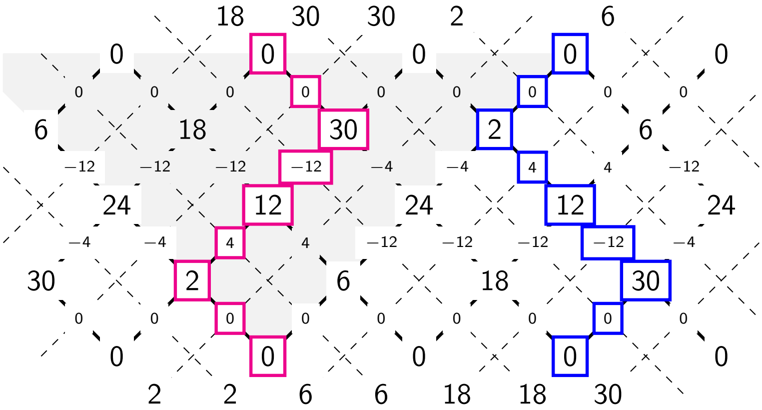
Spherical Heronian frieze patterns



Katie Waddle
University of Michigan

Frieze patterns: an algebraic phenomenon

Survey: [Morier-Genoud, 2015]

Definition [Coxeter, 1971, Conway and Coxeter, 1973]

A frieze pattern is a bi-infinite array of numbers bordered by a row of 0's then a row of 1's, such that for each 2×2 block $\begin{smallmatrix} & b & \\ a & & c \\ & d & \end{smallmatrix}$, we have $ac - bd = 1$.

$$
\begin{array}{cccccccccccccc}
\cdots & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & & x_1 & & & & & & & & & & & & \cdots \\
\cdots & & & x_2 & & & & & & & & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
$$

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$$
\begin{array}{cccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & x_1 & & \frac{x_2+1}{x_1} & & & & & & & & & & \cdots \\
\cdots & & x_2 & & & & & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
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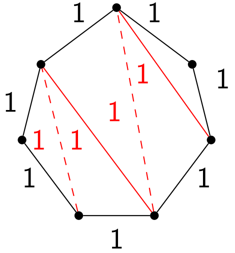
Frieze patterns: a combinatorial interpretation

Theorem [Coxeter, 1971, Conway and Coxeter, 1973]

Integral frieze patterns of width $m = n - 3$ are in bijection with triangulations of a convex n-gon.



	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

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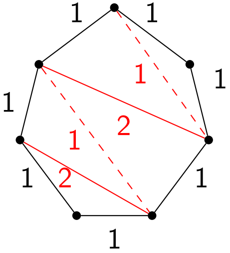

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

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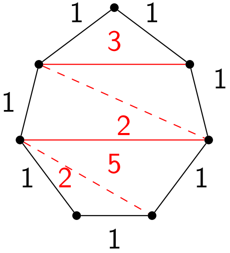

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

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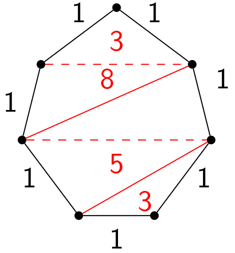

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

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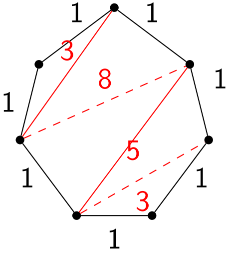

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

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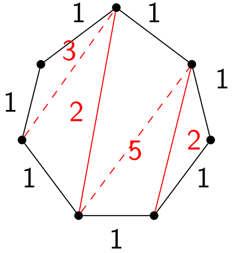

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
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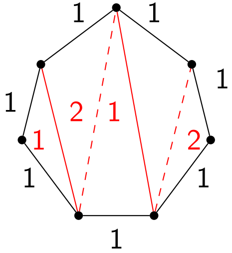

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
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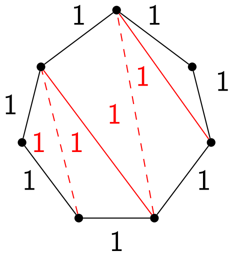

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
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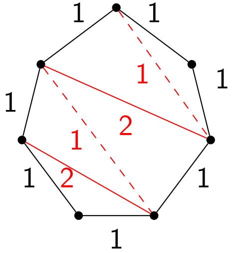

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

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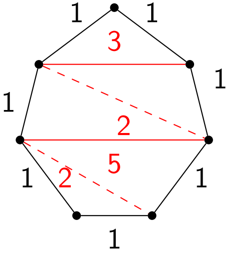

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

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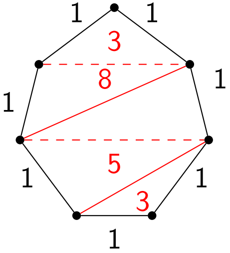

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
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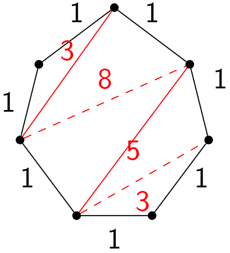


```
   1     1     1     1     1     1     1     1
1     2     3     2     1     3     3     1
   1     5     5     1     2     8     2     1
1     2     8     2     1     5     5     1
   1     3     3     1     2     3     2     1
1     1     1     1     1     1     1     1
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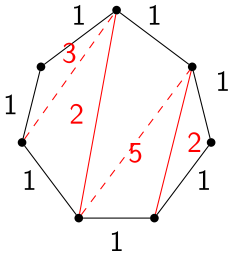

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
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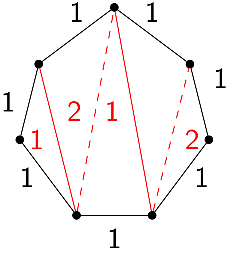

	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
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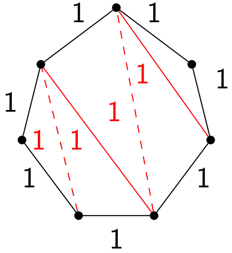
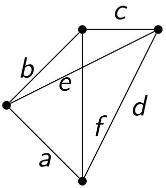


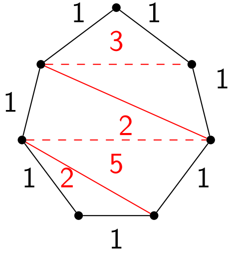
	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

Frieze patterns: towards geometry

Theorem [Ptolemy]

When a quadrilateral as labeled below is inscribed in a circle, $ac + bd = ef$.





	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
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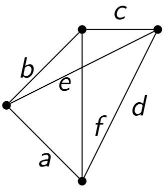
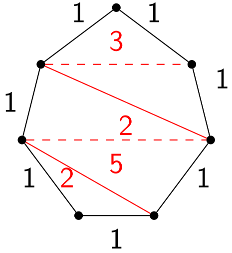
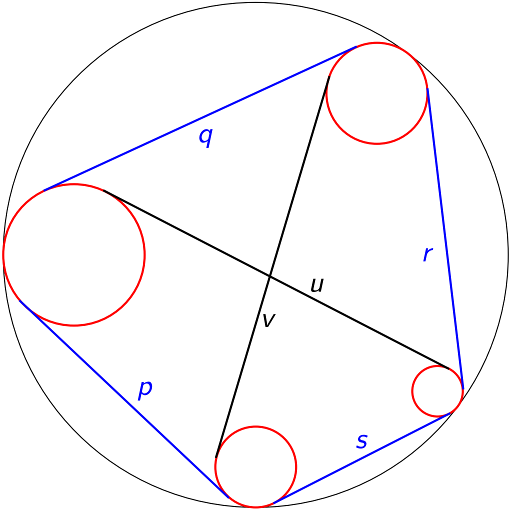




	1		1		1		1		1		1		1		1
1		2		3		2		1		3		3		1	
	1		5		5		1		2		8		2		1
1		2		8		2		1		5		5		1	
	1		3		3		1		2		3		2		1
1		1		1		1		1		1		1		1	

Frieze patterns: actual geometry

Theorem [Casey, 1866]



$$uv = pr + qs$$

Distance geometry: the setup

Definition

Distance geometry is the study of point configurations via measurements of pairwise distances between the points.

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Question

Let $\mathbf{V}_2$ be 2D Euclidean space. What $O(n)$ collection of measurements uniquely determines a set of n points in $\mathbf{V}_2$, considered up to oriented isometry?

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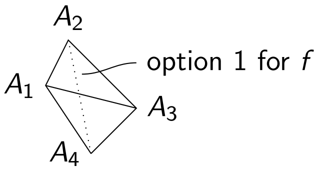
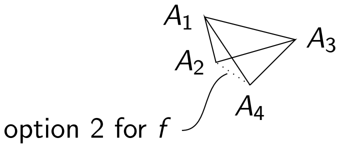
Can we use data corresponding to a triangulation?





Distance geometry: two solution approaches

How to supplement triangulation data?

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↙

add bracing edges

Distance geometry: two solution approaches

How to supplement triangulation data?

add bracing edges

measure oriented areas of triangles

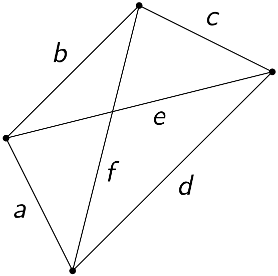
Distance geometry: the Cayley-Menger determinant

What algebraic relations hold for n points in $\mathbf{V}_2$?

Distance geometry: the Cayley-Menger determinant

What algebraic relations hold for n points in $\mathbf{V}_2$?

Quadrilateral:



Distance geometry: the Cayley-Menger determinant

What algebraic relations hold for n points in $\mathbf{V}_2$?

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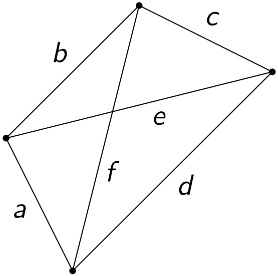

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & b & e & a \\ 1 & b & 0 & c & f \\ 1 & e & c & 0 & d \\ 1 & a & f & d & 0 \end{bmatrix} = 0$$

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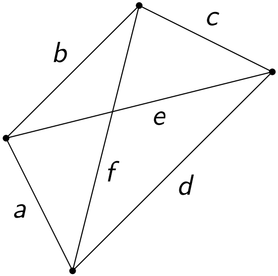

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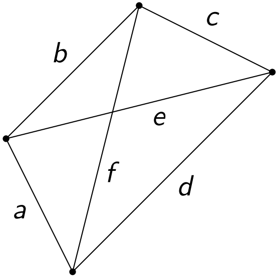

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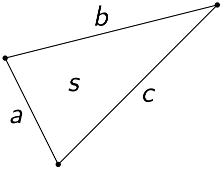
Triangle [Heron, 60]:

$$(\underbrace{4 \cdot \text{area of } A_1 A_2 A_3}_{S})^2 = - \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & b & e \\ 1 & b & 0 & c \\ 1 & e & c & 0 \end{bmatrix}$$

Distance geometry: unique determination



Proposition [Fomin and Setiabrata, 2020]

For a triangle in $\mathbf{V}_2$, the measurements a, b, c, s satisfy

$$s^2 = -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc = H(a, b, c).$$

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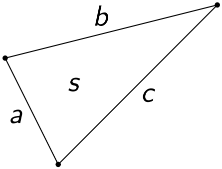

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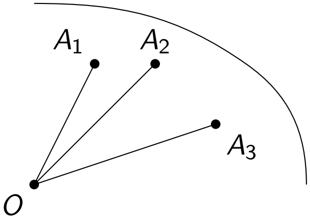

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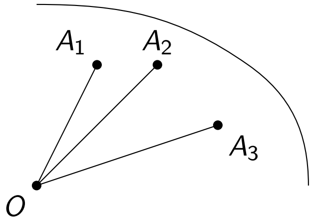

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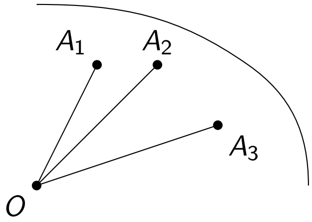

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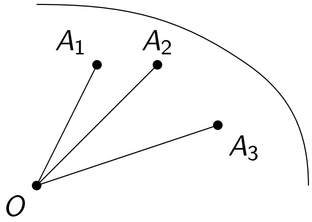

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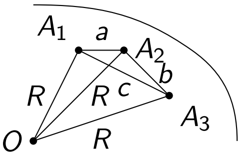
Definition

$S^K(A_1, A_2, A_3) = \frac{12}{R} V(OA_1A_2A_3)$

Distance geometry: unique determination on **S**



Proposition [W., 2024+]

For a triangle on **S** ($K = \frac{1}{R^2}$) the measurements a, b, c, s satisfy

$$(S^K)^2 = -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc - Kabc = H^K(a, b, c)$$

$$= \frac{1}{2R^2} \det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & c & R^2 \\ 1 & a & 0 & b & R^2 \\ 1 & c & b & 0 & R^2 \\ 1 & R^2 & R^2 & R^2 & 0 \end{bmatrix}.$$

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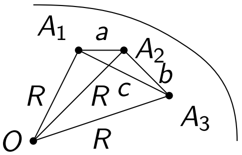

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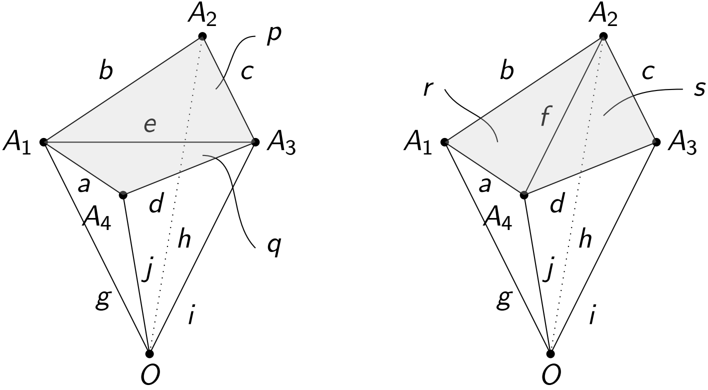
Such a triangle is unique up to orientation preserving isometry.

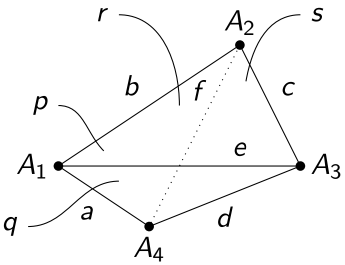
Distance geometry: algebraic formulas

Question

Given measurement data corresponding to a triangulation of an n-gon (squared side distances and S^K measurements), what are the (algebraic) formulas for all the other data?



Distance geometry: algebraic formulas (Euclidean case!)



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$$s^2 = H(c, f, d)$$

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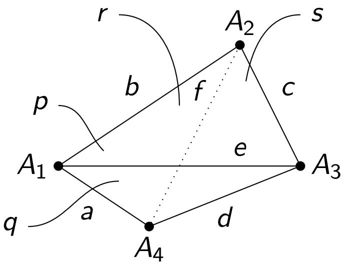

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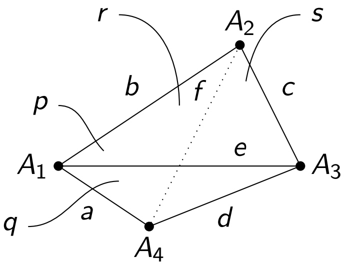

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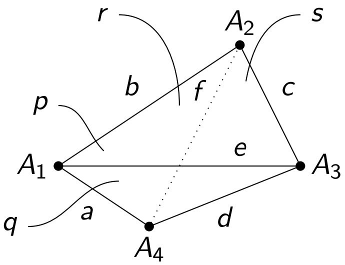

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Distance geometry: algebraic formulas (spherical case!)

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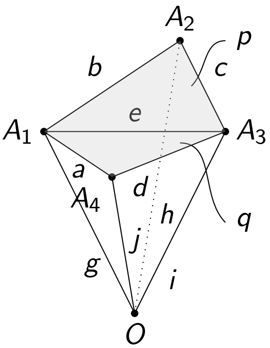

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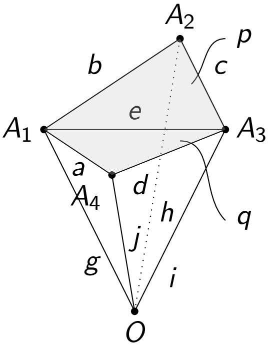

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$$p + q = r + s + \frac{K}{2}(dp + cq - es)$$
$$p + q = r + s + \frac{K}{2}(fq - dr - as)$$

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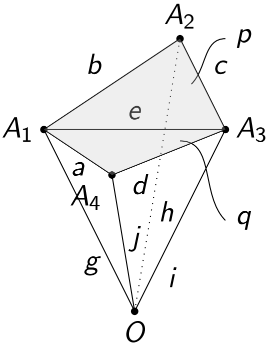

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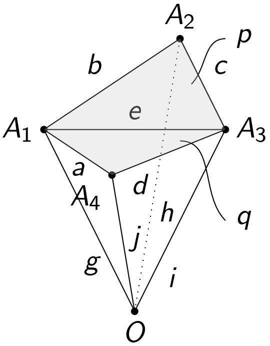

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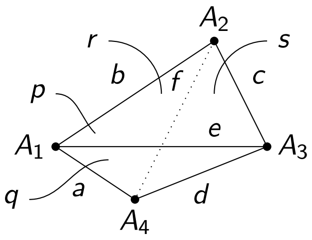
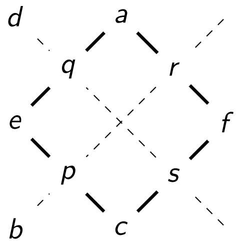
Spherical Heronian friezes: an algebraic phenomenon





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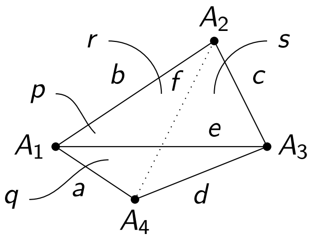
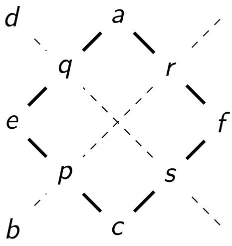




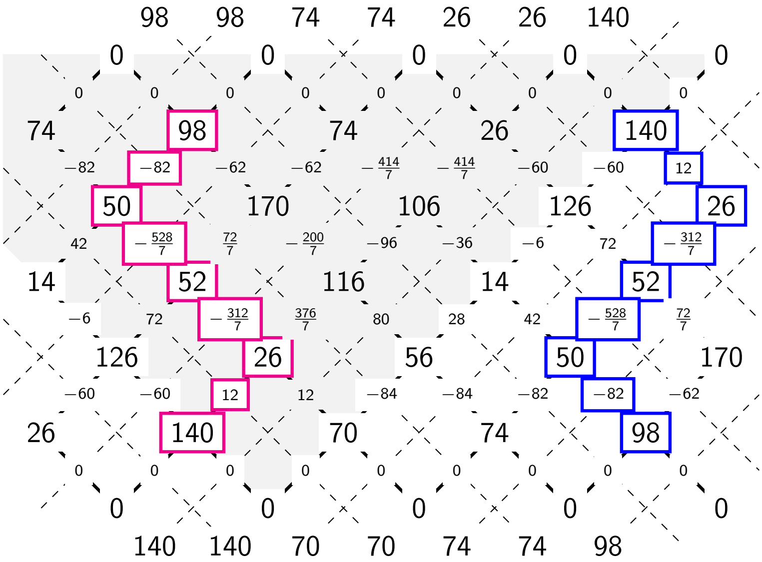
Proposition [W., 2024+]

Let (a, b, c, d, e, p, q) be a 7-tuple satisfying $(*)$.
Assuming $e \notin \left\{0, \frac{4}{K}\right\}$, there exist unique $f, r, s \in \mathbb{C}$ such that $(a, b, c, d, e, f, p, q, r, s)$ is a spherical Heronian diamond. Namely,

$$f = \frac{(p+q)^2+(a-b+c-d)^2-Ke(a-b)(c-d)}{4e\left(1-\frac{Ke}{4}\right)},$$

$$r = \frac{p\left(e+a-d-\frac{Kae}{2}\right)+q\left(e-c+b-\frac{Kbe}{2}\right)}{2e\left(1-\frac{Ke}{4}\right)} \text{ and } s = \frac{p\left(e-a+d-\frac{Kde}{2}\right)+q\left(e+c-b-\frac{Kce}{2}\right)}{2e\left(1-\frac{Ke}{4}\right)}.$$

Spherical Heronian friezes: an algebraic phenomenon



A spherical Heronian frieze with $K = \frac{1}{7^2}$.

Spherical Heronian friezes: an algebraic phenomenon

Proposition [W., 2024+] ("entries are nice")

Let $\mathbf{z}_\pi$ be the initial data associated with a traversing path π in a spherical Heronian frieze $\mathbf{z}$. Then each entry of $\mathbf{z}$ can be written as a rational function of $\mathbf{z}_\pi$ (with nice denominators).

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Proposition [W., 2024+] ("all come from polygons")

If $\mathbf{z}$ is an order n spherical Heronian frieze of sufficient genericity, there exists a unique n-gon P such that $\mathbf{z}$ arises from P.

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Theorem [W., 2024+] ("always get periodicity")

Let $\mathbf{z}_\pi$ be a sufficiently generic collection of numbers associated to a traversing path π. Assume these numbers satisfy $(*)$. Propagate outwards using the formulas to obtain a spherical Heronian frieze. Then $\mathbf{z}$ is periodic (glide symmetry) and exhibits a version of the Laurent phenomenon.

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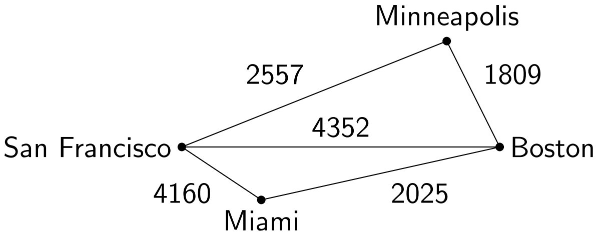
What next?

Question

One application of spherical Heronian friezes is to measuring and computing distances on a globe.



What if we have a point (or points) on a different sphere, like a satellite?