Pop, Crackle, Snap (and Pow)

some facets of shards

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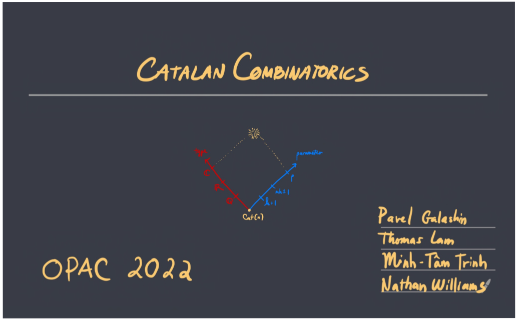
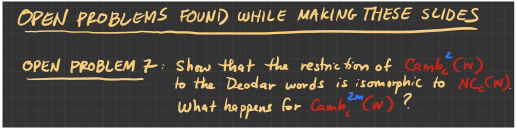
JON MCCAMMOND AND OPAC





There is a natural poset which encodes the factorizations of a central element of a finite Coxeter group into products of squares of the dual generators with each (squared) generator appearing exactly once.

Jon and Dan Margalit (back in the early 2000s) created the following wish list of results:

(1) embeds into the $\mathbb{Z}^m$ cube, $m = \binom{n}{2}$.
(2) clean statement of which elements are involved.
(3) clean statement of which permutations label max chains.
(4) rank vector.
(5) euler characteristic
(6) moebius function.
(7) recursive structure.
(8) some other purely combinatorial model
(9) transitivity on chains under natural 2,3 mods.

REAL, CENTRAL HYPERPLANE ARRANGEMENTS

A hyperplane H is a linear codimension-1 subspace of $\mathbb{R}^n$.

A central hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes.

Central $\supset$ Simplicial $\supset$ Reflection

INTRODUCTION

1. Edelman's Regions
2. Reading's Shards
3. Salvetti's Loops
4. Poset Embeddings
5. Snap = Crackle Pop

EDELMAN'S REGIONS

EDELMAN'S POSET OF REGIONS

Write $\mathcal{R}$ for the set of connected components (the regions) of the complement $\mathbb{R}^n \setminus \mathcal{H}$.

Fix B a base region in $\mathcal{R}$.

For $C \in \mathcal{R}$, define $\mathrm{inv}_{\mathcal{H},B}(C)$ to be the hyperplanes in $\mathcal{H}$ separating B from C.

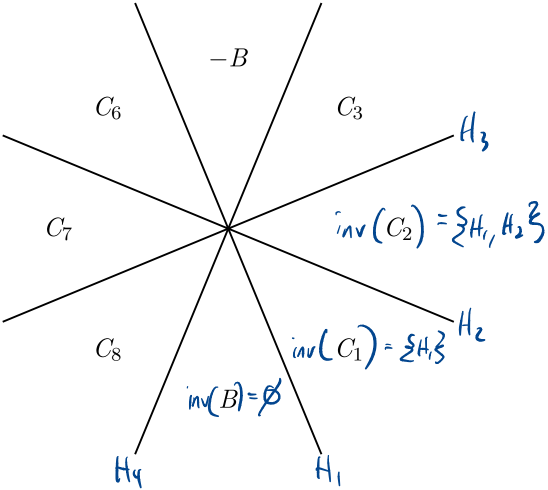
The map $C \mapsto \mathrm{inv}_{\mathcal{H},B}(C) \subseteq \mathcal{H}$ is injective.

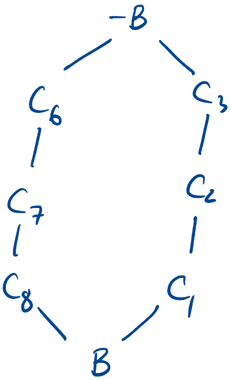
Definition (Edelman)

The poset of regions $\mathrm{Weak}(\mathcal{H}, B)$ has elements $\mathcal{R}$ and relations

$$C \leq D \text{ iff } \mathrm{inv}_{\mathcal{H},B}(C) \subseteq \mathrm{inv}_{\mathcal{H},B}(D).$$

EDELMAN'S POSET OF REGIONS





POSET OF REGIONS FOR $\mathcal{H}$ SIMPLICIAL

Theorem (Bjorner, Edelman, Ziegler)

$\mathrm{Weak}(\mathcal{H}, B)$ is a lattice for every $B \in \mathcal{R}$ iff $\mathcal{H}$ is simplicial.

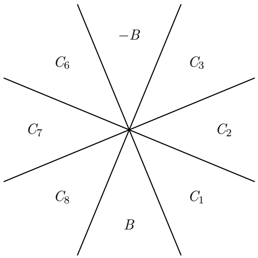
$\wedge$ $\vee$

POP

Definition

The pop-stack sorting operator $\mathrm{Pop} : \mathcal{R} \to \mathcal{R}$ is

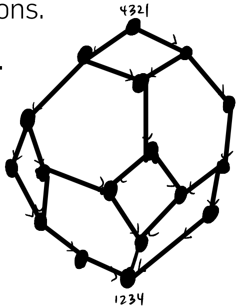
$$\mathrm{Pop}(C) := \bigwedge_{D \lessdot C} D.$$



POSET OF REGIONS FOR $\mathcal{H}$ REFLECTION

For $\mathcal{H}$ the reflection arrangement of a finite Coxeter group W, $\mathrm{Weak}(\mathcal{H}, B)$ is the oriented Cayley graph of W when generated using simple reflections.

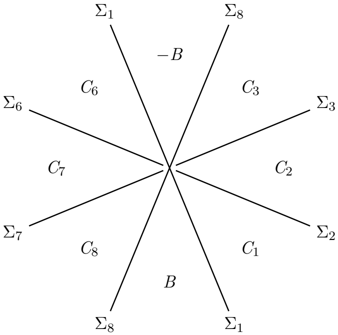
For $W = S_n$, $\mathrm{Weak}(\mathcal{H}, B)$ is the 1-skeleton of the permutahedron.



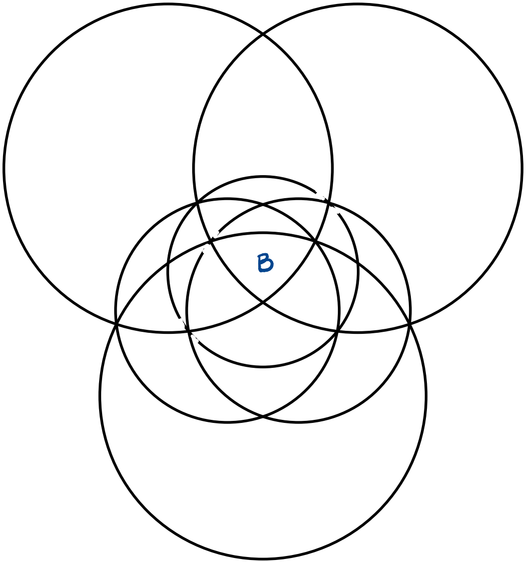
READING'S SHARDS

SHARDS

Reading cut the hyperplanes in an arrangement $\mathcal{H}$ into pieces called shards.



Write $Ш(\mathcal{H}, B)$ for the set of shards. Each cover relation $C \xrightarrow{e} C'$ in $\mathrm{Weak}(\mathcal{H}, B)$ is labeled by the unique shard $\Sigma(e)$ separating the region C from the region C'.


B

SHARDS FOR $\mathcal{H}$ SIMPLICIAL

Write $\mathrm{inv}_{Ш,B}(C)$ to be the shards in $\mathcal{H}$ separating Pop(C) from C.

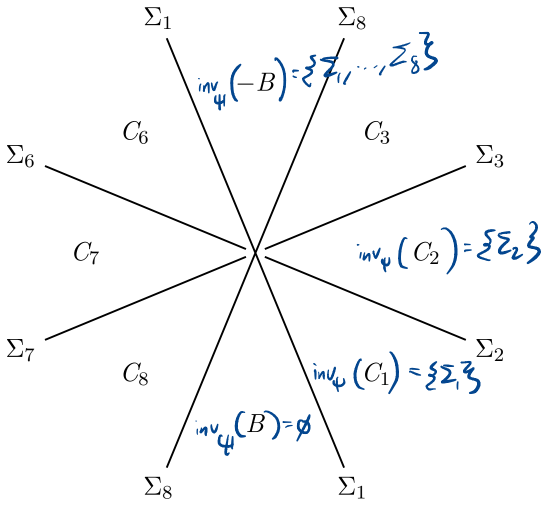
The map $C \mapsto \mathrm{inv}_{Ш,B}(C) \subseteq Ш$ is injective.

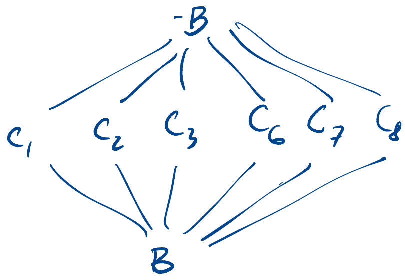
Definition (Reading)

The shard intersection order $\mathrm{Shard}(\mathcal{H}, B)$ has elements $\mathcal{R}$ and relations

$$C \preceq D \text{ iff } \mathrm{inv}_{Ш,B}(C) \subseteq \mathrm{inv}_{Ш,B}(D).$$

SHARDS FOR $\mathcal{H}$ SIMPLICIAL

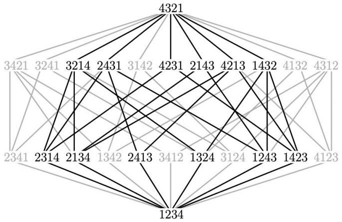




SHARDS FOR $\mathcal{H}$ REFLECTION

For $\mathcal{H}$ the reflection arrangement of a finite Coxeter group W, a sublattice of $\mathrm{Shard}(\mathcal{H}, B)$ recovers the W-noncrossing partition lattice.

Reading used this to give a uniform proof that the W-noncrossing partition lattice is actually a lattice—and weaker than the corresponding Cambrian lattice.



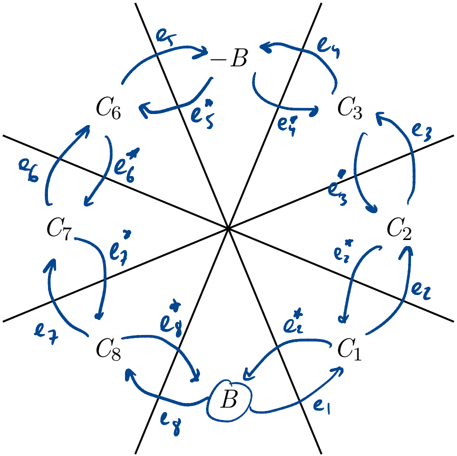
SALVETTI'S LOOPS

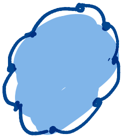
THE SALVETTI COMPLEX

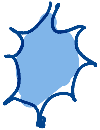
Definition

The Salvetti complex $\mathrm{Sal}(\mathcal{H})$ is defined by gluing together oriented dual zonotopes for $\mathcal{H}$ along compatible faces—one zonotope for each choice of base region B, oriented from B to $-B$.

- 0-cells: one for each region in $\mathcal{R}$
- 1-cells: two for each cover relation $C \lessdot C'$ in $\mathrm{Weak}(\mathcal{H}, B)$, $C \xrightarrow{e} C'$ and $C' \xrightarrow{e^*} C$
- 2-cells: one for each rank 2 intersection


−B
C6
C3
C7
C2
C8
C1
B
e1
e2
e3
e4
e5
e6
e7
e8





THE SALVETTI COMPLEX

Write $\mathbb{C}^n \setminus \mathcal{H}_\mathbb{C}$ for the complexified hyperplane complement of $\mathcal{H}$.

Theorem (Salvetti)

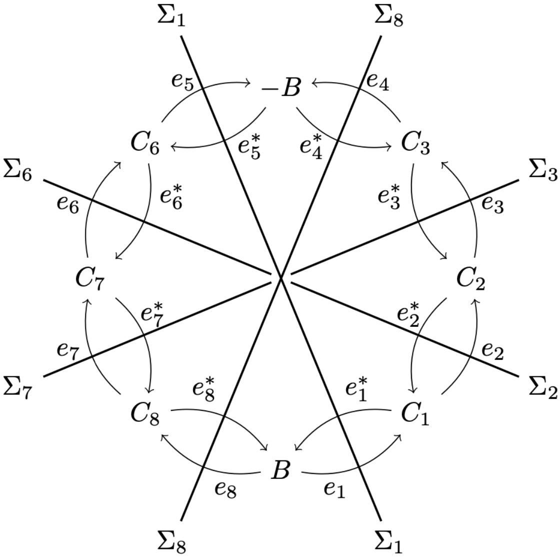
$$\pi_1(\mathrm{Sal}(\mathcal{H}), B) = \pi_1(\mathbb{C}^n \setminus \mathcal{H}_\mathbb{C}, x_B).$$

If $C \xrightarrow{e} C'$ is a cover in $\mathrm{Weak}(\mathcal{H}, B)$, define a loop $\ell_e \in \pi_1(\mathrm{Sal}(\mathcal{H}), B)$ by

$$\ell_e := \mathrm{gal}(B, C) \cdot ee^* \cdot \mathrm{gal}(B, C)^{-1} \in \pi_1(\mathrm{Sal}(\mathcal{H}), B).$$

$\pi_1(\mathrm{Sal}(\mathcal{H}), B)$ is generated by the loops $\mathcal{L}_{\mathrm{edge}}$, the set of all such ℓ_e.

One family of relations for each 2-cell.


Σ_1
Σ_8
e_5
$-B$
e_4
C_6
e_5^*
e_4^*
C_3
Σ_6
Σ_3
e_6
e_6^*
e_3^*
e_3
C_7
C_2
e_7^*
e_2^*
e_7
e_2
Σ_7
Σ_2
e_8^*
e_1^*
C_8
C_1
B
e_8
e_1
Σ_8
Σ_1

SHARDVETTI

When are two loops ℓ_e and ℓ_f homotopic?

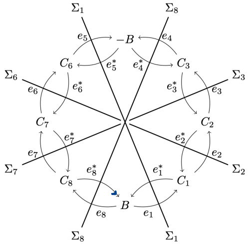
Theorem (Defant, W.)

For a real central arrangement $\mathcal{H}$,

$$\ell_e \simeq \ell_f \text{ iff } \Sigma(e) = \Sigma(f).$$

$$e_8 e_8^* \overset{?}{=} e_1 e_2 e_3 e_4 e_4^* e_3^{-1} e_2^{-1} e_1^{-1} = e_8 e_7 e_6 e_5 e_4^* e_3^{-1} e_2^{-1} e_1^{-1}$$

$$\overset{\checkmark}{=} e_8 e_8^* e_1 e_2 e_3 e_3^{-1} e_2^{-1} e_1^{-1}$$



So label Salvetti's loops by Reading's shards $\mathcal{L}_{\mathrm{III}}$.

THE PURE SHARD MONOID

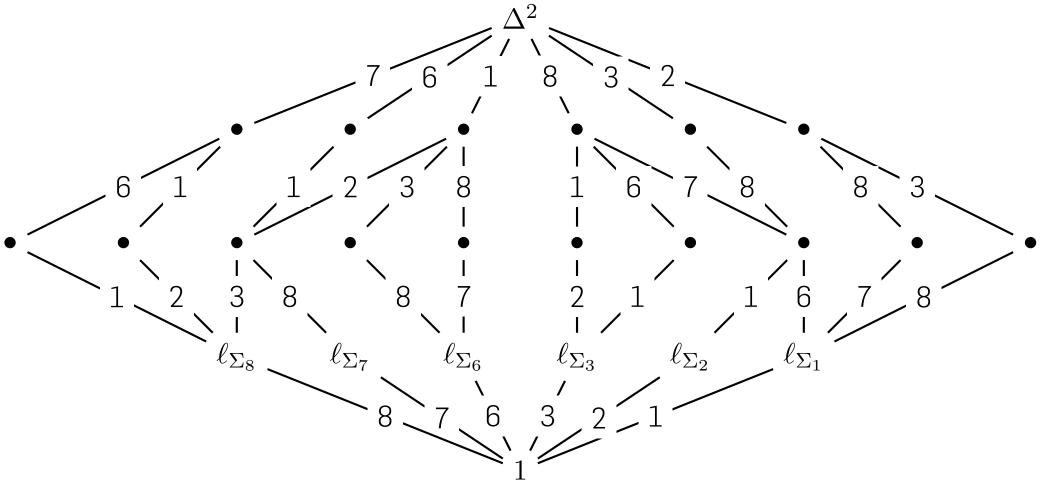
Definition

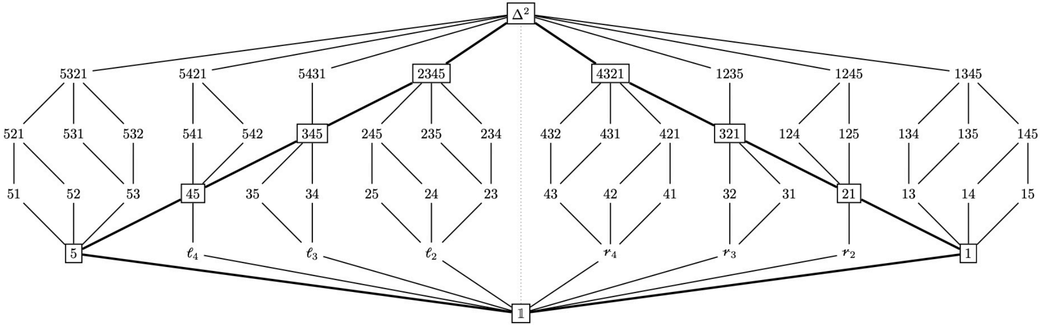
The pure shard monoid $\mathbf{P}^+(\mathcal{H}, B)$ ~~$\subset \pi_1(\mathrm{Sal}(\mathcal{H}), B)$~~ is generated by $\mathcal{L}_{\mathrm{III}}$.

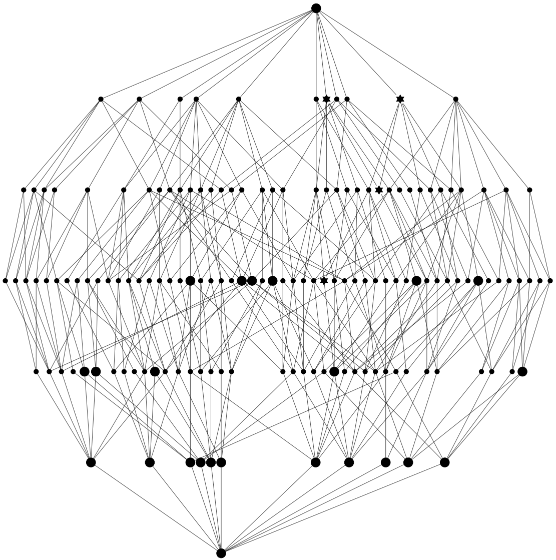
$\mathbf{P}^+(\mathcal{H}, B)$ is ordered by $p \leq p'$ if p is a prefix of p'.

The full twist Δ^2 lies in the center of $\pi_1(\mathrm{Sal}(\mathcal{H}), B)$.

Claim: the interval $[1, \Delta^2]_{\mathbf{P}^+}$ is an analogue of $\mathrm{Weak}(\mathcal{H}, B)$ and $\mathrm{Shard}(\mathcal{H}, B)$.


Δ^2
7 6 1 8 3 2
6 1 1 2 3 8 1 6 7 8 8 3
1 2 3 8 8 7 2 1 1 6 7 8
ℓ_{Σ_8} ℓ_{Σ_7} ℓ_{Σ_6} ℓ_{Σ_3} ℓ_{Σ_2} ℓ_{Σ_1}
8 7 6 3 2 1
1


Δ^2
5321
5421
5431
2345
4321
1235
1245
1345
521
531
532
541
542
345
245
235
234
432
431
421
321
124
125
134
135
145
51
52
53
45
35
34
25
24
23
43
42
41
32
31
21
13
14
15
5
ℓ_4
ℓ_3
ℓ_2
r_4
r_3
r_2
1
$\mathbb{1}$



POSET EMBEDDINGS

POSETS

poset	description	height	atoms
$\mathrm{Weak}(\mathcal{H}, B)$	tall and slender	$\mathcal{H}$	rank
$\mathrm{Shard}(\mathcal{H}, B)$	short and wide	rank	Ψ
$[1,\Delta^2]_{\mathbf{P}^+}$	tall and wide	$\mathcal{H}$	Ψ

POW: AN ORDERED ANALOGUE OF $\mathrm{inv}_{\mathcal{H},B}(C)$

Fix $\mathcal{H}$ central. For $C \in \mathcal{R}$ and a positive minimal gallery

$$B = C_0 \xrightarrow{e_1} C_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} C_{k-1} \xrightarrow{e_k} C_k = C,$$

define Pow : $\mathcal{R} \to \mathbf{P}^+(\mathcal{H}, B)$

$$\mathrm{Pow}(C) := \ell_{\Sigma(e_k)} \ell_{\Sigma(e_{k-1})} \cdots \ell_{\Sigma(e_1)}.$$

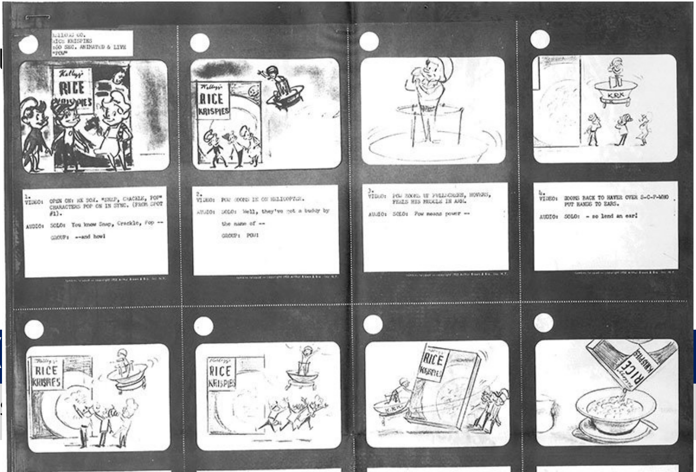
Theorem (Defant, W.)

Pow is a poset embedding of $\mathrm{Weak}(\mathcal{H}, B)$ in $[1, \Delta^2]_{\mathbf{P}^+}$.

POW: AN ORDERED ANALOGUE OF $\mathrm{inv}_{\mathcal{H},B}(C)$

Fix $\mathcal{H}$ central

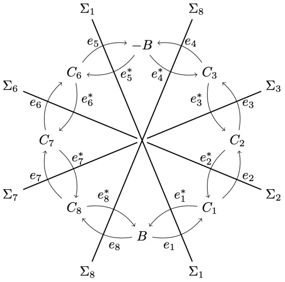
define Pow :

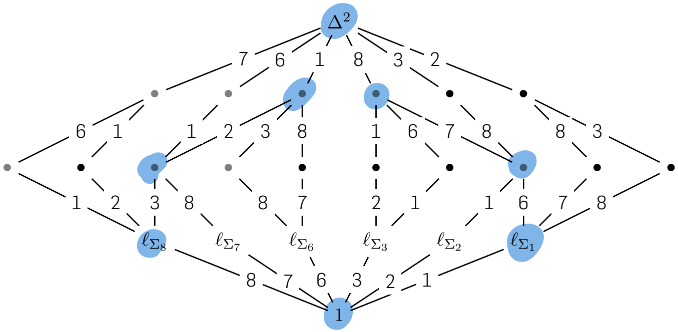


"Lost in the shuffle, however, was a fourth Rice Krispies elf named Pow! His short life is a time-capsule of an era when everyone was dreaming big."

Theorem (

Pow is a po


Σ_1
Σ_8
Σ_6
Σ_3
Σ_7
Σ_2
Σ_8
Σ_1
$-B$
B
C_1
C_2
C_3
C_6
C_7
C_8
e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8
e_1^* e_2^* e_3^* e_4^* e_5^* e_6^* e_7^* e_8^*


Δ^2
1
ℓ_{Σ_8} ℓ_{Σ_7} ℓ_{Σ_6} ℓ_{Σ_3} ℓ_{Σ_2} ℓ_{Σ_1}

CRACKLE: AN ORDERED ANALOGUE OF $\mathrm{inv}_{\mathrm{III},B}(C)$

Fix $\mathcal{H}$ simplicial. For $C \in \mathcal{R}$ and a positive minimal gallery

$$\mathrm{Pop}(C) = E_0 \xrightarrow{e_1} E_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} E_{k-1} \xrightarrow{e_k} E_k = C$$

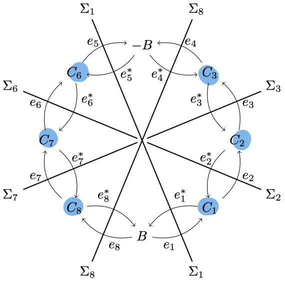
define Crackle: $\mathcal{R} \to \pi_1(\mathrm{Sal}(\mathcal{H}), B)$ by

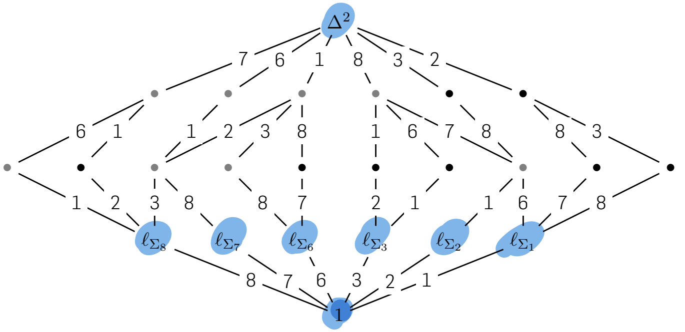
$$\mathrm{Crackle}(C) := \ell_{\Sigma(e_k)} \ell_{\Sigma(e_{k-1})} \cdots \ell_{\Sigma(e_1)}.$$

Crackle generalizes Salvetti's loops ℓ_Σ beyond 1-cells.

Theorem

Crackle is a poset embedding of $\mathrm{Shard}(\mathcal{H}, B)$ into the interval $[1, \Delta^2]_{\mathbf{P}^+}$.


Σ_1 Σ_8
e_5 $-B$ e_4
C_6 e_5^* e_4^* C_3
Σ_6 e_6^* e_3^* Σ_3
e_6 e_3
C_7 C_2
e_7^* e_2^*
e_7 e_2
Σ_7 e_8^* e_1^* Σ_2
C_8 C_1
B
e_8 e_1
Σ_8 Σ_1


Δ^2
7 6 1 8 3 2
6 1 1 2 3 8 1 6 7 8 8 3
1 2 3 8 8 7 2 1 1 6 7 8
ℓ_{Σ_8} ℓ_{Σ_7} ℓ_{Σ_6} ℓ_{Σ_3} ℓ_{Σ_2} ℓ_{Σ_1}
8 7 6 3 2 1
1

SNAP = CRACKLE·POP

BRAID GROUPS

Fix $\mathcal{H}$ a reflection arrangement of a finite Coxeter group W.

$\mathbf{P}(W) := \pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B)$ is the pure braid group of W

$\mathbf{B}(W) := \pi_1((\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})/W, x_B)$ is the braid group of W.

The Coxeter group W fits into an exact sequence with its braid and pure braid groups:

$$1 \to \mathbf{P}(W) \to \mathbf{B}(W) \to W \to 1.$$

Write $\mathbf{w}$ for the usual lift of $w \in W$ to $\mathbf{B}^+(W)$.

SALVETTI'S LOOPS

Theorem (Defant, W.)

For a real central arrangement $\mathcal{H}$,

$$\ell_e \simeq \ell_f \text{ iff } \Sigma(e) = \Sigma(f).$$

We rephrase when $\mathcal{H}$ is the reflection arrangement of W.

Corollary

Suppose $u, v \in W$ and $s, t \in S$ satisfy $u \lessdot us$ and $v \lessdot vt$. Then

$$\mathbf{usu}^{-1} = \mathbf{vtv}^{-1} \text{ iff } \Sigma(\, u \lessdot us\,) = \Sigma(\, v \lessdot vt\,).$$

SNAP: EMBEDDING SHARD IN WEAK

For $w \in W$, write

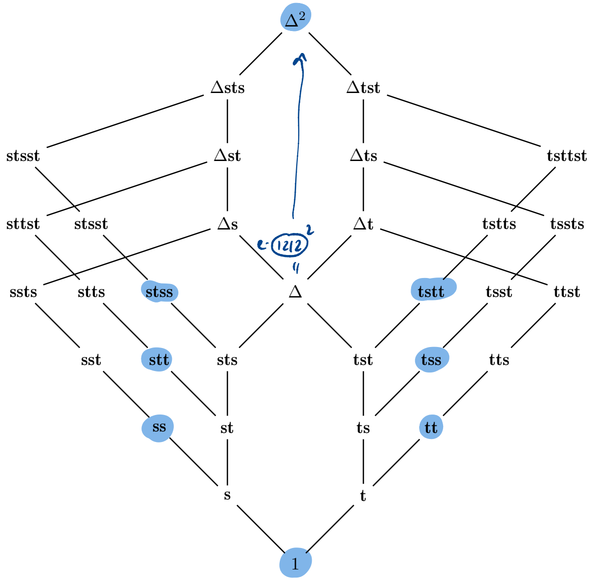
- $\mathrm{des}(w)$ for the right descent set of w,
- $w_\circ(\mathrm{des}(w))$ for the longest element of the parabolic subgroup of W generated by $\mathrm{des}(w)$, and
- $\mathbf{w}$ and $\mathbf{w}_\circ(\mathrm{des}(w))$ for the usual lifts of w and $w_\circ(\mathrm{des}(w))$ to $\mathbf{B}^+(W)$.

Define

$$\mathsf{Snap}(w) := \mathsf{Pop}(\mathbf{w}) \cdot (\mathbf{w}_\circ(\mathrm{des}(w)))^2.$$

Theorem (Defant, W.)

The map Snap is a poset embedding from $\mathrm{Shard}(W)$ into $[1, \Delta^2]_{\mathbf{B}^+}$.



Δ^2
Δsts
Δtst
stsst
Δst
Δts
tsttst
sttst
stsst
Δs
Δt
tstts
tssts
ssts
stts
stss
Δ
tstt
tsst
ttst
sst
stt
sts
tst
tss
tts
ss
st
ts
tt
s
t
1

Interpret everything in $\mathbf{B}(W)$ (since $\mathbf{P}^+(W) \subseteq \mathbf{B}(W)$):

- $\text{Pop}(\mathbf{w}) = \mathbf{w} \cdot \mathbf{w}_\circ(\text{des}(w))^{-1}$
- $\text{Crackle}(w) = \text{Pop}(\mathbf{w}) \cdot (\mathbf{w}_\circ(\text{des}(w)))^2 \cdot \text{Pop}(\mathbf{w})^{-1}$
- $\text{Snap}(w) := \text{Pop}(\mathbf{w}) \cdot (\mathbf{w}_\circ(\text{des}(w)))^2$

Corollary

$$\text{Snap}(w) = \text{Crackle}(w) \cdot \text{Pop}(\mathbf{w}).$$