A Diagram-Like Basis for the Multiset Partition Algebra

(Part of my thesis work under the supervision of Rosa Orellana)

MSU Combinatorics and Graph Theory seminar

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I. Products of Diagrams

II. Motivation and Dualities

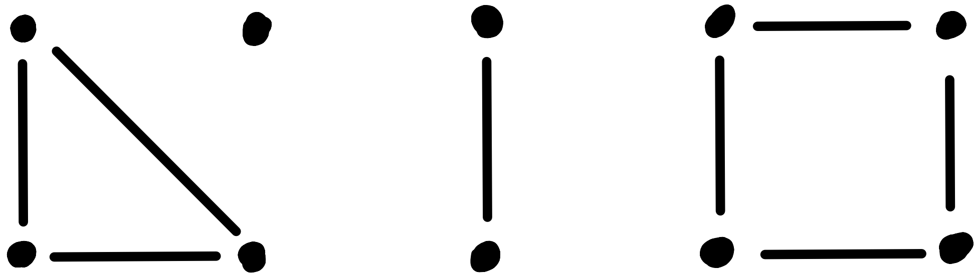
III. The Partition Algebra and the Diagram Basis

IV. The Multiset partition Algebra and a Diagram-Like Basis

V. Representations, Subalgebras, and Generators

A Monoid Structure on Diagrams

An example of what we'll call a <u>partition diagram</u>:



<u>Key features</u>:

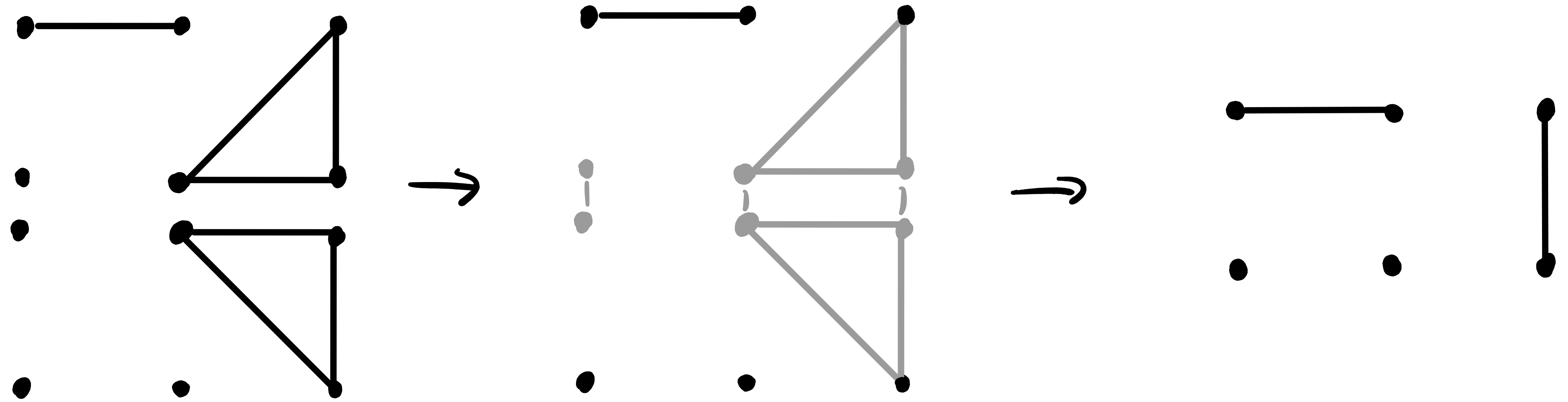
- Has r labeled vertices on top and bottom for some $r > 0$
- The vertices are grouped into connected components by edges

A Monoid Structure on Diagrams

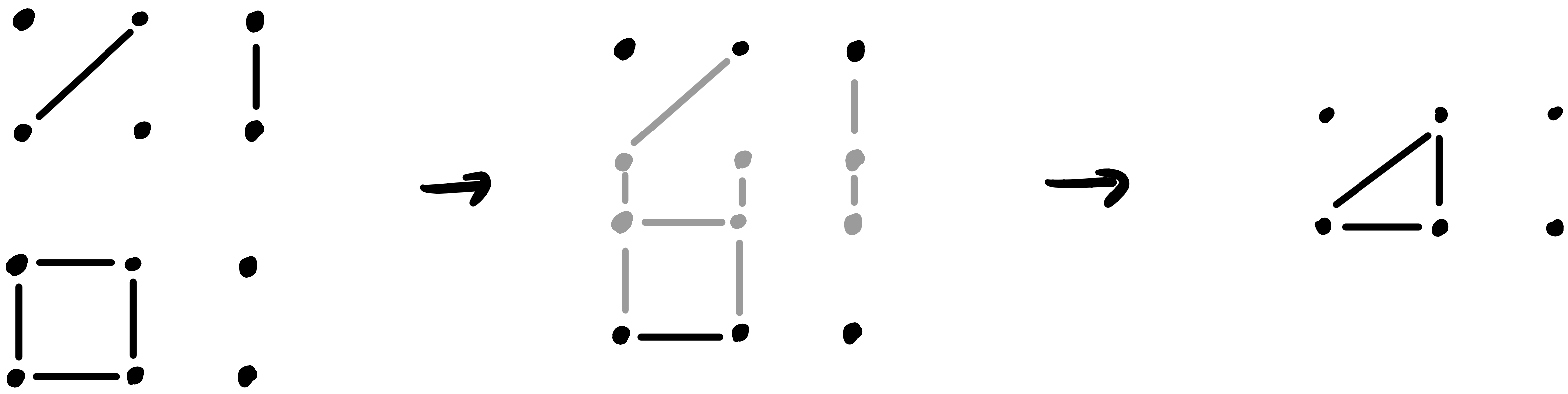
A multiplication formula:

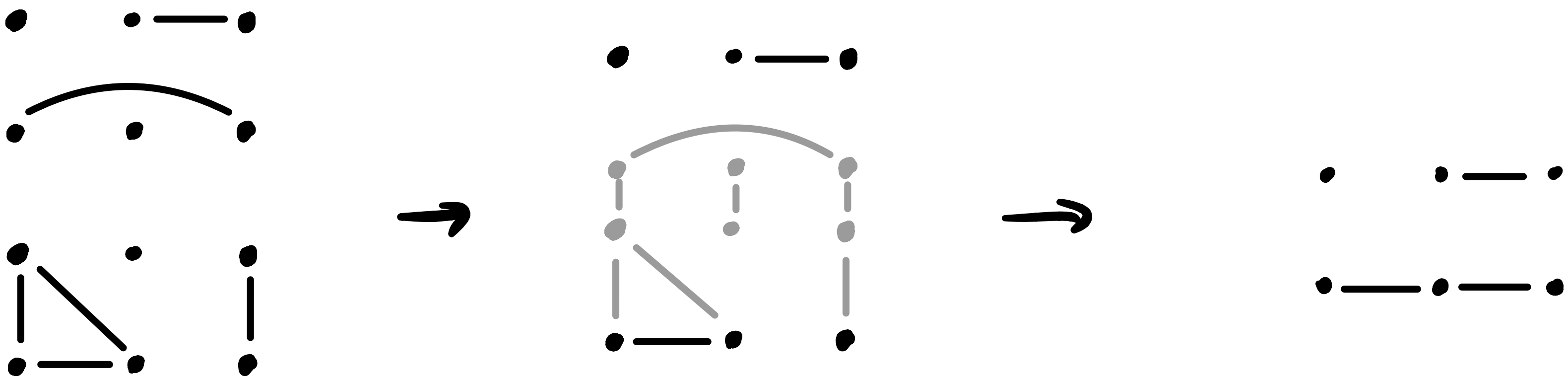
i) Put the first diagram on top of the second, identifying corresponding vertices in the middle

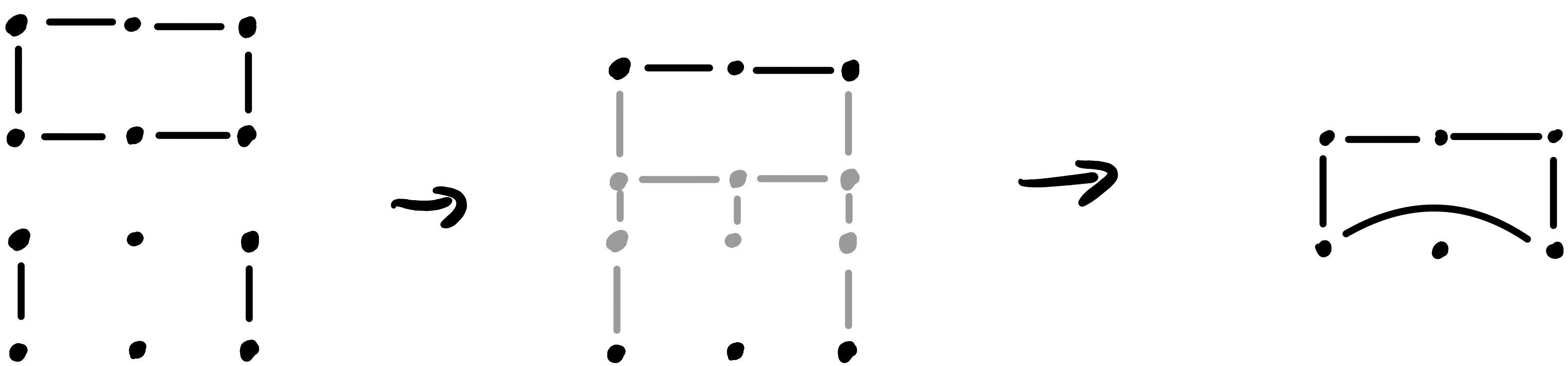
ii) Restrict to the top and bottom, preserving which vertices are connected in the larger diagram



A Monoid Structure on Diagrams







Schur-Weyl Duality

V_n : an n-dimensional $\mathbb{C}$-vector space

$GL_n(\mathbb{C})$: group of $n \times n$ invertible matrices over $\mathbb{C}$

$GL_n(\mathbb{C})$ acts on V_n by the usual matrix-vector multiplication

$V_n^{\otimes r}$: the r^{th} tensor power of V_n. You can think of elements here as $v_1 \otimes v_2 \otimes \cdots \otimes v_r$ with each $v_i \in V_n$
(really they're linear combinations of these but we only need to work with a basis)

$GL_n(\mathbb{C})$ also acts on $V_n^{\otimes r}$ in the following way:

$$A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_r) = (Av_1) \otimes (Av_2) \otimes \cdots \otimes (Av_r)$$

Schur-Weyl Duality

S_r : The symmetric group on r symbols

S_r also acts on $V_n^{\otimes r}$ by permuting the factors:

$$\sigma . (v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}$$

$$GL_n(\mathbb{C}) \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

Natural question: How do these two actions interact with each other?

Schur-Weyl Duality

$$GL_n(\mathbb{C}) \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

It's not too hard to see these actions commute i.e.

$$\sigma.\left(A.\left(v_1 \otimes \cdots \otimes v_r\right)\right) = A.\left(\sigma.\left(v_1 \otimes \cdots \otimes v_r\right)\right)$$

The more interesting fact is that they are mutual centralizers (the S_r action gives <u>all</u> the maps that commute with the $GL_n(\mathbb{C})$ action and vice versa)

This is called Schur-Weyl duality, first discovered by Schur and then popularized by Weyl who used it to classify U_n and GL_n representations

Schur-Weyl Duality

Main point:

This duality connects the representation theory of the two objects, letting us better understand both by studying either.

More precisely:

As a consequence, we have a decomposition

$$V_n^{\otimes r} \cong \bigoplus_{\lambda} E^{\lambda} \otimes S^{\lambda}$$

as a $GL_n(\mathbb{C}) \times S_r$ module, giving us e.g.

- A correspondence between irreducible $GL_n(\mathbb{C})$ and S_r modules where the multiplicity of one is the dimension of the other
- The same coefficients show up studying $E^{\lambda} \otimes E^{\mu}$ and $(S^{\lambda} \otimes S^{\mu})\uparrow_{S_{|\lambda|} \times S_{|\mu|}}^{S_{|\lambda|+|\mu|}}$

The Partition Algebra

We can restrict the action of $GL_n(\mathbb{C})$ to just the $n \times n$ permutation matrices

$$\begin{array}{ccccc} GL_n(\mathbb{C}) & & & & ? \\ \Big| & \searrow & & \swarrow & \Big| \\ & & V_n^{\otimes r} & & \\ \Big| & \nearrow & & \nwarrow & \Big| \\ S_n & & & & S_r \end{array}$$

The object that completes this picture is what is called the Partition Algebra

The Partition Algebra

What does this S_n action look like?

Write $e_1, \ldots, e_n$ for a basis of V_n, then for $\sigma \in S_n$

$$\sigma . e_i = e_{\sigma(i)}$$

Write $\underline{i} = (i_1, \ldots, i_r)$ with $1 \le i_1, \ldots, i_r \le n$,

then $e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_r}$ for all such $\underline{i}$ forms a basis of $V_n^{\otimes r}$.

For $\sigma \in S_n$,

$$\sigma . e_{\underline{i}} = (\sigma e_{i_1}) \otimes \cdots \otimes (\sigma e_{i_r}) = e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_r)} = e_{\sigma(\underline{i})}$$

where $\sigma(\underline{i}) = (\sigma(i_1), \ldots, \sigma(i_r))$

The Partition Algebra

Given this action of S_n, how do we determine what maps commute with it?

$\mathrm{End}(V_n^{\otimes r})$: the space of linear maps $V_n^{\otimes r} \to V_n^{\otimes r}$

$\mathrm{End}_{S_n}(V_n^{\otimes r})$: the maps in $\mathrm{End}(V_n^{\otimes r})$ which commute with the S_n action

We're looking for a basis of $\mathrm{End}_{S_n}(V_n^{\otimes r})$

The Partition Algebra

Generally for $M \in \mathrm{End}(V_n^{\otimes r})$, we can describe it by:

$$M e_{\underline{i}} = \sum_{\underline{j}} M_{\underline{j}}^{\underline{i}} e_{\underline{i}}$$

The condition $M \in \mathrm{End}_{S_n}(V_n^{\otimes r})$ amounts to

$$\sigma M e_{\underline{i}} = M \sigma e_{\underline{i}}$$

$$\sum_{\underline{j}} M_{\underline{j}}^{\underline{i}} e_{\sigma(\underline{j})} = \sum_{\underline{j}} M_{\underline{j}}^{\sigma(\underline{i})} e_{\underline{i}}$$

Comparing the coefficient on $e_{\sigma(\underline{j})}$ tells us that

$$M_{\underline{j}}^{\underline{i}} = M_{\sigma(\underline{j})}^{\sigma(\underline{i})} \quad \text{for all } \underline{i}, \underline{j}, \sigma.$$

The Partition Algebra

Visualizing some of these conditions for $\mathrm{End}_{S_3}(V_3^{\otimes 2})$:

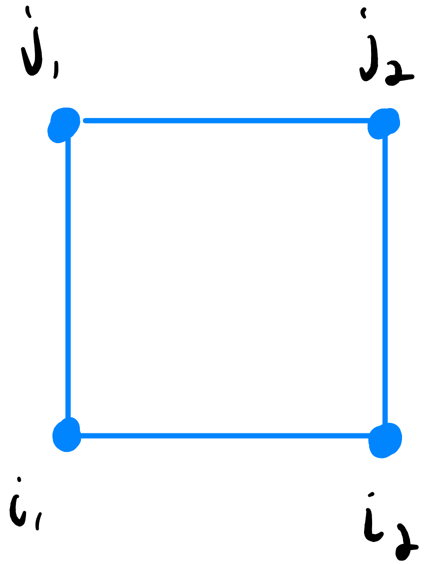
$$
\begin{array}{c|ccccccccc}
\underline{i} \backslash \underline{j} & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
\hline
11 & a & & & & & & & & \\
12 & & & b & & & & & & c \\
13 & & b & & & & & & & \\
21 & & & & & & b & & & c \\
22 & & & c & & a & & & & \\
23 & c & & & b & & & & & \\
31 & & & & & c & & & b & \\
32 & c & & & & & & b & & \\
33 & & & & & & & & & a
\end{array}
$$

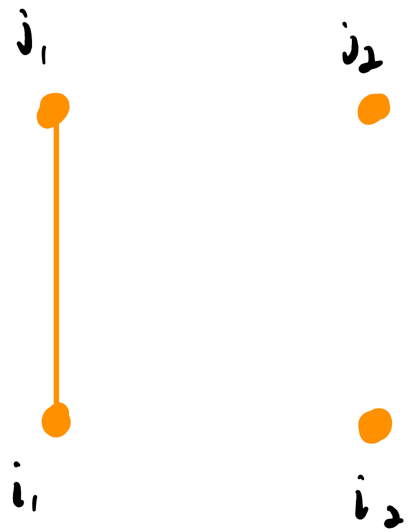
$\underline{\sigma}$			
ε	(11, 11)	(12, 13)	(12, 33)
(12)	22, 22	21, 23	21, 33
(13)	33, 33	32, 31	32, 11
(23)	11, 11	13, 12	13, 22
(123)	22, 22	23, 21	23, 11
(132)	33, 33	31, 32	31, 22

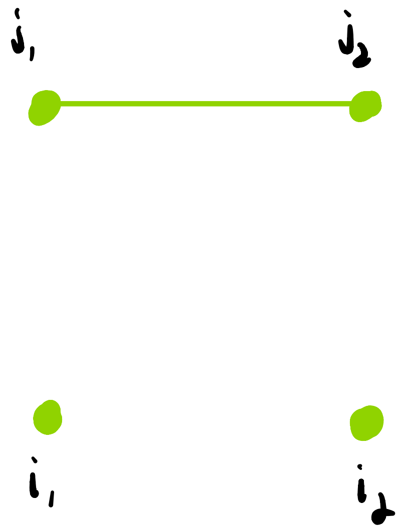
Each of these orbits represents a basis element, so how can we compactly represent these orbits?

The Partition Algebra

σ	$(i_1 i_2, j_1 j_2)$		
ε	(11, 11)	(12, 13)	(12, 33)
(12)	22, 22	21, 23	21, 33
(13)	33, 33	32, 31	32, 11
(23)	11, 11	13, 12	13, 22
(123)	22, 22	23, 21	23, 11
(132)	33, 33	31, 32	31, 22

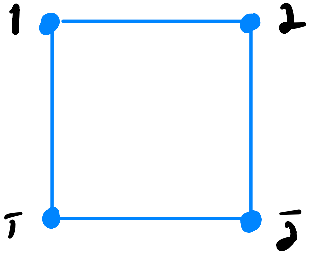




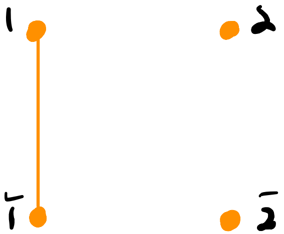


The Partition Algebra

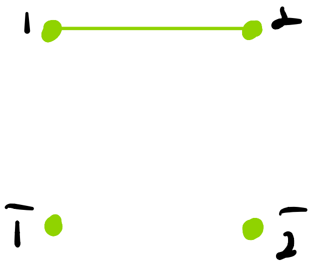
If we label these graphs with $1, \dots, r$ on top and $\bar{1}, \dots, \bar{r}$ on the bottom, we get set partitions from connected components



$$\longrightarrow \{\{1, 2, \bar{1}, \bar{2}\}\}$$



$$\longrightarrow \{\{1, \bar{1}\}, \{2\}, \{\bar{2}\}\}$$

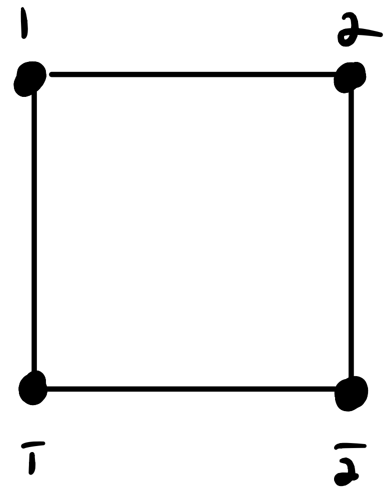


$$\longrightarrow \{\{1, 2\}, \{\bar{1}\}, \{\bar{2}\}\}$$

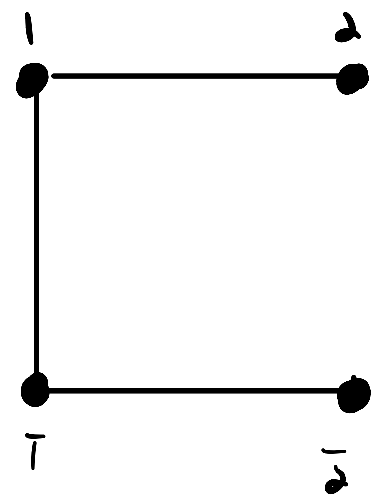
Write Δ_r for the set of these set partitions of $[r] \cup [\bar{r}]$

The Partition Algebra

These graphs representing an orbit are not unique. E.g.



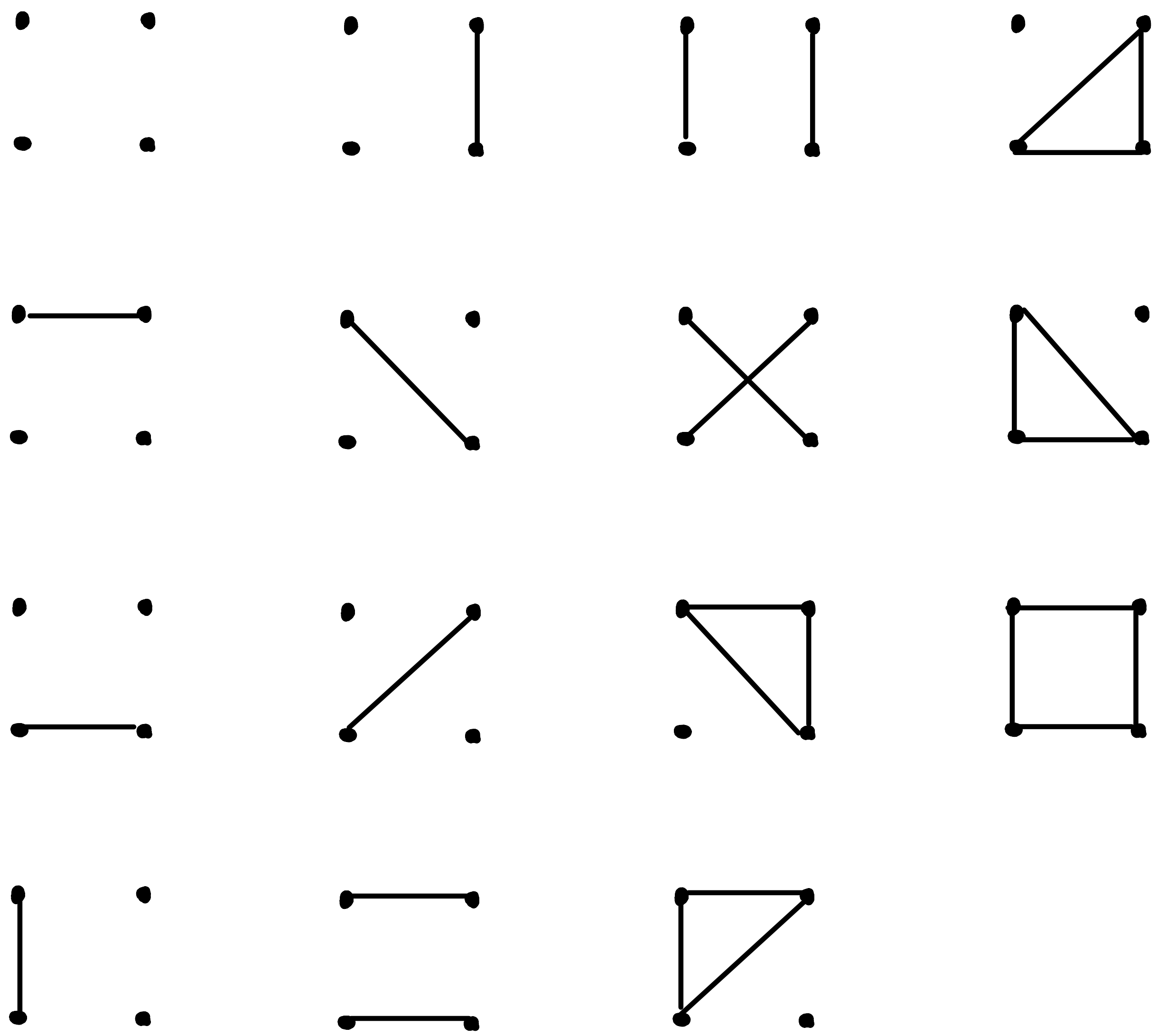
and



represent the same orbit. However, all the graphs representing a particular orbit have the same connected components. So we define a **diagram** as an equivalence class of graphs on the vertices $[r] \cup [\bar{r}]$ with the same connected components.

The Partition Algebra

For example, $\mathrm{End}_{S_4}(V_4^{\otimes 2})$ has a basis indexed by:



(need $n \geq 2r$ for all the diagrams to appear)

The Partition Algebra

We'll now call $\mathrm{End}_{S_n}(V_n^{\otimes r})$ the partition algebra $P_r(n)$ (introduced by P. Martin in 1990s)

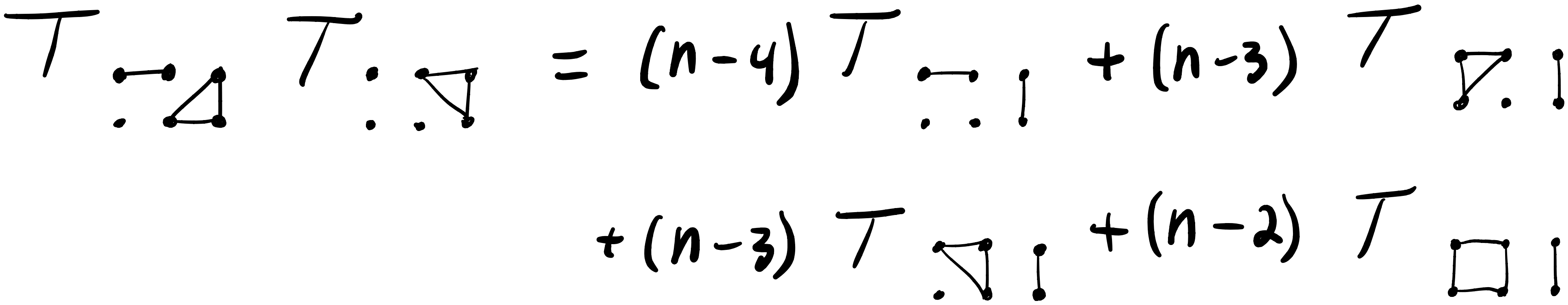
The basis obtained this way is called the orbit basis, which we'll write as

$$\{T_\pi : \pi \in \Delta_r\}$$

This basis does a great job capturing the vector space structure of $P_r(n)$, but it doesn't do much to elucidate the algebra structure.

The Partition Algebra

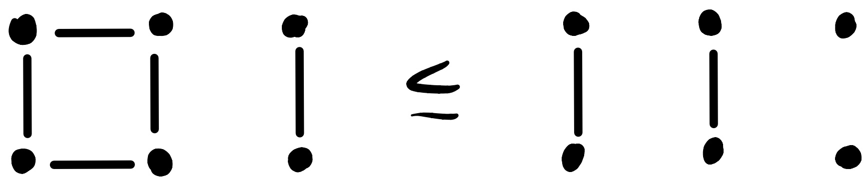
An example of orbit basis multiplication:



The Partition Algebra

Given set partitions $\pi = \{A_1, \dots, A_s\}$, $\nu = \{B_1, \dots, B_t\}$, we say that ν is a coarsening of π and write $\nu \leq \pi$ if each A_i is contained in some B_j.

$$\{\{1, 2, \bar{1}, \bar{2}\}, \{3, \bar{3}\}\} \leq \{\{1, \bar{1}\}, \{2, \bar{2}\}, \{3\}, \{\bar{3}\}\}$$

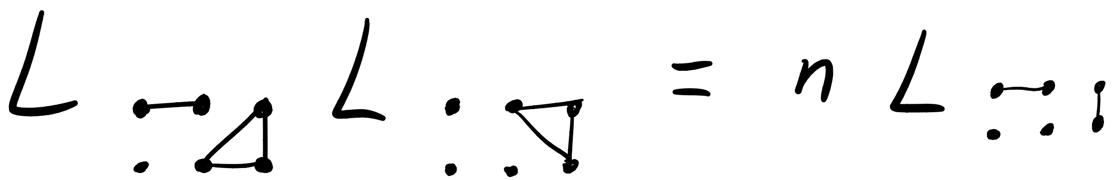


The Partition Algebra

There is another basis $\{L_\pi\}$ called the diagram basis given by

$$L_\pi = \sum_{\nu \leq \pi} T_\nu$$

Revisiting the two diagrams from the earlier multiplication example, we see the diagram basis multiplication looks more like our nice multiplication from earlier:



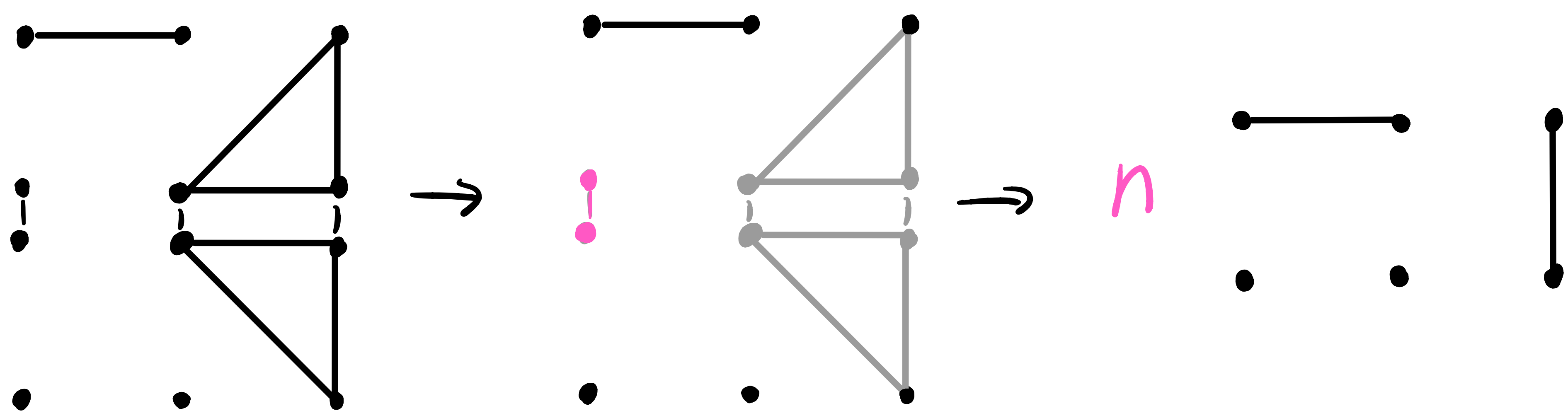
The Partition Algebra

The formula:

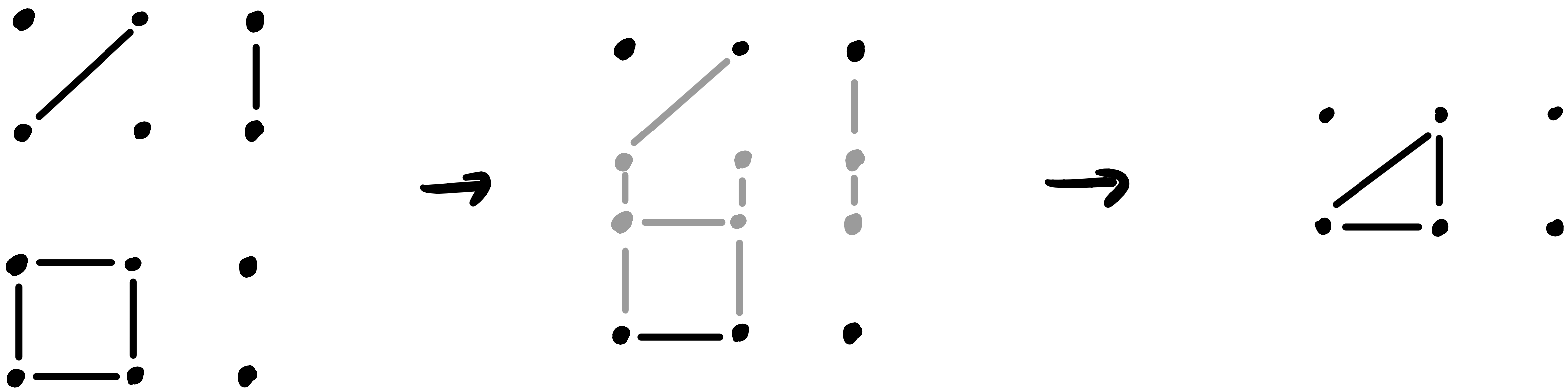
i) Put the first diagram on top of the second, identifying corresponding vertices in the middle

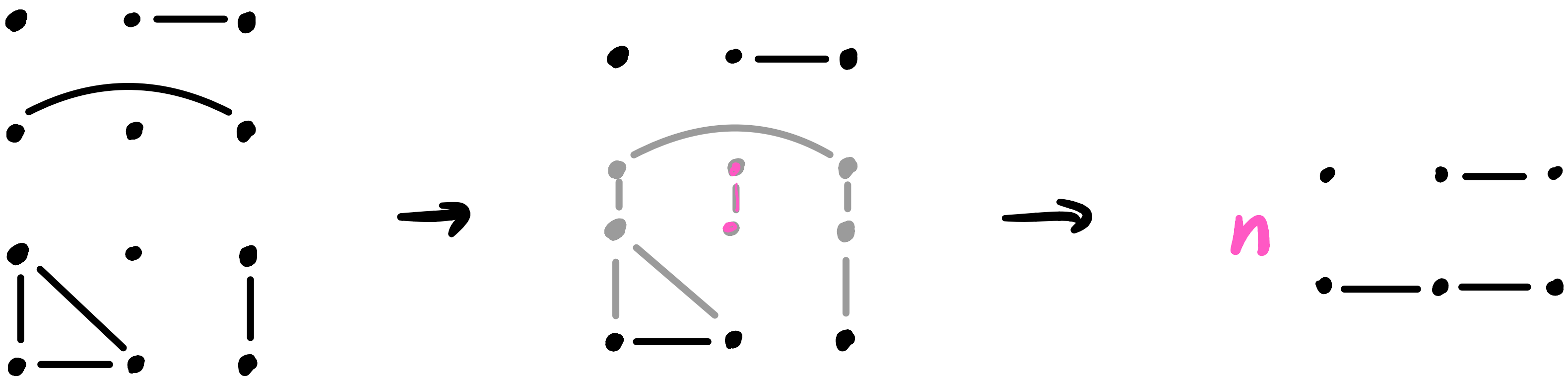
ii) Restrict to the top and bottom, preserving which vertices are connected in the larger diagram

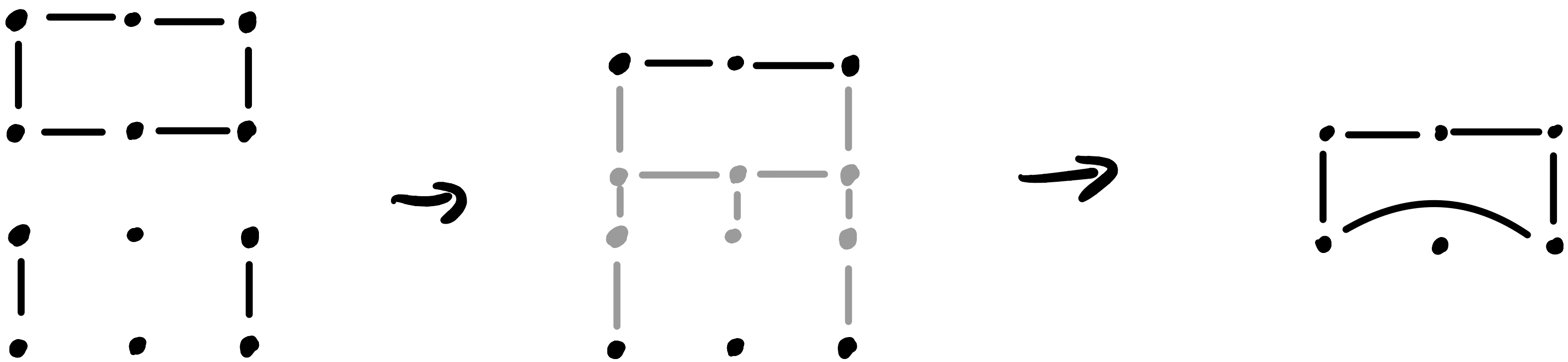
iii) Record a coefficient of n^c where c is the number of components stranded in the middle of the larger diagram.



The Partition Algebra

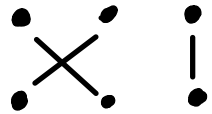
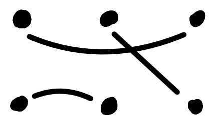
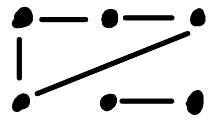
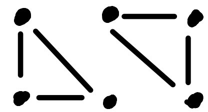
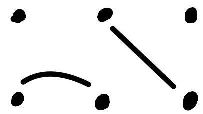
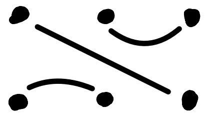






The Partition Algebra

$$G \curvearrowright V_n^{\otimes r} \curvearrowleft A$$

G	A	Typical Element	
GL_n	$\mathbb{C}S_r$		
O_n	Brauer Algebra ($B_r(n)$)		(matchings)
$G(m,p,n)$	Tanabe Algebra ($T_{m,p,r}(n)$)		(subtle, but akin to #top $\equiv$ #bottom (mod m))
S_n	Partition Algebra		
$U_q(\mathfrak{gl}_2)$	Motzkin Algebra		(components of size ≤ 2, non-crossing)
$U_q(\mathfrak{sl}_2)$	Temperley-Lieb Algebra		(non-crossing matchings)

Howe Duality

$V_{n,k}$: The space of $n \times k$ matrices over $\mathbb{C}$

$\mathcal{P}^r(V_{n,k})$: The space of homogeneous polynomial forms of degree r on $V_{n,k}$

Think of the monomials like $x_{\underline{i}\,\underline{j}} = x_{i_1 j_1} \cdots x_{i_r j_r}$ with

$1 \le i_1, \ldots, i_r \le n$ and $1 \le j_1, \ldots, j_r \le k$

where the indeterminate x_{ij} picks out entry (i,j) in the matrix.

$$x_{12}\, x_{13}\, x_{22} \left(\begin{bmatrix} 3 & 2 & 1 & 5 \\ 6 & 3 & 7 & 1 \end{bmatrix} \right) = 2 \cdot 1 \cdot 3$$

Howe Duality

A matrix $A \in GL_n(\mathbb{C})$ acts on $f \in P^r(V_{n,k})$ by

$$(A.f)(X) = f(A^{-1}X)$$

In 1980s, Roger Howe determined the centralizer:

$$GL_n(\mathbb{C}) \circlearrowright P^r(V_{n,k}) \circlearrowleft GL_k(\mathbb{C})$$

where $B \in GL_k(\mathbb{C})$ acts by

$$(B.f)(X) = f(XB)$$

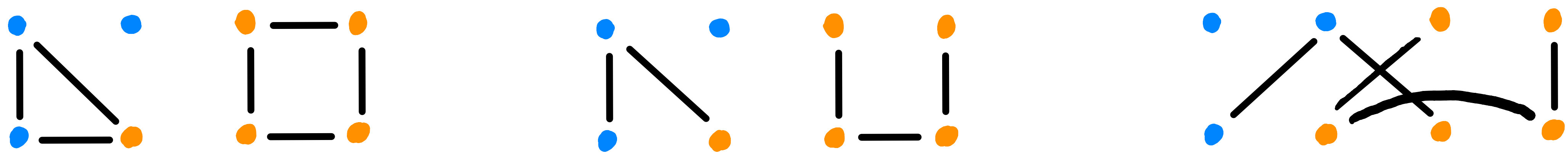
Howe Duality

$$\begin{array}{ccccc} GL_n(\mathbb{C}) & & & & ? \\ | & & \mathcal{P}^r(V_{n,k}) & & | \\ S_n & & & & GL_k(\mathbb{C}) \end{array}$$

The Multiset Partition Algebra

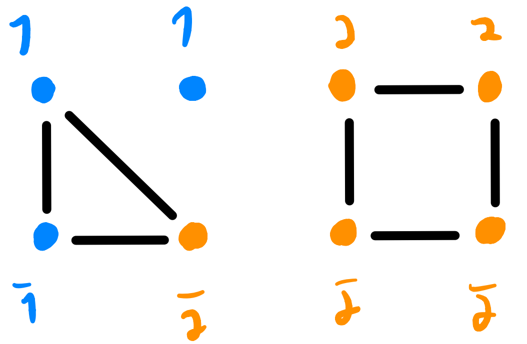
Orellana and Zabrocki (2020) examined $\mathrm{End}_{S_n}(P^r(V_{n,k}))$, describing an orbit basis for it and dubbing it $MP_{r,k}(n)$, the multiset partition algebra

This basis is indexed by partition diagrams whose vertices are colored from a set of k colors, with identically colored vertices among the top or bottom indistinguishable:



The Multiset Partition Algebra

Like before, these diagrams represent partitions, but this time repitition is allowed (indicated by the double brackets):



$$\longrightarrow \{\{ \{\{1, \bar{1}, \bar{2}\}\}, \{\{1\}\}, \{\{2, 2, \bar{2}, \bar{2}\}\} \}\}$$

We'll write $\tilde{\Delta}_{r,k}$ for these multiset partitions with entries from $[k] \cup [\bar{k}]$ with r unbarred and r barred entries. We'll write $\tilde{\pi}$ or $\tilde{\nu}$ for a particular such multiset partition.

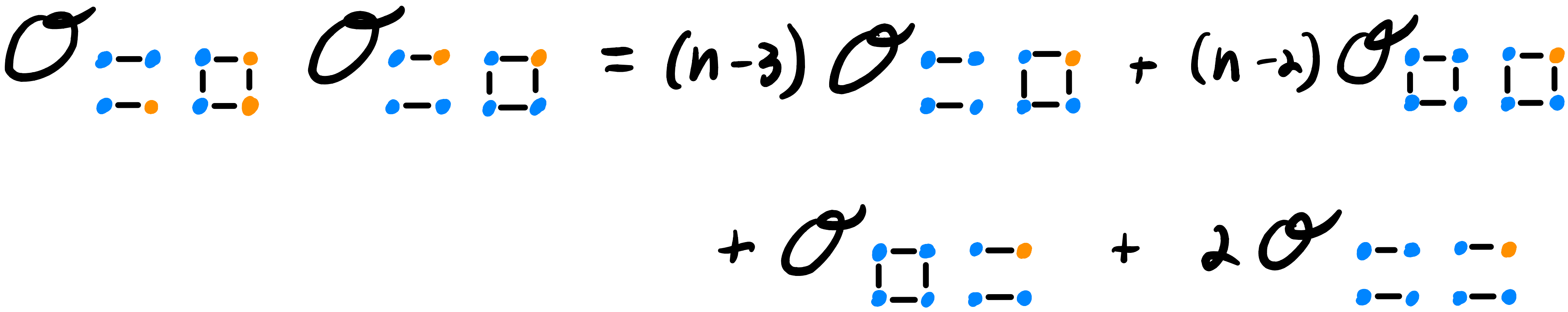
Note, I will consistently use these colors:

1 = 

2 = 

The Multiset Partition Algebra

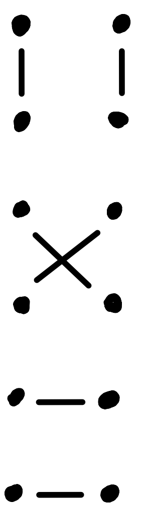
Writing $\{\mathcal{O}_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Delta}_{r,k}\}$ for the orbit basis obtained by Orellana and Zabrocki, an example of its multiplication is:



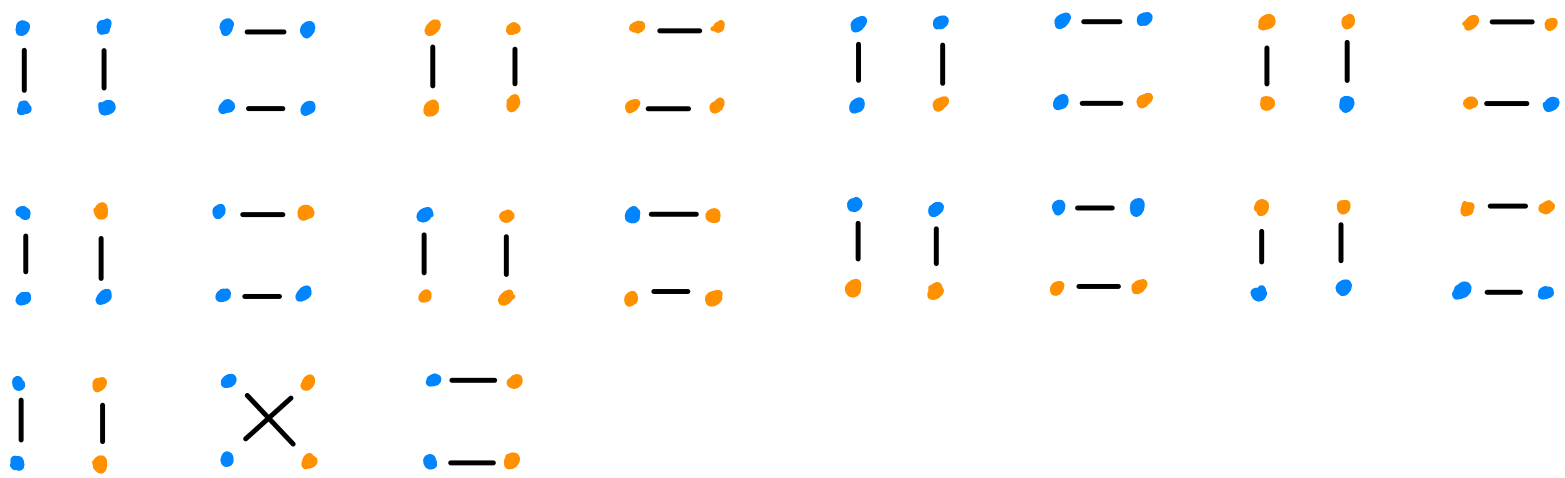
The Multiset Partition Algebra

Let $A_r(n) \subseteq P_r(n)$, and define a new algebra $\tilde{A}_{r,k}(n)$ called the corresponding Painted algebra with basis $\{D_{\tilde{\pi}} : \tilde{\pi}$ obtained by coloring the vertices of a diagram in $A_r(n)\}$

$B_2(n)$

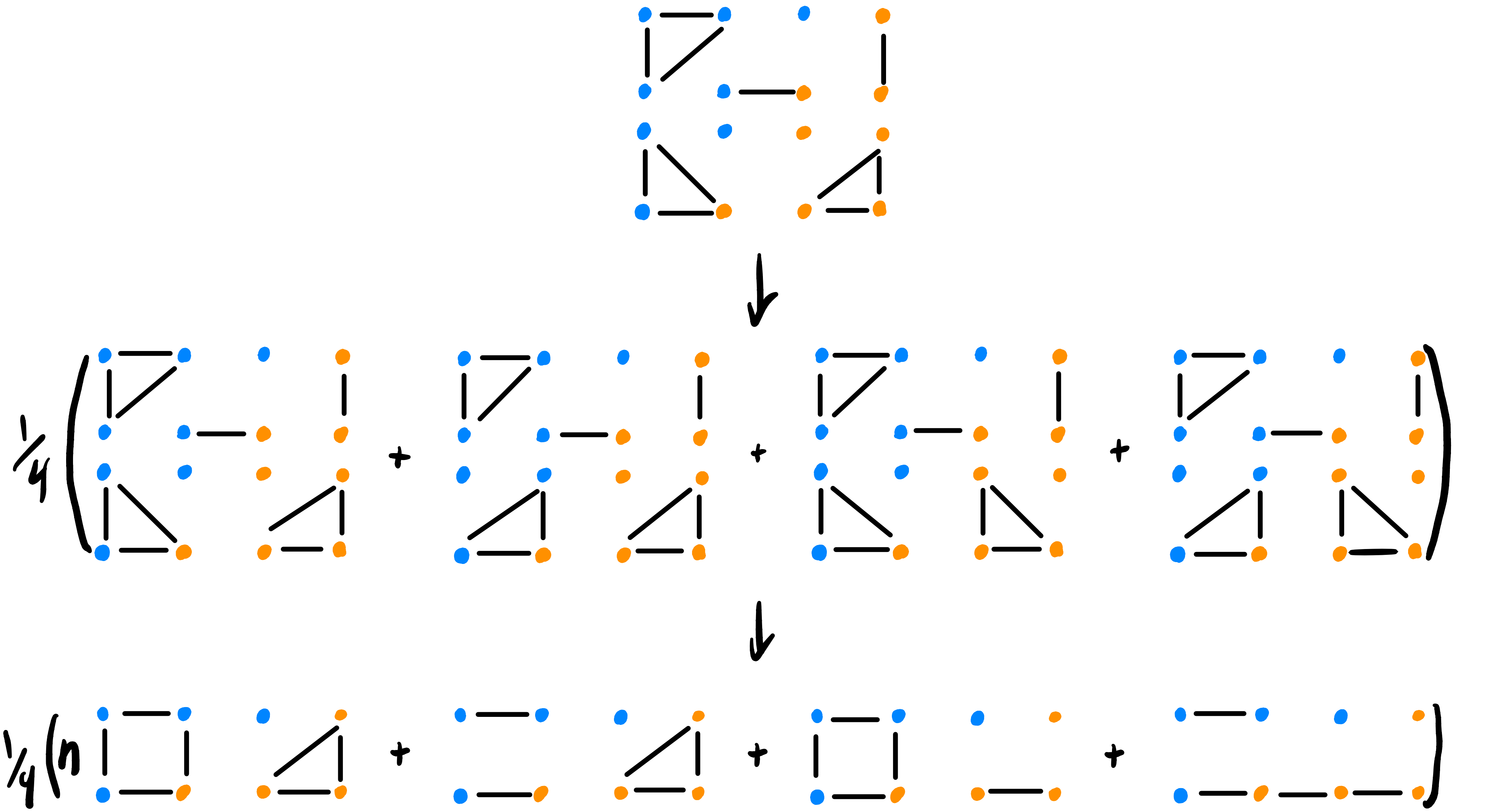


$\tilde{B}_{2,2}(n)$



The Multiset Partition Algebra

If the multiplicities of colors in the bottom of one diagram match those in the top of the other, their product is given by the following averaging. Otherwise it is zero.



The Multiset Partition Algebra

Theorem There is an isomorphism

$$\varphi : MP_{r,k}(n) \longrightarrow \tilde{P}_{r,k}(n)$$

The Multiset Partition Algebra

Idea of proof

We need to establish that $\varphi(\mathcal{O}_{\tilde{\tilde{\pi}}}\mathcal{O}_{\tilde{\nu}}) = \varphi(\mathcal{O}_{\tilde{\tilde{\pi}}})\varphi(\mathcal{O}_{\tilde{\nu}})$.

After algebraic manipulation, this comes down to handling a sum of the form

$$\sum_{\sigma \in S_r} \sum_{\gamma} 1$$

where γ is a set partition of $[r] \cup [\tilde{r}] \cup [\tilde{\tilde{r}}]$ subject to conditions depending on $\tilde{\tilde{\pi}}$, $\tilde{\nu}$ and σ. It turns out:

- The set of σ for which there exists a γ is a nice product of subgroups
- The number of γ is the same for any σ and can be enumerated via an orbit-stabilizer argument.

The Multiset Partition Algebra

The change-of-basis $\mathcal{O}_{\tilde{\pi}} \to D_{\tilde{\pi}}$ is given by:

$$D_{\tilde{\pi}} = \sum_{\tilde{\nu} \leq \tilde{\pi}} \frac{c_{\tilde{\nu}, \tilde{\pi}}}{w(\tilde{\nu})} \mathcal{O}_{\tilde{\nu}}$$

where $c_{\tilde{\nu}, \tilde{\pi}}$ is, for a fixed π which can be painted to get $\tilde{\pi}$, the number of ν such that $\nu \leq \pi$ and ν can be painted to obtain $\tilde{\nu}$,

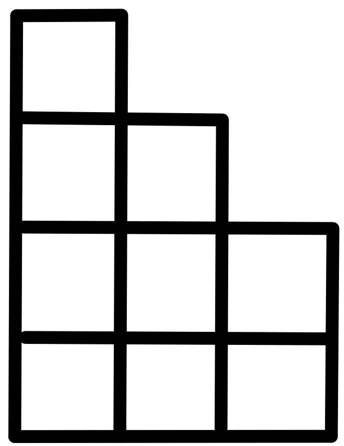
$w(\tilde{\nu})$ is a coefficient depending on $\tilde{\nu}$

This basis $\{D_{\tilde{\pi}}\}$ is the diagram-like basis.

Representations

A partition λ of n is a weakly decreasing sequence $(\lambda_1, \ldots, \lambda_\ell)$ of positive integers which sum to n. We write $\lambda \vdash n$ for such a partition

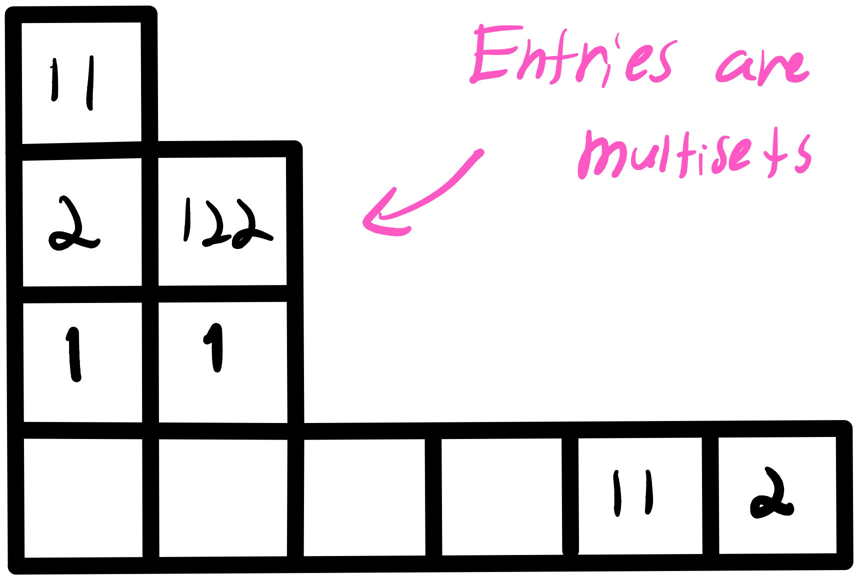
The Young diagram of λ is an array of left-justified boxes with λ_i boxes in the i^{th} row from the bottom. E.g.:

$(3, 3, 2, 1) \longleftrightarrow$ 

Representations

A multiset partition tableau of shape λ is a filling of λ's Young diagram like so:

Only the first row has empty boxes – at least as many as λ_2



Entries are multisets

Write $MSPT(\lambda, r, k)$ for the set of these tableaux with a total of r numbers from $[k]$.

Representations

Order multisets by the last-letter order

$11 < 2$ $\qquad$ $12 < 22$ $\qquad$ $22 < 122$

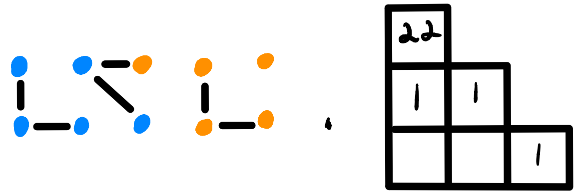
A semistandard multiset partition tableau has rows weakly increasing and columns strictly increasing:

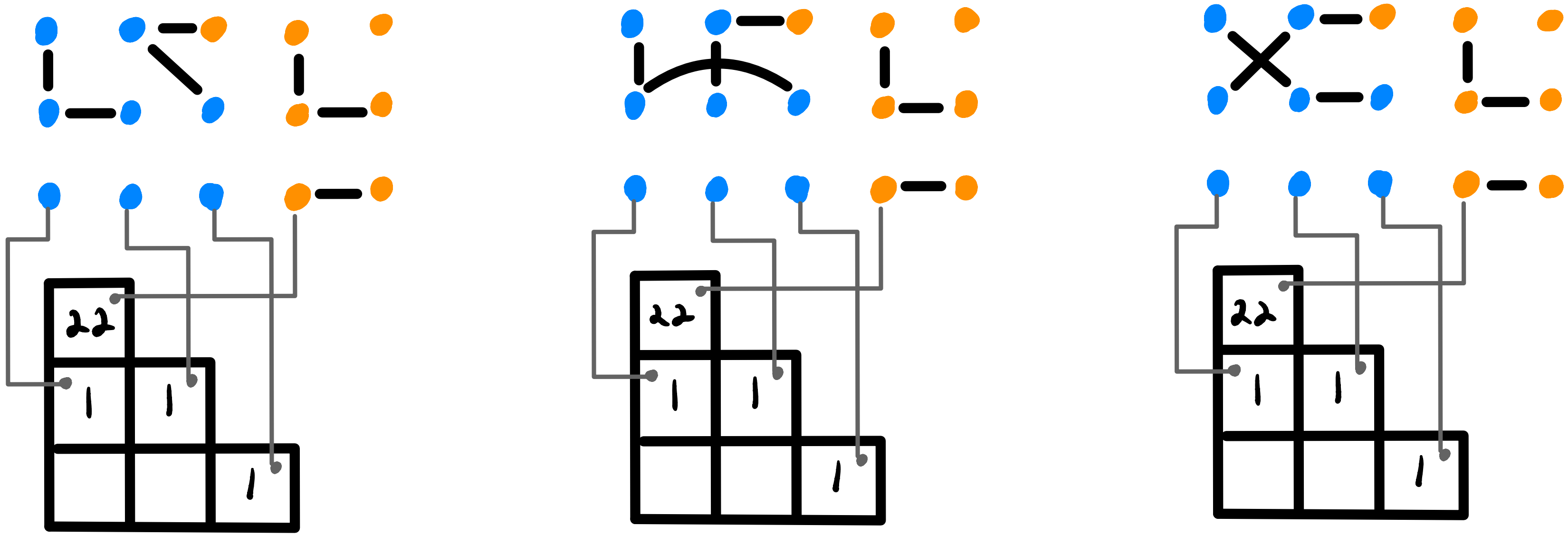
22					
2	12				
1	1				
				11	2

Write $SSMSPT(\lambda, r, \kappa)$ for these

Representations

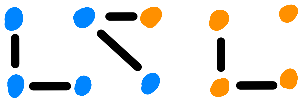
An example of the action ;





X Two blocks above the first row get combined

Representations

 $\cdot$ $\begin{array}{|c|c|c|}\cline{1-1} 22 \\ \cline{1-2} 1 & 1 \\ \hline & & 1 \\ \hline\end{array} = \frac{1}{3}\left(\begin{array}{|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 1 & 12 \\ \hline & & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 12 & 1 \\ \hline & & 2 \\ \hline\end{array} \right)$

Straightening algorithm

$$= \frac{1}{3}\left(\begin{array}{|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 1 & 12 \\ \hline & & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\cline{1-1} 12 \\ \cline{1-2} 1 & 2 \\ \hline & & 2 \\ \hline\end{array} - \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 1 & 12 \\ \hline & & 2 \\ \hline\end{array} \right)$$

Write $\Lambda^{MP_{r,k}(n)} = \{ \lambda \vdash n : SSMSPT(\lambda, r, k) \neq \emptyset \}$

$MP^{\lambda}_{r,k} :=$ Span of $SSMSPT(\lambda, r, k)$ for $\lambda \in \Lambda^{MP_{r,k}(n)}$

$\boxed{\text{Theorem}}$ The $MP^{\lambda}_{r,k}$ for $\lambda \in \Lambda^{MP_{r,k}(n)}$ form a complete set of irreducible representations for $MP_{r,k}(n)$ for $n \geq 2r$.

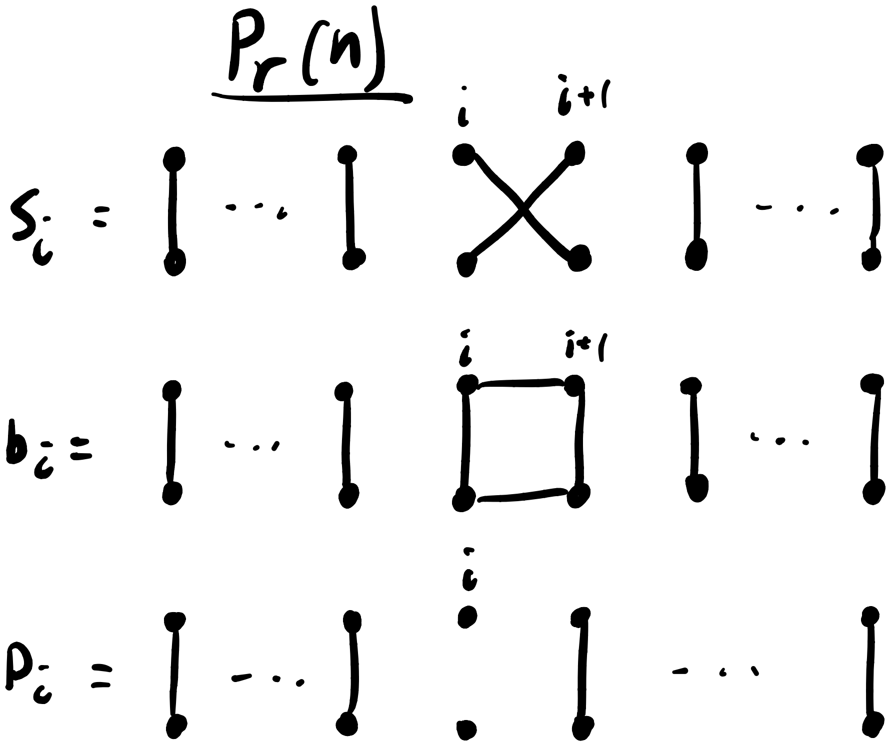
Subalgebras

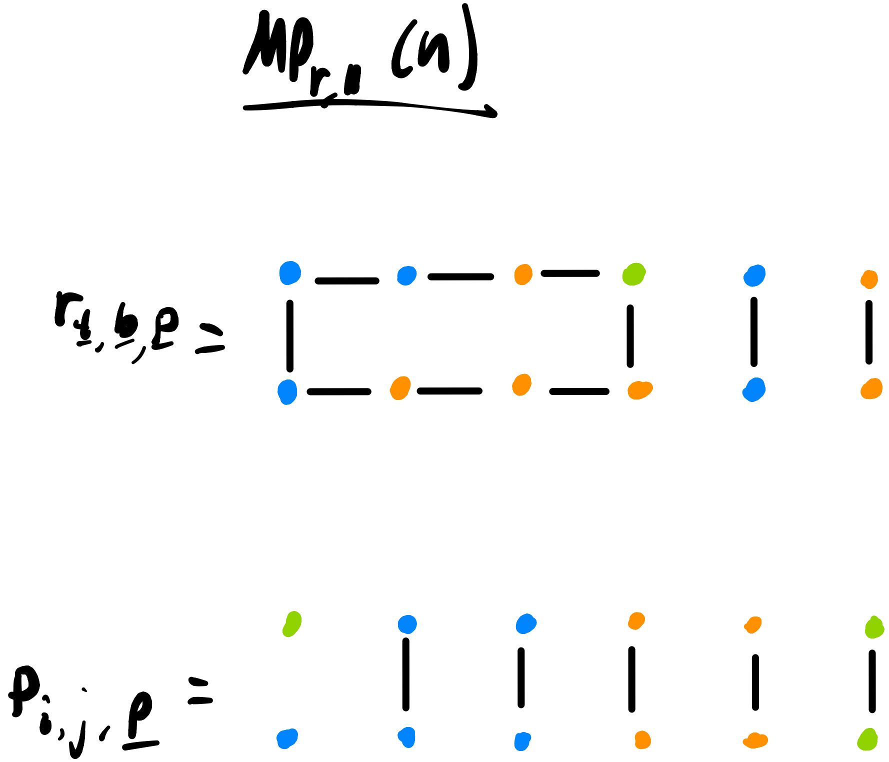
Theorem $\mathrm{End}_{G(m,p,n)}(P^r(V_{n,k})) \cong \widetilde{T}_{r,m,p,k}(n)$

But recall that $\mathrm{End}_{G(m,p,n)}(V_n^{\otimes r}) \cong T_{r,m,p}(n)$

Conjecture If G is a reductive algebraic subgroup of Gl_n and $\mathrm{End}_G(V_n^{\otimes r}) \cong A_r(n)$, then $\mathrm{End}_G(P^r(V_{n,k})) \cong \widetilde{A}_{r,k}(n)$.

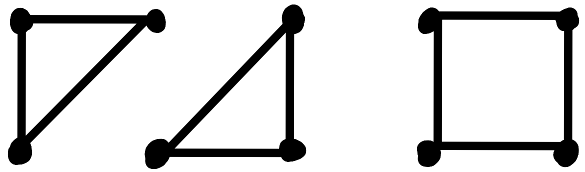
Generators



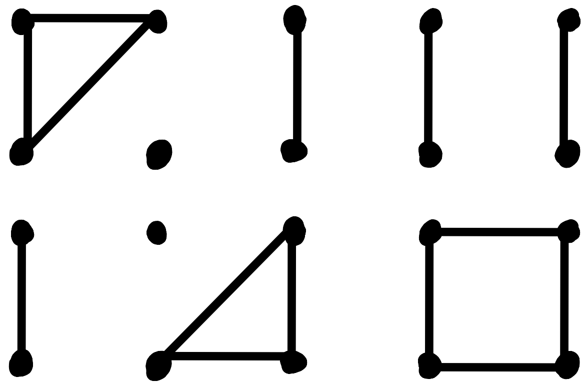


Thank you!

Generators

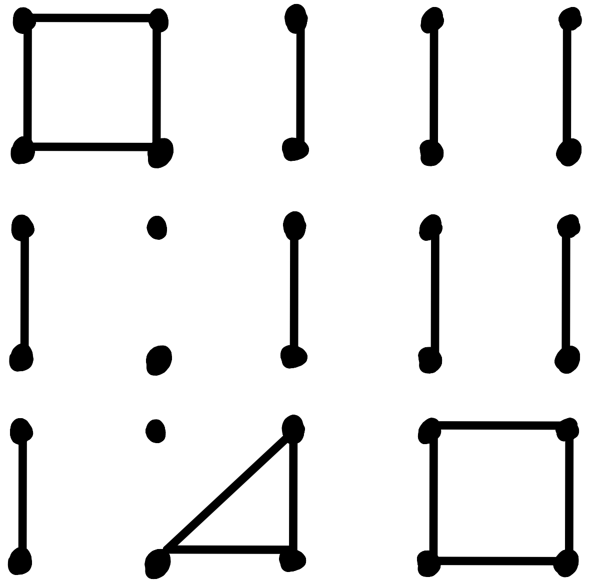


$= \frac{1}{x}$



(i) Factor out "interesting" block

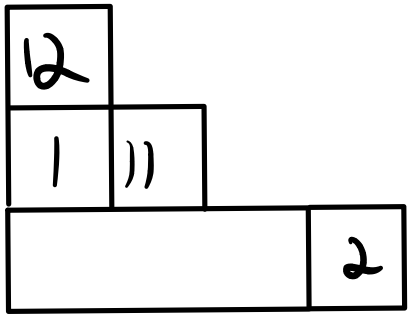
$= \frac{1}{x}$

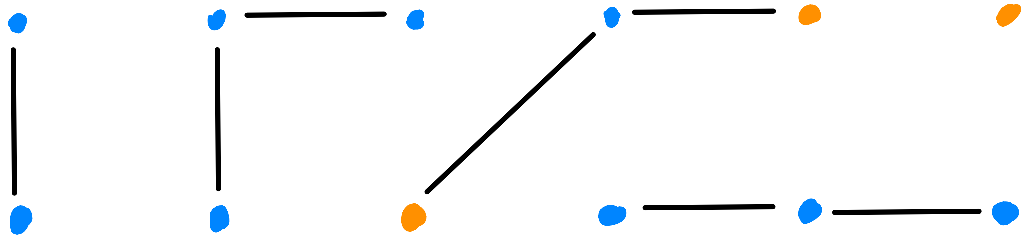


(ii) write diagram with single interesting block in terms of generators

Representations

Tabloid diagram

$T =$ 

$[T] =$ 

Polytabloid element

$$v_T = \sum_{\sigma \in C(T)}' \operatorname{sgn}(\sigma)\,[\sigma \cdot T]$$

$MP_{r,k}^{\lambda}$ is the span of $\bar{v}_T$ in the quotient by elements with fewer than $\lambda_2 + \cdots + \lambda_\ell$ propagating blocks

02 2020 Thm 5.10