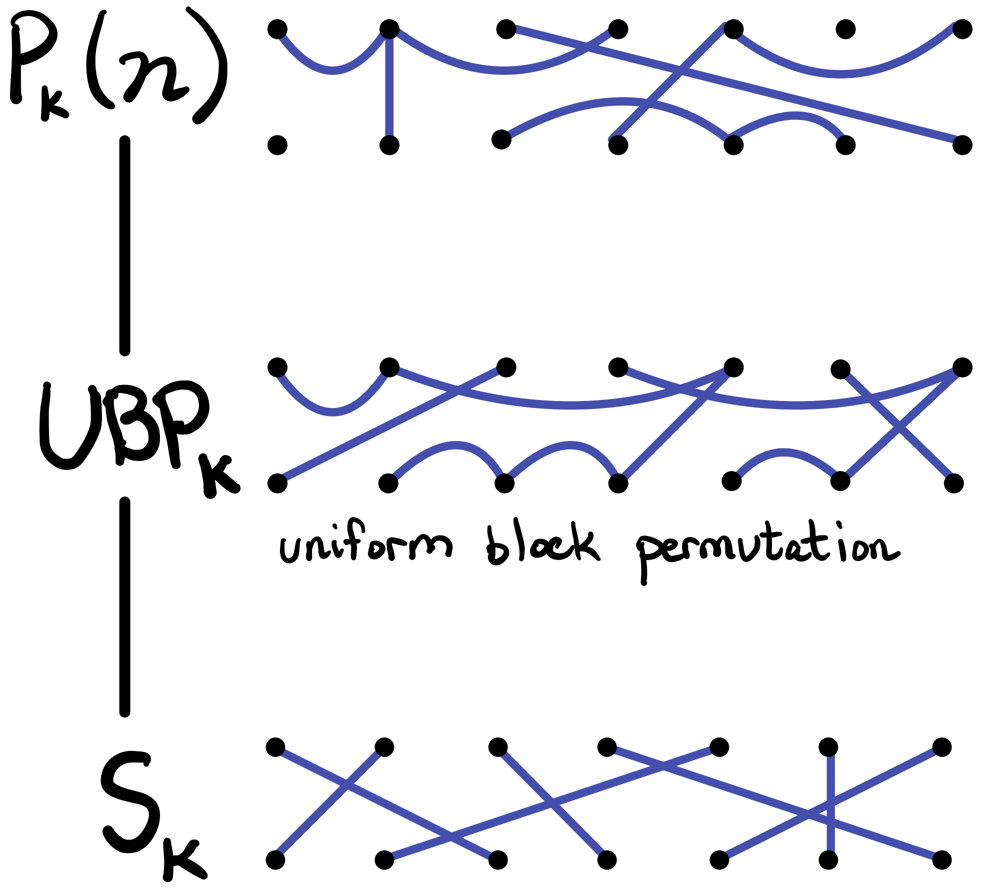

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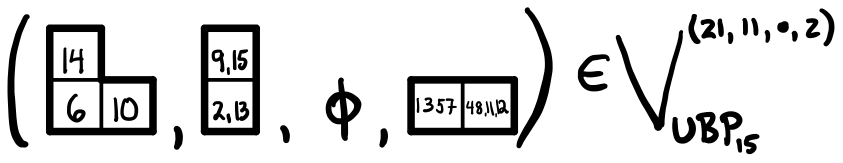

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- UBP_k is a monoid – no parameter n, with S_k maximal subgroup (factorizable inverse semigroup)
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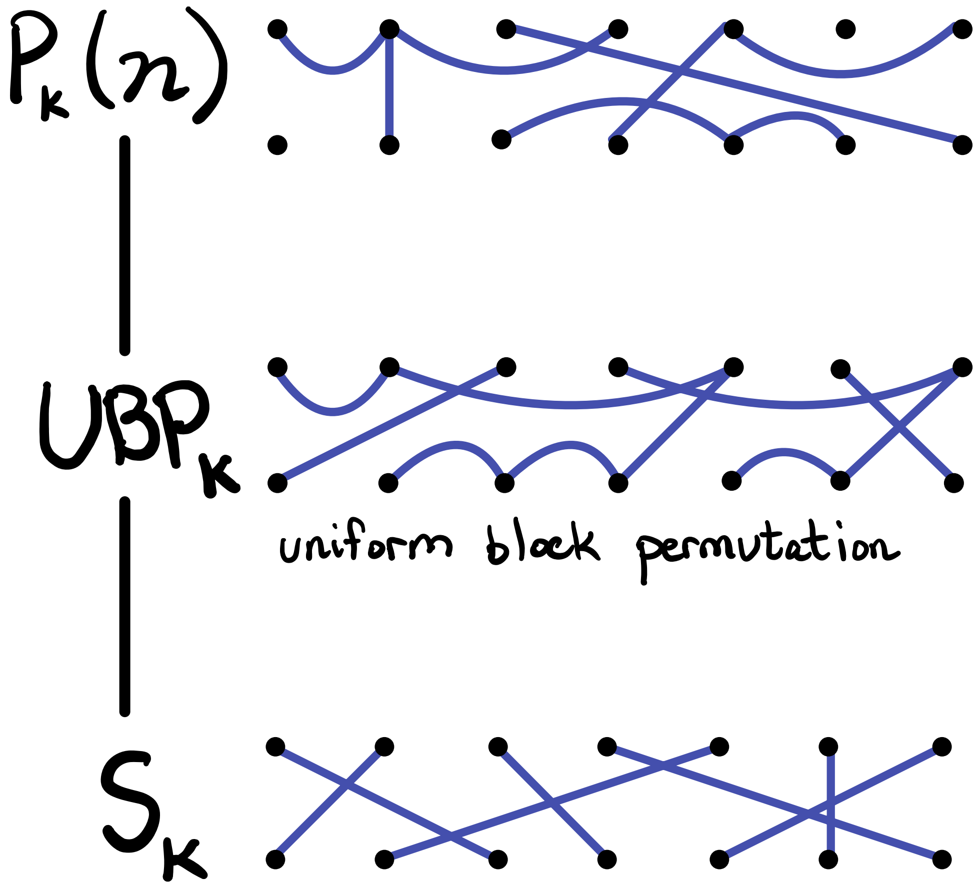
- dimension of irreducibles is equal to number of tuples of set valued tableaux
- explicit formula for the characters in terms of symmetric functions in multiple sets of variables



	0	1	2	3	4	5	6
$\dim P_k(n) =$	1	2	15	877	21147	678570	27644437
$\dim UBP_k =$	1	1	3	16	131	1496	22482

$$\left(\begin{array}{|c|c} \hline 14 & \multicolumn{1}{|c}{} \\ \hline 6 & \multicolumn{1}{|c|}{10} \\ \hline \end{array}, \begin{array}{|c|} \hline 9,15 \\ \hline 2,13 \\ \hline \end{array}, \phi, \begin{array}{|c|c|} \hline 1357 & 48,11,12 \\ \hline \end{array} \right) \in V^{(21,11,\circ,2)}_{UBP_{15}}$$

Summary of results from OSSZ'22

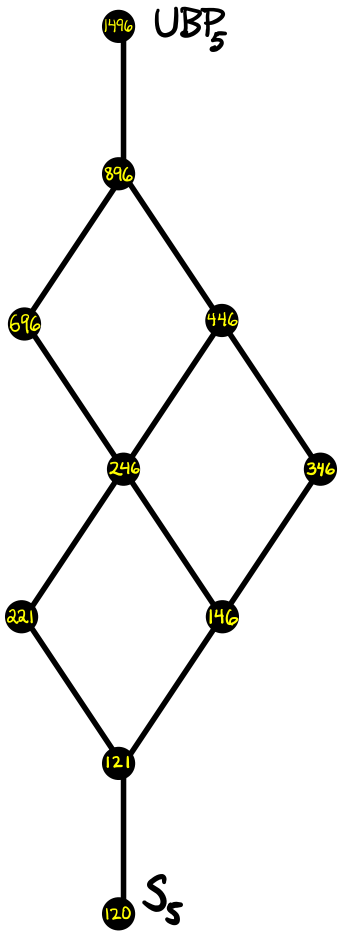
- UBP_k is a monoid – no parameter n, with S_k maximal subgroup
  (factorizable inverse semigroup)
- irreducibles are indexed by sequences of partitions
  $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ $\quad k = |\lambda^{(1)}| + 2|\lambda^{(2)}| + \cdots + k|\lambda^{(k)}|$
- dimension of irreducibles is equal to number of tuples of set valued tableaux
- explicit formula for the characters in terms of symmetric functions in multiple sets of variables
- $$\mathrm{Res}^{UBP_k}_{S_k} V^{\vec{\lambda}}_{UBP_k} \cong \bigoplus_{\mu \vdash k} \left(S^{\mu}\right)^{\oplus b_{\vec{\lambda}\mu}}$$

$$b_{\vec{\lambda}\mu} = \left\langle s_\mu, s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \cdots s_{\lambda^{(k)}}[s_k] \right\rangle$$

Restrict from UBP_k to S_k? Don't know, but ...

$J_\mu = \{$ set partions with sizes of blocks $= \mu \}$ μ partition k

Theorem: Let $\mathcal{A}_k = \{ M \text{ monoid} : S_k \subseteq M \subseteq UBP_k \}$

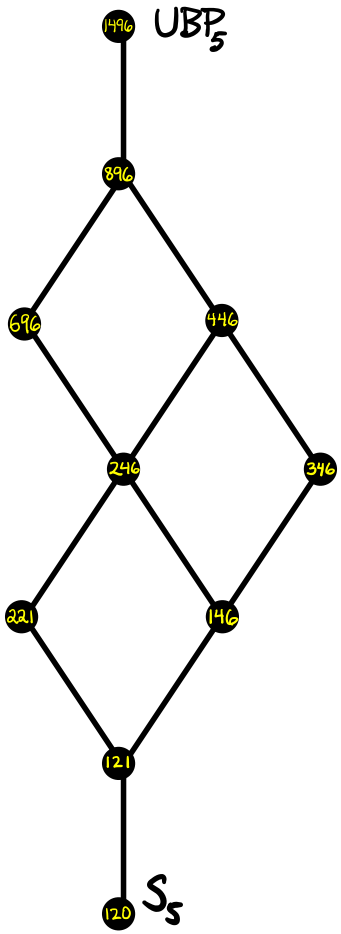


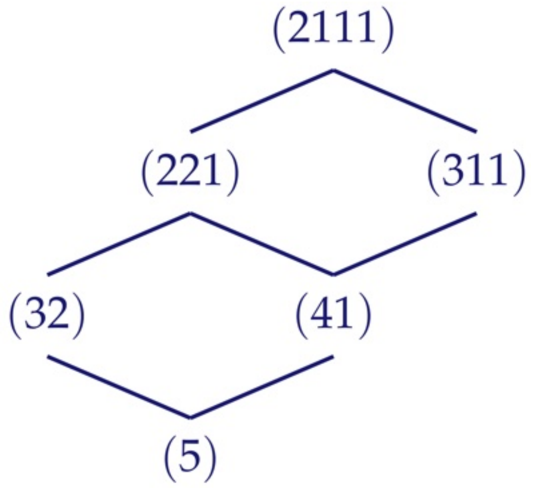
Restrict from UBP_k to S_k? Don't know, but ...

$J_\mu = \{$ set partions with sizes of blocks $= \mu \}$ μ partition k

Theorem: Let $\mathcal{A}_k = \{ M \text{ monoid} : S_k \subseteq M \subseteq UBP_k \}$

- ● $(\mathcal{A}_k, \cup, \cap)$ is a distributive lattice





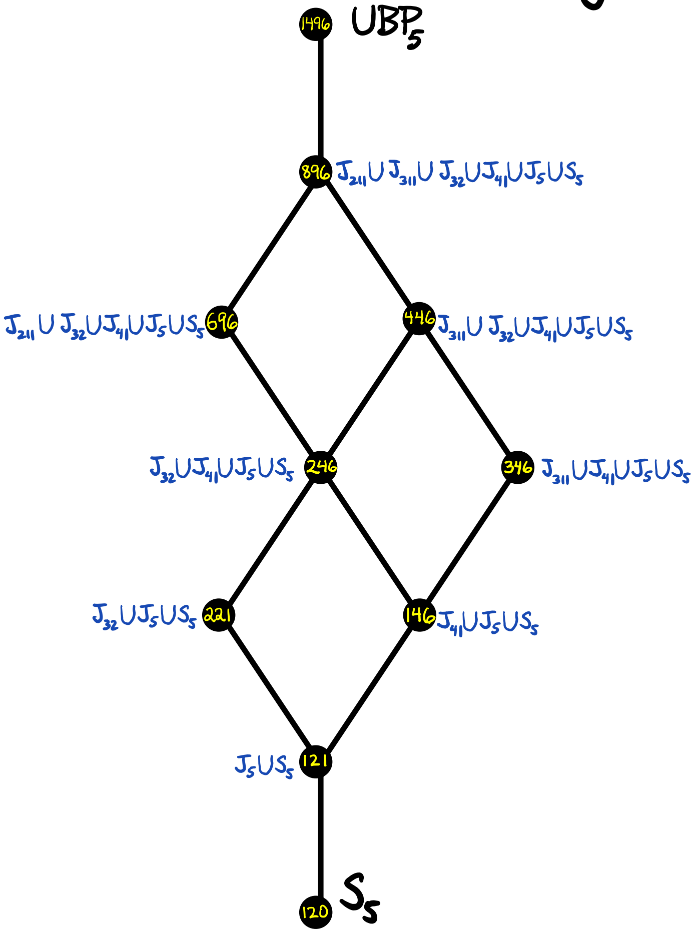
Restrict from UBP_k to S_k? Don't know, but ...

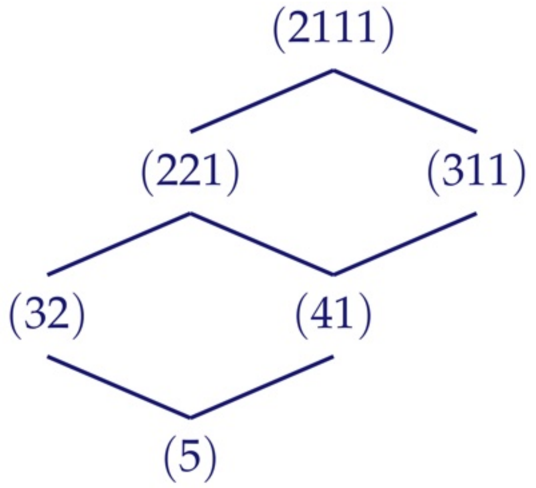
$J_\mu = \{$ set partions with sizes of blocks $= \mu \}$ $\quad$ μ partition k

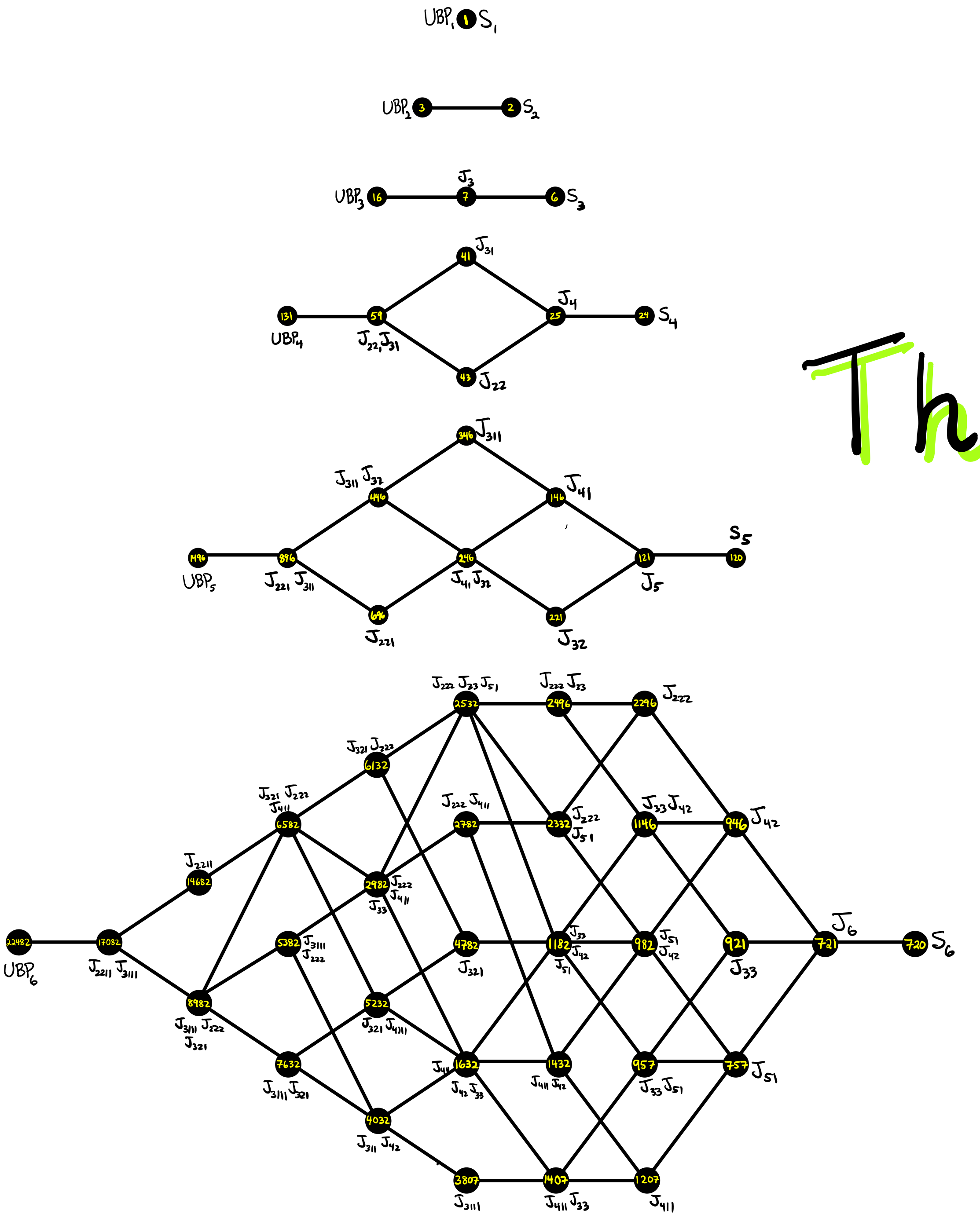
Theorem: Let $\mathcal{A}_k = \{ M \text{ monoid} : S_k \subseteq M \subseteq UBP_k \}$

- $(\mathcal{A}_k, \cup, \cap)$ is a distributive lattice
- Every $M \in \mathcal{A}_k$ is $M = S_k \bigcup_{\mu \in S} J_\mu$ where S order ideal of $(Par_k \setminus \{(1^k)\}, \leq_*)$

with $\lambda \leq_* \mu$ if λ is finer than μ
and $q_\lambda \geq q_\mu$ $\quad$ $q_\lambda :=$ smallest part $\lambda \neq 1$






UBP1 1 S1
UBP2 3 2 S2
UBP3 16 7 J3 6 S3
131 UBP4
59 J22 J31
41 J31
43 J22
25 J4
24 S4
1496 UBP5
896 J221 J311
446 J311 J32
696 J221
346 J311
246 J41 J32
146 J41
221 J32
121 J5
120 S5
22482 UBP6
17082 J2211 J3111
14682 J2211
8982 J3111 J222 J321
6582 J321 J222 J411
5382 J3111 J222
7632 J3111 J321
6132 J321 J222
2982 J222 J411 J33
5232 J321 J4111
4032 J311 J42
2532 J222 J33 J51
2782 J222 J411
4782 J321
1632 J411 J42 J33
3807 J3111
2496 J222 J33
2332 J222 J51
1182 J33 J42 J51
1432 J411 J42
1407 J411 J33
2296 J222
1146 J33 J42
982 J51 J42
957 J33 J51
1207 J411
946 J42
921 J33
757 J51
721 J6
720 S6

Thank you!